Chapter 3

Cohomology

Cohomology is an algebraic variant of homology, the result of a simple dualization in the definition. Not surprisingly, the cohomology groups $H^i(X)$ satisfy axioms much like the axioms for homology, except that induced homomorphisms go in the opposite direction as a result of the dualization. The basic distinction between homology and cohomology is thus that cohomology groups are contravariant functors while homology groups are covariant. In terms of intrinsic information, however, there is not a big difference between homology groups and cohomology groups. The homology groups of a space determine its cohomology groups, and the converse holds at least when the homology groups are finitely generated.

What is a little surprising is that contravariance leads to extra structure in cohomology. This first appears in a natural product, called *cup product*, which makes the cohomology groups of a space into a ring. This is an extremely useful piece of additional structure, and much of this chapter is devoted to studying cup products, which are considerably more subtle than the additive structure of cohomology.

How does contravariance lead to a product in cohomology that is not present in homology? Actually there is a natural product in homology, but it takes the somewhat different form of a map $H_i(X) \times H_j(Y) \longrightarrow H_{i+j}(X \times Y)$ called the *cross product*. If both X and Y are CW complexes, this cross product in homology is induced from a map of cellular chains sending a pair (e^i, e^j) consisting of a cell of X and a cell of Y to the product cell $e^i \times e^j$ in $X \times Y$. The details of the construction are described in §3.B. Taking $X = Y$, we thus have the first half of a hypothetical product

$$H_i(X) \times H_j(X) \longrightarrow H_{i+j}(X \times X) \longrightarrow H_{i+j}(X)$$

The difficulty is in defining the second map. The natural thing would be for this to be induced by a map $X \times X \to X$. The multiplication map in a topological group, or more generally an H–space, is such a map, and the resulting *Pontryagin product* can be quite useful when studying these spaces, as we show in §3.C. But for general X, the only

natural maps $X \times X \to X$ are the projections onto one of the factors, and since these projections collapse the other factor to a point, the resulting product in homology is rather trivial.

With cohomology, however, the situation is better. One still has a cross product $H^i(X) \times H^j(Y) \to H^{i+j}(X \times Y)$ constructed in much the same way as in homology, so one can again take $X = Y$ and get the first half of a product

$$H^i(X) \times H^j(X) \longrightarrow H^{i+j}(X \times X) \longrightarrow H^{i+j}(X)$$

But now by contravariance the second map would be induced by a map $X \to X \times X$, and there is an obvious candidate for this map, the diagonal map $\Delta(x) = (x, x)$. This turns out to work very nicely, giving a well-behaved product in cohomology, the cup product.

Another sort of extra structure in cohomology whose existence is traceable to contravariance is provided by cohomology operations. These make the cohomology groups of a space into a module over a certain rather complicated ring. Cohomology operations lie at a depth somewhat greater than the cup product structure, so we defer their study to §4.L.

The extra layer of algebra in cohomology arising from the dualization in its definition may seem at first to be separating it further from topology, but there are many topological situations where cohomology arises quite naturally. One of these is Poincaré duality, the topic of the third section of this chapter. Another is obstruction theory, covered in §4.3. Characteristic classes in vector bundle theory (see [Milnor & Stasheff 1974] or [VBKT]) provide a further instance.

From the viewpoint of homotopy theory, cohomology is in some ways more basic than homology. As we shall see in §4.3, cohomology has a description in terms of homotopy classes of maps that is very similar to, and in a certain sense dual to, the definition of homotopy groups. There is an analog of this for homology, described in §4.F, but the construction is more complicated.

The Idea of Cohomology

Let us look at a few low-dimensional examples to get an idea of how one might be led naturally to consider cohomology groups, and to see what properties of a space they might be measuring. For the sake of simplicity we consider simplicial cohomology of Δ-complexes, rather than singular cohomology of more general spaces.

Taking the simplest case first, let X be a 1-dimensional Δ-complex, or in other words an oriented graph. For a fixed abelian group G, the set of all functions from vertices of X to G also forms an abelian group, which we denote by $\Delta^0(X; G)$. Similarly the set of all functions assigning an element of G to each edge of X forms an abelian group $\Delta^1(X; G)$. We will be interested in the homomorphism $\delta : \Delta^0(X; G) \to \Delta^1(X; G)$ sending $\varphi \in \Delta^0(X; G)$ to the function $\delta\varphi \in \Delta^1(X; G)$ whose value on an oriented

edge $[v_0, v_1]$ is the difference $\varphi(v_1) - \varphi(v_0)$. For example, X might be the graph formed by a system of trails on a mountain, with vertices at the junctions between trails. The function φ could then assign to each junction its elevation above sea level, in which case $\delta\varphi$ would measure the net change in elevation along the trail from one junction to the next. Or X might represent a simple electrical circuit with φ measuring voltages at the connection points, the vertices, and $\delta\varphi$ measuring changes in voltage across the components of the circuit, represented by edges.

Regarding the map $\delta : \Delta^0(X; G) \to \Delta^1(X; G)$ as a chain complex with 0's before and after these two terms, the homology groups of this chain complex are by definition the simplicial cohomology groups of X, namely $H^0(X; G) = \operatorname{Ker}\delta \subset \Delta^0(X; G)$ and $H^1(X; G) = \Delta^1(X; G) / \operatorname{Im}\delta$. For simplicity we are using here the same notation as will be used for singular cohomology later in the chapter, in anticipation of the theorem that the two theories coincide for Δ-complexes, as we show in §3.1.

The group $H^0(X; G)$ is easy to describe explicitly. A function $\varphi \in \Delta^0(X; G)$ has $\delta\varphi = 0$ iff φ takes the same value at both ends of each edge of X. This is equivalent to saying that φ is constant on each component of X. So $H^0(X; G)$ is the group of all functions from the set of components of X to G. This is a direct product of copies of G, one for each component of X.

The cohomology group $H^1(X; G) = \Delta^1(X; G) / \operatorname{Im}\delta$ will be trivial iff the equation $\delta\varphi = \psi$ has a solution $\varphi \in \Delta^0(X; G)$ for each $\psi \in \Delta^1(X; G)$. Solving this equation means deciding whether specifying the change in φ across each edge of X determines an actual function $\varphi \in \Delta^0(X; G)$. This is rather like the calculus problem of finding a function having a specified derivative, with the difference operator δ playing the role of differentiation. As in calculus, if a solution of $\delta\varphi = \psi$ exists, it will be unique up to adding an element of the kernel of δ, that is, a function that is constant on each component of X.

The equation $\delta\varphi = \psi$ is always solvable if X is a tree since if we choose arbitrarily a value for φ at a basepoint vertex v_0, then if the change in φ across each edge of X is specified, this uniquely determines the value of φ at every other vertex v by induction along the unique path from v_0 to v in the tree. When X is not a tree, we first choose a maximal tree in each component of X. Then, since every vertex lies in one of these maximal trees, the values of ψ on the edges of the maximal trees determine φ uniquely up to a constant on each component of X. But in order for the equation $\delta\varphi = \psi$ to hold, the value of ψ on each edge not in any of the maximal trees must equal the difference in the already-determined values of φ at the two ends of the edge. This condition need not be satisfied since ψ can have arbitrary values on these edges. Thus we see that the cohomology group $H^1(X; G)$ is a direct product of copies of the group G, one copy for each edge of X not in one of the chosen maximal trees. This can be compared with the homology group $H_1(X; G)$ which consists of a direct *sum* of copies of G, one for each edge of X not in one of the maximal trees.

Note that the relation between $H^1(X;G)$ and $H_1(X;G)$ is the same as the relation between $H^0(X;G)$ and $H_0(X;G)$, with $H^0(X;G)$ being a direct product of copies of G and $H_0(X;G)$ a direct sum, with one copy for each component of X in either case.

Now let us move up a dimension, taking X to be a 2-dimensional Δ-complex. Define $\Delta^0(X;G)$ and $\Delta^1(X;G)$ as before, as functions from vertices and edges of X to the abelian group G, and define $\Delta^2(X;G)$ to be the functions from 2-simplices of X to G. A homomorphism $\delta:\Delta^1(X;G)\to\Delta^2(X;G)$ is defined by $\delta\psi([v_0,v_1,v_2]) = \psi([v_0,v_1]) + \psi([v_1,v_2]) - \psi([v_0,v_2])$, a signed sum of the values of ψ on the three edges in the boundary of $[v_0,v_1,v_2]$, just as $\delta\varphi([v_0,v_1])$ for $\varphi \in \Delta^0(X;G)$ was a signed sum of the values of φ on the boundary of $[v_0,v_1]$. The two homomorphisms $\Delta^0(X;G) \xrightarrow{\delta} \Delta^1(X;G) \xrightarrow{\delta} \Delta^2(X;G)$ form a chain complex since for $\varphi \in \Delta^0(X;G)$ we have $\delta\delta\varphi = (\varphi(v_1)-\varphi(v_0))+(\varphi(v_2)-\varphi(v_1))-(\varphi(v_2)-\varphi(v_0)) = 0$. Extending this chain complex by 0's on each end, the resulting homology groups are by definition the cohomology groups $H^i(X;G)$.
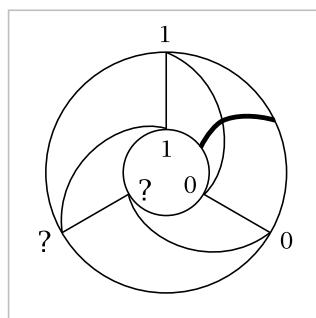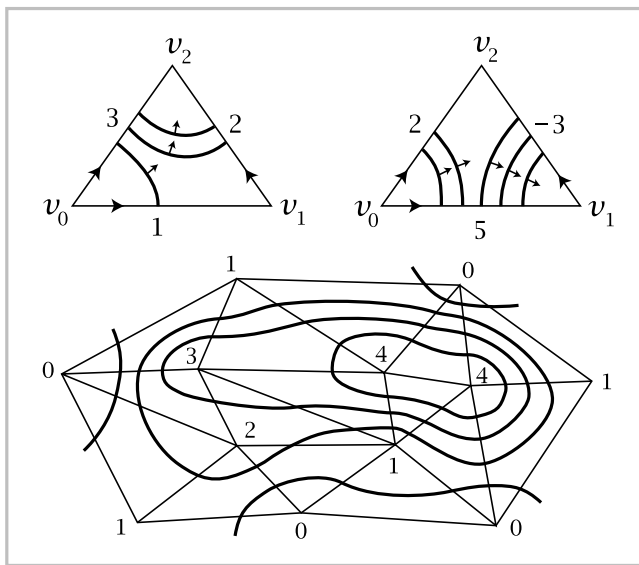
The formula for the map $\delta:\Delta^1(X;G)\to\Delta^2(X;G)$ can be looked at from several different viewpoints. Perhaps the simplest is the observation that $\delta\psi = 0$ iff ψ satisfies the additivity property $\psi([v_0,v_2]) = \psi([v_0,v_1]) + \psi([v_1,v_2])$, where we think of the edge $[v_0,v_2]$ as the sum of the edges $[v_0,v_1]$ and $[v_1,v_2]$. Thus $\delta\psi$ measures the deviation of ψ from being additive.

From another point of view, $\delta\psi$ can be regarded as an obstruction to finding $\varphi \in \Delta^0(X;G)$ with $\psi = \delta\varphi$, for if $\psi = \delta\varphi$ then $\delta\psi = 0$ since $\delta\delta\varphi = 0$ as we saw above. We can think of $\delta\psi$ as a local obstruction to solving $\psi = \delta\varphi$ since it depends only on the values of ψ within individual 2-simplices of X. If this local obstruction vanishes, then ψ defines an element of $H^1(X;G)$ which is zero iff $\psi = \delta\varphi$ has an actual solution. This class in $H^1(X;G)$ is thus the global obstruction to solving $\psi = \delta\varphi$. This situation is similar to the calculus problem of determining whether a given vector field is the gradient vector field of some function. The local obstruction here is the vanishing of the curl of the vector field, and the global obstruction is the vanishing of all line integrals around closed loops in the domain of the vector field.

The condition $\delta\psi = 0$ has an interpretation of a more geometric nature when X is a surface and the group G is $\mathbb{Z}$ or $\mathbb{Z}_2$. Consider first the simpler case $G = \mathbb{Z}_2$. The condition $\delta\psi = 0$ means that the number of times that ψ takes the value 1 on the edges of each 2-simplex is even, either 0 or 2. This means we can associate to ψ a

collection C_ψ of disjoint curves in X crossing the 1-skeleton transversely, such that the number of intersections of C_ψ with each edge is equal to the value of ψ on that edge. If $\psi = \delta\varphi$ for some φ, then the curves of C_ψ divide X into two regions X_0 and X_1 where the subscript indicates the value of φ on all vertices in the region.
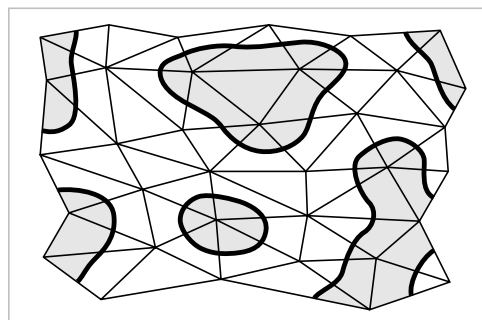
When $G = \mathbb{Z}$ we can refine this construction by building C_ψ from a number of arcs in each 2-simplex, each arc having a transverse orientation, the orientation which agrees or disagrees with the orientation of each edge according to the sign of the value of ψ on the edge, as in the figure at the right. The resulting collection C_ψ of disjoint curves in X can be thought of as something like level curves for a function φ with $\delta\varphi = \psi$, if such a function exists. The value of φ changes by 1 each time a curve of C_ψ is crossed. For example, if X is a disk then we will show that $H^1(X;\mathbb{Z}) = 0$, so $\delta\psi = 0$ implies $\psi = \delta\varphi$ for some φ, hence every transverse curve system C_ψ forms the level curves of a function φ. On the other hand, if X is an annulus then this need no longer be true, as illustrated in the example shown in the figure at the left, where the equation $\psi = \delta\varphi$ obviously has no solution even though $\delta\psi = 0$. By identifying the inner and outer boundary circles of this annulus we obtain a similar example on the torus. Even with $G = \mathbb{Z}_2$ the equation $\psi = \delta\varphi$ has no solution since the curve C_ψ does not separate X into two regions X_0 and X_1.

The key to relating cohomology groups to homology groups is the observation that a function from i-simplices of X to G is equivalent to a homomorphism from the simplicial chain group $\Delta_i(X)$ to G. This is because $\Delta_i(X)$ is free abelian with basis the i-simplices of X, and a homomorphism with domain a free abelian group is uniquely determined by its values on basis elements, which can be assigned arbitrarily. Thus we have an identification of $\Delta^i(X;G)$ with the group $\mathrm{Hom}(\Delta_i(X),G)$ of homomorphisms $\Delta_i(X) \to G$, which is called the *dual group* of $\Delta_i(X)$. There is also a simple relationship of duality between the homomorphism $\delta : \Delta^i(X;G) \to \Delta^{i+1}(X;G)$ and the boundary homomorphism $\partial : \Delta_{i+1}(X) \to \Delta_i(X)$. The general formula for δ is

$$\delta\varphi([v_0, \cdots, v_{i+1}]) = \sum_j (-1)^j \varphi([v_0, \cdots, \hat{v}_j, \cdots, v_{i+1}])$$

and the latter sum is just $\varphi(\partial[v_0, \cdots, v_{i+1}])$. Thus we have $\delta\varphi = \varphi\partial$. In other words, δ sends each $\varphi \in \mathrm{Hom}(\Delta_i(X),G)$ to the composition $\Delta_{i+1}(X) \xrightarrow{\partial} \Delta_i(X) \xrightarrow{\varphi} G$, which in the language of linear algebra means that δ is the dual map of ∂.

Thus we have the algebraic problem of understanding the relationship between the homology groups of a chain complex and the homology groups of the dual complex obtained by applying the functor $C \mapsto \mathrm{Hom}(C,G)$. This is the first topic of the chapter.

3.1 Cohomology Groups

Homology groups $H_n(X)$ are the result of a two-stage process: First one forms a chain complex $\cdots \longrightarrow C_n \xrightarrow{\partial} C_{n-1} \longrightarrow \cdots$ of singular, simplicial, or cellular chains, then one takes the homology groups of this chain complex, $\operatorname{Ker} \partial / \operatorname{Im} \partial$. To obtain the cohomology groups $H^n(X; G)$ we interpolate an intermediate step, replacing the chain groups C_n by the dual groups $\operatorname{Hom}(C_n, G)$ and the boundary maps ∂ by their dual maps δ, before forming the cohomology groups $\operatorname{Ker} \delta / \operatorname{Im} \delta$. The plan for this section is first to sort out the algebra of this dualization process and show that the cohomology groups are determined algebraically by the homology groups, though in a somewhat subtle way. Then after this algebraic excursion we will define the cohomology groups of spaces and show that these satisfy basic properties very much like those for homology. The payoff for all this formal work will begin to be apparent in subsequent sections.

The Universal Coefficient Theorem

Let us begin with a simple example. Consider the chain complex

$$0 \longrightarrow \underset{\underset{C_3}{\|}}{\mathbb{Z}} \xrightarrow{0} \underset{\underset{C_2}{\|}}{\mathbb{Z}} \xrightarrow{2} \underset{\underset{C_1}{\|}}{\mathbb{Z}} \xrightarrow{0} \underset{\underset{C_0}{\|}}{\mathbb{Z}} \longrightarrow 0$$

where $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ is the map $x \mapsto 2x$. If we dualize by taking $\operatorname{Hom}(-, G)$ with $G = \mathbb{Z}$, we obtain the cochain complex

$$0 \longleftarrow \underset{\underset{C_3^*}{\|}}{\mathbb{Z}} \xleftarrow{0} \underset{\underset{C_2^*}{\|}}{\mathbb{Z}} \xleftarrow{2} \underset{\underset{C_1^*}{\|}}{\mathbb{Z}} \xleftarrow{0} \underset{\underset{C_0^*}{\|}}{\mathbb{Z}} \longleftarrow 0$$

In the original chain complex the homology groups are $\mathbb{Z}$'s in dimensions 0 and 3, together with a $\mathbb{Z}_2$ in dimension 1. The homology groups of the dual cochain complex, which are called cohomology groups to emphasize the dualization, are again $\mathbb{Z}$'s in dimensions 0 and 3, but the $\mathbb{Z}_2$ in the 1-dimensional homology of the original complex has shifted up a dimension to become a $\mathbb{Z}_2$ in 2-dimensional cohomology.

More generally, consider any chain complex of finitely generated free abelian groups. Such a chain complex always splits as the direct sum of elementary complexes of the forms $0 \to \mathbb{Z} \to 0$ and $0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to 0$, according to Exercise 43 in §2.2. Applying $\operatorname{Hom}(-, \mathbb{Z})$ to this direct sum of elementary complexes, we obtain the direct sum of the corresponding dual complexes $0 \leftarrow \mathbb{Z} \leftarrow 0$ and $0 \leftarrow \mathbb{Z} \xleftarrow{m} \mathbb{Z} \leftarrow 0$. Thus the cohomology groups are the same as the homology groups except that torsion is shifted up one dimension. We will see later in this section that the same relation between homology and cohomology holds whenever the homology groups are finitely generated, even when the chain groups are not finitely generated. It would also be quite easy to

see in this example what happens if $\text{Hom}(-, \mathbb{Z})$ is replaced by $\text{Hom}(-, G)$, since the dual elementary cochain complexes would then be $0 \leftarrow G \leftarrow 0$ and $0 \leftarrow G \xleftarrow{m} G \leftarrow 0$.

Consider now a completely general chain complex C of free abelian groups

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \longrightarrow \cdots$$

To dualize this complex we replace each chain group C_n by its dual **cochain group** $C_n^* = \text{Hom}(C_n, G)$, the group of homomorphisms $C_n \to G$, and we replace each boundary map $\partial : C_n \to C_{n-1}$ by its dual **coboundary map** $\delta = \partial^* : C_{n-1}^* \to C_n^*$. The reason why δ goes in the opposite direction from ∂, increasing rather than decreasing dimension, is purely formal: For a homomorphism $\alpha : A \to B$, the dual homomorphism $\alpha^* : \text{Hom}(B, G) \to \text{Hom}(A, G)$ is defined by $\alpha^*(\varphi) = \varphi\alpha$, so α^* sends $B \xrightarrow{\varphi} G$ to the composition $A \xrightarrow{\alpha} B \xrightarrow{\varphi} G$. Dual homomorphisms obviously satisfy $(\alpha\beta)^* = \beta^*\alpha^*$, $\mathbb{1}^* = \mathbb{1}$, and $0^* = 0$. In particular, since $\partial\partial = 0$ it follows that $\delta\delta = 0$, and the **cohomology group** $H^n(C; G)$ can be defined as the 'homology group' $\text{Ker}\,\delta / \text{Im}\,\delta$ at C_n^* in the cochain complex

$$\cdots \longleftarrow C_{n+1}^* \xleftarrow{\delta} C_n^* \xleftarrow{\delta} C_{n-1}^* \longleftarrow \cdots$$

Our goal is to show that the cohomology groups $H^n(C; G)$ are determined solely by G and the homology groups $H_n(C) = \text{Ker}\,\partial / \text{Im}\,\partial$. A first guess might be that $H^n(C; G)$ is isomorphic to $\text{Hom}(H_n(C), G)$, but this is overly optimistic, as shown by the example above where H_2 was zero while H^2 was nonzero. Nevertheless, there is a natural map $h : H^n(C; G) \to \text{Hom}(H_n(C), G)$, defined as follows. Denote the cycles and boundaries by $Z_n = \text{Ker}\,\partial \subset C_n$ and $B_n = \text{Im}\,\partial \subset C_n$. A class in $H^n(C; G)$ is represented by a homomorphism $\varphi : C_n \to G$ such that $\delta\varphi = 0$, that is, $\varphi\partial = 0$, or in other words, φ vanishes on B_n. The restriction $\varphi_0 = \varphi \mid Z_n$ then induces a quotient homomorphism $\overline{\varphi}_0 : Z_n / B_n \to G$, an element of $\text{Hom}(H_n(C), G)$. If φ is in $\text{Im}\,\delta$, say $\varphi = \delta\psi = \psi\partial$, then φ is zero on Z_n, so $\varphi_0 = 0$ and hence also $\overline{\varphi}_0 = 0$. Thus there is a well-defined quotient map $h : H^n(C; G) \to \text{Hom}(H_n(C), G)$ sending the cohomology class of φ to $\overline{\varphi}_0$. Obviously h is a homomorphism.

It is not hard to see that h is surjective. The short exact sequence

$$0 \longrightarrow Z_n \longrightarrow C_n \xrightarrow{\partial} B_{n-1} \longrightarrow 0$$

splits since B_{n-1} is free, being a subgroup of the free abelian group C_{n-1}. Thus there is a projection homomorphism $p : C_n \to Z_n$ that restricts to the identity on Z_n. Composing with p gives a way of extending homomorphisms $\varphi_0 : Z_n \to G$ to homomorphisms $\varphi = \varphi_0 p : C_n \to G$. In particular, this extends homomorphisms $Z_n \to G$ that vanish on B_n to homomorphisms $C_n \to G$ that still vanish on B_n, or in other words, it extends homomorphisms $H_n(C) \to G$ to elements of $\text{Ker}\,\delta$. Thus we have a homomorphism $\text{Hom}(H_n(C), G) \to \text{Ker}\,\delta$. Composing this with the quotient map $\text{Ker}\,\delta \to H^n(C; G)$ gives a homomorphism from $\text{Hom}(H_n(C), G)$ to $H^n(C; G)$. If we

follow this map by h we get the identity map on $\mathrm{Hom}(H_n(C), G)$ since the effect of composing with h is simply to undo the effect of extending homomorphisms via p. This shows that h is surjective. In fact it shows that we have a split short exact sequence

$$0 \longrightarrow \mathrm{Ker}\, h \longrightarrow H^n(C; G) \xrightarrow{\ h\ } \mathrm{Hom}(H_n(C), G) \longrightarrow 0$$

The remaining task is to analyze $\mathrm{Ker}\, h$. A convenient way to start the process is to consider not just the chain complex C, but also its subcomplexes consisting of the cycles and the boundaries. Thus we consider the commutative diagram of short exact sequences

(i)
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Z_{n+1} & \longrightarrow & C_{n+1} & \xrightarrow{\ \partial\ } & B_n & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & Z_n & \longrightarrow & C_n & \xrightarrow{\ \partial\ } & B_{n-1} & \longrightarrow & 0
\end{array}
$$

where the vertical boundary maps on Z_{n+1} and B_n are the restrictions of the boundary map in the complex C, hence are zero. Dualizing (i) gives a commutative diagram

(ii)
$$
\begin{array}{ccccccccc}
0 & \longleftarrow & Z_{n+1}^* & \longleftarrow & C_{n+1}^* & \longleftarrow & B_n^* & \longleftarrow & 0 \\
& & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle 0} & & \\
0 & \longleftarrow & Z_n^* & \longleftarrow & C_n^* & \longleftarrow & B_{n-1}^* & \longleftarrow & 0
\end{array}
$$

The rows here are exact since, as we have already remarked, the rows of (i) split, and the dual of a split short exact sequence is a split short exact sequence because of the natural isomorphism $\mathrm{Hom}(A \oplus B, G) \approx \mathrm{Hom}(A, G) \oplus \mathrm{Hom}(B, G)$.

We may view (ii), like (i), as part of a short exact sequence of chain complexes. Since the coboundary maps in the Z_n^* and B_n^* complexes are zero, the associated long exact sequence of homology groups has the form

(iii)
$$\cdots \longleftarrow B_n^* \longleftarrow Z_n^* \longleftarrow H^n(C; G) \longleftarrow B_{n-1}^* \longleftarrow Z_{n-1}^* \longleftarrow \cdots$$

The 'boundary maps' $Z_n^* \to B_n^*$ in this long exact sequence are in fact the dual maps i_n^* of the inclusions $i_n : B_n \to Z_n$, as one sees by recalling how these boundary maps are defined: In (ii) one takes an element of Z_n^*, pulls this back to C_n^*, applies δ to get an element of C_{n+1}^*, then pulls this back to B_n^*. The first of these steps extends a homomorphism $\varphi_0 : Z_n \to G$ to $\varphi : C_n \to G$, the second step composes this φ with ∂, and the third step undoes this composition and restricts φ to B_n. The net effect is just to restrict φ_0 from Z_n to B_n.

A long exact sequence can always be broken up into short exact sequences, and doing this for the sequence (iii) yields short exact sequences

(iv)
$$0 \longleftarrow \mathrm{Ker}\, i_n^* \longleftarrow H^n(C; G) \longleftarrow \mathrm{Coker}\, i_{n-1}^* \longleftarrow 0$$

The group $\mathrm{Ker}\, i_n^*$ can be identified naturally with $\mathrm{Hom}(H_n(C), G)$ since elements of $\mathrm{Ker}\, i_n^*$ are homomorphisms $Z_n \to G$ that vanish on the subgroup B_n, and such homomorphisms are the same as homomorphisms $Z_n / B_n \to G$. Under this identification of

$\operatorname{Ker} i_n^*$ with $\operatorname{Hom}(H_n(C), G)$, the map $H^n(C; G) \to \operatorname{Ker} i_n^*$ in (iv) becomes the map h considered earlier. Thus we can rewrite (iv) as a split short exact sequence

(v) $$0 \longrightarrow \operatorname{Coker} i_{n-1}^* \longrightarrow H^n(C; G) \xrightarrow{h} \operatorname{Hom}(H_n(C), G) \longrightarrow 0$$

Our objective now is to show that the more mysterious term $\operatorname{Coker} i_{n-1}^*$ depends only on $H_{n-1}(C)$ and G, in a natural, functorial way. First let us observe that $\operatorname{Coker} i_{n-1}^*$ would be zero if it were always true that the dual of a short exact sequence was exact, since the dual of the short exact sequence

(vi) $$0 \longrightarrow B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \longrightarrow H_{n-1}(C) \longrightarrow 0$$

is the sequence

(vii) $$0 \longleftarrow B_{n-1}^* \xleftarrow{i_{n-1}^*} Z_{n-1}^* \longleftarrow H_{n-1}(C)^* \longleftarrow 0$$

and if this were exact at B_{n-1}^*, then i_{n-1}^* would be surjective, hence $\operatorname{Coker} i_{n-1}^*$ would be zero. This argument does apply if $H_{n-1}(C)$ happens to be free, since (vi) splits in this case, which implies that (vii) is also split exact. So in this case the map h in (v) is an isomorphism. However, in the general case it is easy to find short exact sequences whose duals are not exact. For example, if we dualize $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}_n \to 0$ by applying $\operatorname{Hom}(-, \mathbb{Z})$ we get $0 \leftarrow \mathbb{Z} \xleftarrow{n} \mathbb{Z} \leftarrow 0 \leftarrow 0$ which fails to be exact at the left-hand $\mathbb{Z}$, precisely the place we are interested in for $\operatorname{Coker} i_{n-1}^*$.

We might mention in passing that the loss of exactness at the left end of a short exact sequence after dualization is in fact all that goes wrong, in view of the following:

Exercise. If $A \to B \to C \to 0$ is exact, then dualizing by applying $\operatorname{Hom}(-, G)$ yields an exact sequence $A^* \leftarrow B^* \leftarrow C^* \leftarrow 0$.

However, we will not need this fact in what follows.

The exact sequence (vi) has the special feature that both B_{n-1} and Z_{n-1} are free, so (vi) can be regarded as a free resolution of $H_{n-1}(C)$, where a **free resolution** of an abelian group H is an exact sequence

$$\cdots \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \longrightarrow 0$$

with each F_n free. If we dualize this free resolution by applying $\operatorname{Hom}(-, G)$, we may lose exactness, but at least we get a chain complex — or perhaps we should say 'cochain complex', but algebraically there is no difference. This dual complex has the form

$$\cdots \longleftarrow F_2^* \xleftarrow{f_2^*} F_1^* \xleftarrow{f_1^*} F_0^* \xleftarrow{f_0^*} H^* \longleftarrow 0$$

Let us use the temporary notation $H^n(F; G)$ for the homology group $\operatorname{Ker} f_{n+1}^* / \operatorname{Im} f_n^*$ of this dual complex. Note that the group $\operatorname{Coker} i_{n-1}^*$ that we are interested in is $H^1(F; G)$ where F is the free resolution in (vi). Part (b) of the following lemma therefore shows that $\operatorname{Coker} i_{n-1}^*$ depends only on $H_{n-1}(C)$ and G.

Lemma 3.1. (a) *Given free resolutions F and F' of abelian groups H and H', then every homomorphism $\alpha : H \to H'$ can be extended to a chain map from F to F':*

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0 \\
& & \downarrow{\alpha_2} & & \downarrow{\alpha_1} & & \downarrow{\alpha_0} & & \downarrow{\alpha} & & \\
\cdots & \longrightarrow & F_2' & \xrightarrow{f_2'} & F_1' & \xrightarrow{f_1'} & F_0' & \xrightarrow{f_0'} & H' & \longrightarrow & 0
\end{array}
$$

Furthermore, any two such chain maps extending α are chain homotopic.

(b) *For any two free resolutions F and F' of H, there are canonical isomorphisms $H^n(F; G) \approx H^n(F'; G)$ for all n.*

Proof: The α_i's will be constructed inductively. Since the F_i's are free, it suffices to define each α_i on a basis for F_i. To define α_0, observe that surjectivity of f_0' implies that for each basis element x of F_0 there exists $x' \in F_0'$ such that $f_0'(x') = \alpha f_0(x)$, so we define $\alpha_0(x) = x'$. We would like to define α_1 in the same way, sending a basis element $x \in F_1$ to an element $x' \in F_1'$ such that $f_1'(x') = \alpha_0 f_1(x)$. Such an x' will exist if $\alpha_0 f_1(x)$ lies in $\mathrm{Im}\, f_1' = \mathrm{Ker}\, f_0'$, which it does since $f_0' \alpha_0 f_1 = \alpha f_0 f_1 = 0$. The same procedure defines all the subsequent α_i's.

If we have another chain map extending α given by maps $\alpha_i' : F_i \to F_i'$, then the differences $\beta_i = \alpha_i - \alpha_i'$ define a chain map extending the zero map $\beta : H \to H'$. It will suffice to construct maps $\lambda_i : F_i \to F_{i+1}'$ defining a chain homotopy from β_i to 0, that is, with $\beta_i = f_{i+1}' \lambda_i + \lambda_{i-1} f_i$. The λ_i's are constructed inductively by a procedure much like the construction of the α_i's. When $i = 0$ we let $\lambda_{-1} : H \to F_0'$ be zero, and then the desired relation becomes $\beta_0 = f_1' \lambda_0$. We can achieve this by letting λ_0 send a basis element x to an element $x' \in F_1'$ such that $f_1'(x') = \beta_0(x)$. Such an x' exists since $\mathrm{Im}\, f_1' = \mathrm{Ker}\, f_0'$ and $f_0' \beta_0(x) = \beta f_0(x) = 0$. For the inductive step we wish to define λ_i to take a basis element $x \in F_i$ to an element $x' \in F_{i+1}'$ such that $f_{i+1}'(x') = \beta_i(x) - \lambda_{i-1} f_i(x)$. This will be possible if $\beta_i(x) - \lambda_{i-1} f_i(x)$ lies in $\mathrm{Im}\, f_{i+1}' = \mathrm{Ker}\, f_i'$, which will hold if $f_i'(\beta_i - \lambda_{i-1} f_i) = 0$. Using the relation $f_i' \beta_i = \beta_{i-1} f_i$ and the relation $\beta_{i-1} = f_i' \lambda_{i-1} + \lambda_{i-2} f_{i-1}$ which holds by induction, we have

$$
\begin{aligned}
f_i'(\beta_i - \lambda_{i-1} f_i) &= f_i' \beta_i - f_i' \lambda_{i-1} f_i \\
&= \beta_{i-1} f_i - f_i' \lambda_{i-1} f_i = (\beta_{i-1} - f_i' \lambda_{i-1}) f_i = \lambda_{i-2} f_{i-1} f_i = 0
\end{aligned}
$$

as desired. This finishes the proof of (a).

The maps α_n constructed in (a) dualize to maps $\alpha_n^* : F_n'^* \to F_n^*$ forming a chain map between the dual complexes F'^* and F^*. Therefore we have induced homomorphisms on cohomology $\alpha^* : H^n(F'; G) \to H^n(F; G)$. These do not depend on the choice of α_n's since any other choices α_n' are chain homotopic, say via chain homotopies λ_n, and then α_n^* and $\alpha_n'^*$ are chain homotopic via the dual maps λ_n^* since the dual of the relation $\alpha_i - \alpha_i' = f_{i+1}' \lambda_i + \lambda_{i-1} f_i$ is $\alpha_i^* - \alpha_i'^* = \lambda_i^* f_{i+1}'^* + f_i^* \lambda_{i-1}^*$.

The induced homomorphisms $\alpha^* : H^n(F'; G) \to H^n(F; G)$ satisfy $(\beta\alpha)^* = \alpha^* \beta^*$ for a composition $H \xrightarrow{\alpha} H' \xrightarrow{\beta} H''$ with a free resolution F'' of H'' also given, since

one can choose the compositions $\beta_n \alpha_n$ of extensions α_n of α and β_n of β as an extension of $\beta\alpha$. In particular, if we take α to be an isomorphism and β to be its inverse, with $F'' = F$, then $\alpha^* \beta^* = (\beta\alpha)^* = \mathbb{1}$, the latter equality coming from the obvious extension of $\mathbb{1}: H \to H$ by the identity map of F. The same reasoning shows $\beta^* \alpha^* = \mathbb{1}$, so α^* is an isomorphism. Finally, if we specialize further, taking α to be the identity but with two different free resolutions F and F', we get a canonical isomorphism $\mathbb{1}^*: H^n(F'; G) \to H^n(F; G)$. $\qquad\square$

Every abelian group H has a free resolution of the form $0 \to F_1 \to F_0 \to H \to 0$, with $F_i = 0$ for $i > 1$, obtainable in the following way. Choose a set of generators for H and let F_0 be a free abelian group with basis in one-to-one correspondence with these generators. Then we have a surjective homomorphism $f_0: F_0 \to H$ sending the basis elements to the chosen generators. The kernel of f_0 is free, being a subgroup of a free abelian group, so we can let F_1 be this kernel with $f_1: F_1 \to F_0$ the inclusion, and we can then take $F_i = 0$ for $i > 1$. For this free resolution we obviously have $H^n(F; G) = 0$ for $n > 1$, so this must also be true for all free resolutions. Thus the only interesting group $H^n(F; G)$ is $H^1(F; G)$. As we have seen, this group depends only on H and G, and the standard notation for it is $\mathrm{Ext}(H, G)$. This notation arises from the fact that $\mathrm{Ext}(H, G)$ has an interpretation as the set of isomorphism classes of extensions of G by H, that is, short exact sequences $0 \to G \to J \to H \to 0$, with a natural definition of isomorphism between such exact sequences. This is explained in books on homological algebra, for example [Brown 1982], [Hilton & Stammbach 1970], or [MacLane 1963]. However, this interpretation of $\mathrm{Ext}(H, G)$ is rarely needed in algebraic topology.

Summarizing, we have established the following algebraic result:

Theorem 3.2. *If a chain complex C of free abelian groups has homology groups $H_n(C)$, then the cohomology groups $H^n(C; G)$ of the cochain complex $\mathrm{Hom}(C_n, G)$ are determined by split exact sequences*

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(C), G) \longrightarrow H^n(C; G) \overset{h}{\longrightarrow} \mathrm{Hom}(H_n(C), G) \longrightarrow 0 \qquad\square$$

This is known as the **universal coefficient theorem for cohomology** because it is formally analogous to the universal coefficient theorem for homology in §3.A which expresses homology with arbitrary coefficients in terms of homology with $\mathbb{Z}$ coefficients.

Computing $\mathrm{Ext}(H, G)$ for finitely generated H is not difficult using the following three properties:

- $\mathrm{Ext}(H \oplus H', G) \approx \mathrm{Ext}(H, G) \oplus \mathrm{Ext}(H', G)$.
- $\mathrm{Ext}(H, G) = 0$ if H is free.
- $\mathrm{Ext}(\mathbb{Z}_n, G) \approx G/nG$.

The first of these can be obtained by using the direct sum of free resolutions of H and H' as a free resolution for $H \oplus H'$. If H is free, the free resolution $0 \to H \to H \to 0$

yields the second property, while the third comes from dualizing the free resolution $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}_n \to 0$ to produce an exact sequence

$$0 \longleftarrow \mathrm{Ext}(\mathbb{Z}_n, G) \longleftarrow \mathrm{Hom}(\mathbb{Z}, G) \xleftarrow{\;n\;} \mathrm{Hom}(\mathbb{Z}, G) \longleftarrow \mathrm{Hom}(\mathbb{Z}_n, G) \longleftarrow 0$$
$$\| \qquad\qquad \| \qquad\qquad \|$$
$$G/nG \longleftarrow\!\!\!\!\!\!\!\!\!\!\!\!\!\!\! G \xleftarrow{\quad n \quad} G$$

In particular, these three properties imply that $\mathrm{Ext}(H, \mathbb{Z})$ is isomorphic to the torsion subgroup of H if H is finitely generated. Since $\mathrm{Hom}(H, \mathbb{Z})$ is isomorphic to the free part of H if H is finitely generated, we have:

Corollary 3.3. *If the homology groups H_n and H_{n-1} of a chain complex C of free abelian groups are finitely generated, with torsion subgroups $T_n \subset H_n$ and $T_{n-1} \subset H_{n-1}$, then $H^n(C; \mathbb{Z}) \approx (H_n/T_n) \oplus T_{n-1}$.* $\qquad\square$

It is useful in many situations to know that the short exact sequences in the universal coefficient theorem are natural, meaning that a chain map α between chain complexes C and C' of free abelian groups induces a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(C), G) & \longrightarrow & H^n(C; G) & \xrightarrow{\;h\;} & \mathrm{Hom}(H_n(C), G) & \longrightarrow & 0 \\
& & \big\uparrow{\scriptstyle(\alpha_*)^*} & & \big\uparrow{\scriptstyle\alpha^*} & & \big\uparrow{\scriptstyle(\alpha_*)^*} & & \\
0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(C'), G) & \longrightarrow & H^n(C'; G) & \xrightarrow{\;h\;} & \mathrm{Hom}(H_n(C'), G) & \longrightarrow & 0
\end{array}
$$

This is apparent if one just thinks about the construction; one obviously obtains a map between the short exact sequences (iv) containing $\mathrm{Ker}\, i_n^*$ and $\mathrm{Coker}\, i_{n-1}^*$, the identification $\mathrm{Ker}\, i_n^* = \mathrm{Hom}(H_n(C), G)$ is certainly natural, and the proof of Lemma 3.1 shows that $\mathrm{Ext}(H, G)$ depends naturally on H.

However, the splitting in the universal coefficient theorem is not natural since it depends on the choice of the projections $p : C_n \to Z_n$. An exercise at the end of the section gives a topological example showing that the splitting in fact cannot be natural.

The naturality property together with the five-lemma proves:

Corollary 3.4. *If a chain map between chain complexes of free abelian groups induces an isomorphism on homology groups, then it induces an isomorphism on cohomology groups with any coefficient group G.* $\qquad\square$

One could attempt to generalize the algebraic machinery of the universal coefficient theorem by replacing abelian groups by modules over a chosen ring R and Hom by Hom_R, the R-module homomorphisms. The key fact about abelian groups that was needed was that subgroups of free abelian groups are free. Submodules of free R-modules are free if R is a principal ideal domain, so in this case the generalization is automatic. One obtains natural split short exact sequences

$$0 \to \mathrm{Ext}_R(H_{n-1}(C), G) \to H^n(C; G) \xrightarrow{\;h\;} \mathrm{Hom}_R(H_n(C), G) \to 0$$

where C is a chain complex of free R-modules with boundary maps R-module ho-momorphisms, and the coefficient group G is also an R-module. If R is a field, for example, then R-modules are always free and so the Ext_R term is always zero since we may choose free resolutions of the form $0 \to F_0 \to H \to 0$.

It is interesting to note that the proof of Lemma 3.1 on the uniqueness of free res-olutions is valid for modules over an arbitrary ring R. Moreover, every R-module H has a free resolution, which can be constructed in the following way. Choose a set of generators for H as an R-module, and let F_0 be a free R-module with basis in one-to-one correspondence with these generators. Thus we have a surjective homomorphism $f_0 : F_0 \to H$ sending the basis elements to the chosen generators. Now repeat the pro-cess with $\mathrm{Ker}\, f_0$ in place of H, constructing a homomorphism $f_1 : F_1 \to F_0$ sending a basis for a free R-module F_1 onto generators for $\mathrm{Ker}\, f_0$. And inductively, construct $f_n : F_n \to F_{n-1}$ with image equal to $\mathrm{Ker}\, f_{n-1}$ by the same procedure.

By Lemma 3.1 the groups $H^n(F; G)$ depend only on H and G, not on the free resolution F. The standard notation for $H^n(F; G)$ is $\mathrm{Ext}_R^n(H, G)$. For sufficiently complicated rings R the groups $\mathrm{Ext}_R^n(H, G)$ can be nonzero for $n > 1$. In certain more advanced topics in algebraic topology these Ext_R^n groups play an essential role.

A final remark about the definition of $\mathrm{Ext}_R^n(H, G)$: By the Exercise stated earlier, exactness of $F_1 \to F_0 \to H \to 0$ implies exactness of $F_1^* \leftarrow F_0^* \leftarrow H^* \leftarrow 0$. This means that $H^0(F; G)$ as defined above is zero. Rather than having $\mathrm{Ext}_R^0(H, G)$ be automati-cally zero, it is better to define $H^n(F; G)$ as the nth homology group of the complex $\cdots \leftarrow F_1^* \leftarrow F_0^* \leftarrow 0$ with the term H^* omitted. This can be viewed as defining the groups $H^n(F; G)$ to be unreduced cohomology groups. With this slightly modified definition we have $\mathrm{Ext}_R^0(H, G) = H^0(F; G) = H^* = \mathrm{Hom}_R(H, G)$ by the exactness of $F_1^* \leftarrow F_0^* \leftarrow H^* \leftarrow 0$. The real reason why unreduced Ext groups are better than re-duced groups is perhaps to be found in certain exact sequences involving Ext and Hom derived in §3.F, which would not work with the Hom terms replaced by zeros.

Cohomology of Spaces

Now we return to topology. Given a space X and an abelian group G, we define the group $C^n(X; G)$ of **singular n-cochains with coefficients in G** to be the dual group $\mathrm{Hom}(C_n(X), G)$ of the singular chain group $C_n(X)$. Thus an n-cochain $\varphi \in C^n(X; G)$ assigns to each singular n-simplex $\sigma : \Delta^n \to X$ a value $\varphi(\sigma) \in G$. Since the singular n-simplices form a basis for $C_n(X)$, these values can be chosen arbitrarily, hence n-cochains are exactly equivalent to functions from singular n-simplices to G.

The **coboundary map** $\delta : C^n(X; G) \to C^{n+1}(X; G)$ is the dual ∂^*, so for a cochain $\varphi \in C^n(X; G)$, its coboundary $\delta \varphi$ is the composition $C_{n+1}(X) \xrightarrow{\partial} C_n(X) \xrightarrow{\varphi} G$. This means that for a singular $(n + 1)$-simplex $\sigma : \Delta^{n+1} \to X$ we have

$$\delta \varphi(\sigma) = \sum_i (-1)^i \varphi(\sigma \,|\, [v_0, \cdots, \hat{v}_i, \cdots, v_{n+1}])$$

It is automatic that $\delta^2 = 0$ since δ^2 is the dual of $\partial^2 = 0$. Therefore we can define the **cohomology group** $H^n(X; G)$ **with coefficients in G** to be the quotient $\operatorname{Ker}\delta / \operatorname{Im}\delta$ at $C^n(X; G)$ in the cochain complex

$$\cdots \longleftarrow C^{n+1}(X; G) \xleftarrow{\ \delta\ } C^n(X; G) \xleftarrow{\ \delta\ } C^{n-1}(X; G) \longleftarrow \cdots \longleftarrow C^0(X; G) \longleftarrow 0$$

Elements of $\operatorname{Ker}\delta$ are **cocycles**, and elements of $\operatorname{Im}\delta$ are **coboundaries**. For a cochain φ to be a cocycle means that $\delta\varphi = \varphi\partial = 0$, or in other words, φ vanishes on boundaries.

Since the chain groups $C_n(X)$ are free, the algebraic universal coefficient theorem takes on the topological guise of split short exact sequences

$$0 \longrightarrow \operatorname{Ext}(H_{n-1}(X), G) \longrightarrow H^n(X; G) \longrightarrow \operatorname{Hom}(H_n(X), G) \longrightarrow 0$$

which describe how cohomology groups with arbitrary coefficients are determined purely algebraically by homology groups with $\mathbb{Z}$ coefficients. For example, if the homology groups of X are finitely generated then Corollary 3.3 tells how to compute the cohomology groups $H^n(X; \mathbb{Z})$ from the homology groups.

When $n = 0$ there is no Ext term, and the universal coefficient theorem reduces to an isomorphism $H^0(X; G) \approx \operatorname{Hom}(H_0(X), G)$. This can also be seen directly from the definitions. Since singular 0-simplices are just points of X, a cochain in $C^0(X; G)$ is an arbitrary function $\varphi : X \to G$, not necessarily continuous. For this to be a cocycle means that for each singular 1-simplex $\sigma : [v_0, v_1] \to X$ we have $\delta\varphi(\sigma) = \varphi(\partial\sigma) = \varphi(\sigma(v_1)) - \varphi(\sigma(v_0)) = 0$. This is equivalent to saying that φ is constant on path-components of X. Thus $H^0(X; G)$ is all the functions from path-components of X to G. This is the same as $\operatorname{Hom}(H_0(X), G)$.

Likewise in the case of $H^1(X; G)$ the universal coefficient theorem gives an isomorphism $H^1(X; G) \approx \operatorname{Hom}(H_1(X), G)$ since $\operatorname{Ext}(H_0(X), G) = 0$, the group $H_0(X)$ being free. If X is path-connected, $H_1(X)$ is the abelianization of $\pi_1(X)$ and we can identify $\operatorname{Hom}(H_1(X), G)$ with $\operatorname{Hom}(\pi_1(X), G)$ since G is abelian.

The universal coefficient theorem has a simpler form if we take coefficients in a field F for both homology and cohomology. In §2.2 we defined the homology groups $H_n(X; F)$ as the homology groups of the chain complex of free F-modules $C_n(X; F)$, where $C_n(X; F)$ has basis the singular n-simplices in X. The dual complex $\operatorname{Hom}_F(C_n(X; F), F)$ of F-module homomorphisms is the same as $\operatorname{Hom}(C_n(X), F)$ since both can be identified with the functions from singular n-simplices to F. Hence the homology groups of the dual complex $\operatorname{Hom}_F(C_n(X; F), F)$ are the cohomology groups $H^n(X; F)$. In the generalization of the universal coefficient theorem to the case of modules over a principal ideal domain, the Ext_F terms vanish since F is a field, so we obtain isomorphisms

$$H^n(X; F) \approx \operatorname{Hom}_F(H_n(X; F), F)$$

Thus, with field coefficients, cohomology is the exact dual of homology. Note that when $F = \mathbb{Z}_p$ or $\mathbb{Q}$ we have $\mathrm{Hom}_F(H, G) = \mathrm{Hom}(H, G)$, the group homomorphisms, for arbitrary F-modules G and H.

For the remainder of this section we will go through the main features of singular homology and check that they extend without much difficulty to cohomology.

Reduced Groups. Reduced cohomology groups $\tilde{H}^n(X; G)$ can be defined by dualizing the augmented chain complex $\cdots \to C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$, then taking Ker / Im. As with homology, this gives $\tilde{H}^n(X; G) = H^n(X; G)$ for $n > 0$, and the universal coefficient theorem identifies $\tilde{H}^0(X; G)$ with $\mathrm{Hom}(\tilde{H}_0(X), G)$. We can describe the difference between $\tilde{H}^0(X; G)$ and $H^0(X; G)$ more explicitly by using the interpretation of $H^0(X; G)$ as functions $X \to G$ that are constant on path-components. Recall that the augmentation map $\varepsilon : C_0(X) \to \mathbb{Z}$ sends each singular 0-simplex σ to 1, so the dual map ε^* sends a homomorphism $\varphi : \mathbb{Z} \to G$ to the composition $C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \xrightarrow{\varphi} G$, which is the function $\sigma \mapsto \varphi(1)$. This is a constant function $X \to G$, and since $\varphi(1)$ can be any element of G, the image of ε^* consists of precisely the constant functions. Thus $\tilde{H}^0(X; G)$ is all functions $X \to G$ that are constant on path-components modulo the functions that are constant on all of X.

Relative Groups and the Long Exact Sequence of a Pair. To define relative groups $H^n(X, A; G)$ for a pair (X, A) we first dualize the short exact sequence

$$0 \to C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X, A) \to 0$$

by applying $\mathrm{Hom}(-, G)$ to get

$$0 \longleftarrow C^n(A; G) \xleftarrow{i^*} C^n(X; G) \xleftarrow{j^*} C^n(X, A; G) \longleftarrow 0$$

where by definition $C^n(X, A; G) = \mathrm{Hom}(C_n(X, A), G)$. This sequence is exact by the following direct argument. The map i^* restricts a cochain on X to a cochain on A. Thus for a function from singular n-simplices in X to G, the image of this function under i^* is obtained by restricting the domain of the function to singular n-simplices in A. Every function from singular n-simplices in A to G can be extended to be defined on all singular n-simplices in X, for example by assigning the value 0 to all singular n-simplices not in A, so i^* is surjective. The kernel of i^* consists of cochains taking the value 0 on singular n-simplices in A. Such cochains are the same as homomorphisms $C_n(X, A) = C_n(X)/C_n(A) \to G$, so the kernel of i^* is exactly $C^n(X, A; G) = \mathrm{Hom}(C_n(X, A), G)$, giving the desired exactness. Notice that we can view $C^n(X, A; G)$ as the functions from singular n-simplices in X to G that vanish on simplices in A, since the basis for $C_n(X)$ consisting of singular n-simplices in X is the disjoint union of the simplices with image contained in A and the simplices with image not contained in A.

Relative coboundary maps $\delta : C^n(X, A; G) \to C^{n+1}(X, A; G)$ are obtained as restrictions of the absolute δ's, so relative cohomology groups $H^n(X, A; G)$ are defined. The

fact that the relative cochain group is a subgroup of the absolute cochains, namely the cochains vanishing on chains in A, means that relative cohomology is conceptually a little simpler than relative homology.

The maps i^* and j^* commute with δ since i and j commute with ∂, so the preceding displayed short exact sequence of cochain groups is part of a short exact sequence of cochain complexes, giving rise to an associated long exact sequence of cohomology groups

$$\cdots \longrightarrow H^n(X,A;G) \xrightarrow{\ j^*\ } H^n(X;G) \xrightarrow{\ i^*\ } H^n(A;G) \xrightarrow{\ \delta\ } H^{n+1}(X,A;G) \longrightarrow \cdots$$

By similar reasoning one obtains a long exact sequence of reduced cohomology groups for a pair (X,A) with A nonempty, where $\tilde{H}^n(X,A;G) = H^n(X,A;G)$ for all n, as in homology. Taking A to be a point x_0, this exact sequence gives an identification of $\tilde{H}^n(X;G)$ with $H^n(X,x_0;G)$.

More generally there is a long exact sequence for a triple (X,A,B) coming from the short exact sequences

$$0 \longleftarrow C^n(A,B;G) \xleftarrow{\ i^*\ } C^n(X,B;G) \xleftarrow{\ j^*\ } C^n(X,A;G) \longleftarrow 0$$

The long exact sequence of reduced cohomology can be regarded as the special case that B is a point.

As one would expect, there is a duality relationship between the connecting homomorphisms $\delta : H^n(A;G) \to H^{n+1}(X,A;G)$ and $\partial : H_{n+1}(X,A) \to H_n(A)$. This takes the form of the commutative diagram shown at the right. To verify commutativity, recall how the two connecting homomorphisms are defined, via the diagrams

$$\begin{array}{ccc} H^n(A;G) & \xrightarrow{\ \delta\ } & H^{n+1}(X,A;G) \\ \big\downarrow h & & \big\downarrow h \\ \mathrm{Hom}\,(H_n(A),G) & \xrightarrow{\ \partial^*\ } & \mathrm{Hom}\,(H_{n+1}(X,A),G) \end{array}$$

$$\begin{array}{ccc} C^{n+1}(X;G) & \longleftarrow & C^{n+1}(X,A;G) \\ \uparrow & \diagup\nearrow & \\ C^n(A;G) & \longleftarrow & C^n(X;G) \end{array} \qquad \begin{array}{ccc} C_{n+1}(X) & \longrightarrow & C_{n+1}(X,A) \\ \big\downarrow & \diagdown & \\ C_n(A) & \longrightarrow & C_n(X) \end{array}$$

The connecting homomorphisms are represented by the dashed arrows, which are well-defined only when the chain and cochain groups are replaced by homology and cohomology groups. To show that $h\delta = \partial^* h$, start with an element $\alpha \in H^n(A;G)$ represented by a cocycle $\varphi \in C^n(A;G)$. To compute $\delta(\alpha)$ we first extend φ to a cochain $\overline{\varphi} \in C^n(X;G)$, say by letting it take the value 0 on singular simplices not in A. Then we compose $\overline{\varphi}$ with $\partial : C_{n+1}(X) \to C_n(X)$ to get a cochain $\overline{\varphi}\partial \in C^{n+1}(X;G)$, which actually lies in $C^{n+1}(X,A;G)$ since the original φ was a cocycle in A. This cochain $\overline{\varphi}\partial \in C^{n+1}(X,A;G)$ represents $\delta(\alpha)$ in $H^{n+1}(X,A;G)$. Now we apply the map h, which simply restricts the domain of $\overline{\varphi}\partial$ to relative cycles in $C_{n+1}(X,A)$, that is, $(n+1)$-chains in X whose boundary lies in A. On such chains we have $\overline{\varphi}\partial = \varphi\partial$ since the extension of φ to $\overline{\varphi}$ is irrelevant. The net result of all this is that $h\delta(\alpha)$

is represented by $\varphi\partial$. Let us compare this with $\partial^* h(\alpha)$. Applying h to φ restricts its domain to cycles in A. Then applying ∂^* composes with the map which sends a relative $(n+1)$-cycle in X to its boundary in A. Thus $\partial^* h(\alpha)$ is represented by $\varphi\partial$ just as $h\delta(\alpha)$ was, and so the square commutes.

Induced Homomorphisms. Dual to the chain maps $f_\sharp : C_n(X) \to C_n(Y)$ induced by $f : X \to Y$ are the cochain maps $f^\sharp : C^n(Y;G) \to C^n(X;G)$. The relation $f_\sharp \partial = \partial f_\sharp$ dualizes to $\delta f^\sharp = f^\sharp \delta$, so $f^\sharp$ induces homomorphisms $f^* : H^n(Y;G) \to H^n(X;G)$. In the relative case a map $f : (X,A) \to (Y,B)$ induces $f^* : H^n(Y,B;G) \to H^n(X,A;G)$ by the same reasoning, and in fact f induces a map between short exact sequences of cochain complexes, hence a map between long exact sequences of cohomology groups, with commuting squares. The properties $(fg)^\sharp = g^\sharp f^\sharp$ and $\mathbb{1}^\sharp = \mathbb{1}$ imply $(fg)^* = g^* f^*$ and $\mathbb{1}^* = \mathbb{1}$, so $X \mapsto H^n(X;G)$ and $(X,A) \mapsto H^n(X,A;G)$ are contravariant functors, the 'contra' indicating that induced maps go in the reverse direction.

The algebraic universal coefficient theorem applies also to relative cohomology since the relative chain groups $C_n(X,A)$ are free, and there is a naturality statement: A map $f : (X,A) \to (Y,B)$ induces a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(X,A),G) & \longrightarrow & H^n(X,A;G) & \xrightarrow{\ h\ } & \mathrm{Hom}(H_n(X,A),G) & \longrightarrow & 0 \\
& & \big\uparrow{\scriptstyle (f_*)^*} & & \big\uparrow{\scriptstyle f^*} & & \big\uparrow{\scriptstyle (f_*)^*} & & \\
0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(Y,B),G) & \longrightarrow & H^n(Y,B;G) & \xrightarrow{\ h\ } & \mathrm{Hom}(H_n(Y,B),G) & \longrightarrow & 0
\end{array}
$$

This follows from the naturality of the algebraic universal coefficient sequences since the vertical maps are induced by the chain maps $f_\sharp : C_n(X,A) \to C_n(Y,B)$. When the subspaces A and B are empty we obtain the absolute forms of these results.

Homotopy Invariance. The statement is that if $f \simeq g : (X,A) \to (Y,B)$, then $f^* = g^* : H^n(Y,B) \to H^n(X,A)$. This is proved by direct dualization of the proof for homology. From the proof of Theorem 2.10 we have a chain homotopy P satisfying $g_\sharp - f_\sharp = \partial P + P\partial$. This relation dualizes to $g^\sharp - f^\sharp = P^* \delta + \delta P^*$, so P^* is a chain homotopy between the maps $f^\sharp, g^\sharp : C^n(Y;G) \to C^n(X;G)$. This restricts also to a chain homotopy between $f^\sharp$ and $g^\sharp$ on relative cochains, the cochains vanishing on singular simplices in the subspaces B and A. Since $f^\sharp$ and $g^\sharp$ are chain homotopic, they induce the same homomorphism $f^* = g^*$ on cohomology.

Excision. For cohomology this says that for subspaces $Z \subset A \subset X$ with the closure of Z contained in the interior of A, the inclusion $i : (X - Z, A - Z) \hookrightarrow (X,A)$ induces isomorphisms $i^* : H^n(X,A;G) \to H^n(X - Z, A - Z;G)$ for all n. This follows from the corresponding result for homology by the naturality of the universal coefficient theorem and the five-lemma. Alternatively, if one wishes to avoid appealing to the universal coefficient theorem, the proof of excision for homology dualizes easily to cohomology by the following argument. In the proof for homology there were chain maps $\iota : C_n(A+B) \to C_n(X)$ and $\rho : C_n(X) \to C_n(A+B)$ such that $\rho\iota = \mathbb{1}$ and $\mathbb{1} - \iota\rho = \partial D + D\partial$ for a chain homotopy D. Dualizing by taking $\mathrm{Hom}(-,G)$, we have maps

ρ^* and ι^* between $C^n(A+B;G)$ and $C^n(X;G)$, and these induce isomorphisms on cohomology since $\iota^*\rho^* = \mathbb{1}$ and $\mathbb{1} - \rho^*\iota^* = D^*\delta + \delta D^*$. By the five-lemma, the maps $C^n(X,A;G) \to C^n(A+B,A;G)$ also induce isomorphisms on cohomology. There is an obvious identification of $C^n(A+B,A;G)$ with $C^n(B,A\cap B;G)$, so we get isomorphisms $H^n(X,A;G)) \approx H^n(B,A\cap B;G)$ induced by the inclusion $(B,A\cap B) \hookrightarrow (X,A)$.

Axioms for Cohomology. These are exactly dual to the axioms for homology. Restricting attention to CW complexes again, a (reduced) **cohomology theory** is a sequence of contravariant functors $\tilde{h}^n$ from CW complexes to abelian groups, together with natural coboundary homomorphisms $\delta : \tilde{h}^n(A) \to \tilde{h}^{n+1}(X/A)$ for CW pairs (X,A), satisfying the following axioms:

(1) If $f \simeq g : X \to Y$, then $f^* = g^* : \tilde{h}^n(Y) \to \tilde{h}^n(X)$.
(2) For each CW pair (X,A) there is a long exact sequence

$$\cdots \xrightarrow{\delta} \tilde{h}^n(X/A) \xrightarrow{q^*} \tilde{h}^n(X) \xrightarrow{i^*} \tilde{h}^n(A) \xrightarrow{\delta} \tilde{h}^{n+1}(X/A) \xrightarrow{q^*} \cdots$$

where i is the inclusion and q is the quotient map.
(3) For a wedge sum $X = \bigvee_\alpha X_\alpha$ with inclusions $i_\alpha : X_\alpha \hookrightarrow X$, the product map $\prod_\alpha i_\alpha^* : \tilde{h}^n(X) \to \prod_\alpha \tilde{h}^n(X_\alpha)$ is an isomorphism for each n.

We have already seen that the first axiom holds for singular cohomology. The second axiom follows from excision in the same way as for homology, via isomorphisms $\tilde{H}^n(X/A;G) \approx H^n(X,A;G)$. Note that the third axiom involves direct product, rather than the direct sum appearing in the homology version. This is because of the natural isomorphism $\mathrm{Hom}(\bigoplus_\alpha A_\alpha, G) \approx \prod_\alpha \mathrm{Hom}(A_\alpha, G)$, which implies that the cochain complex of a disjoint union $\coprod_\alpha X_\alpha$ is the direct product of the cochain complexes of the individual X_α's, and this direct product splitting passes through to cohomology groups. The same argument applies in the relative case, so we get isomorphisms $H^n(\coprod_\alpha X_\alpha, \coprod_\alpha A_\alpha; G) \approx \prod_\alpha H^n(X_\alpha, A_\alpha; G)$. The third axiom is obtained by taking the A_α's to be basepoints x_α and passing to the quotient $\coprod_\alpha X_\alpha / \coprod_\alpha x_\alpha = \bigvee_\alpha X_\alpha$.

The relation between reduced and unreduced cohomology theories is the same as for homology, as described in §2.3.

Simplicial Cohomology. If X is a Δ-complex and $A \subset X$ is a subcomplex, then the simplicial chain groups $\Delta_n(X,A)$ dualize to simplicial cochain groups $\Delta^n(X,A;G) = \mathrm{Hom}(\Delta_n(X,A),G)$, and the resulting cohomology groups are by definition the simplicial cohomology groups $H_\Delta^n(X,A;G)$. Since the inclusions $\Delta_n(X,A) \subset C_n(X,A)$ induce isomorphisms $H_n^\Delta(X,A) \approx H_n(X,A)$, Corollary 3.4 implies that the dual maps $C^n(X,A;G) \to \Delta^n(X,A;G)$ also induce isomorphisms $H^n(X,A;G) \approx H_\Delta^n(X,A;G)$.

Cellular Cohomology. For a CW complex X this is defined via the cellular cochain complex formed by the horizontal sequence in the following diagram, where coefficients in a given group G are understood, and the cellular coboundary maps d_n are

the compositions $\delta_n j_n$, making the triangles commute. Note that $d_n d_{n-1} = 0$ since $j_n \delta_{n-1} = 0$.

$$
\begin{array}{c}
& & 0 \\
& H^{n-1}(X^{n-1}) & \nearrow \\
j_{n-1} \nearrow & \searrow \delta_{n-1} \\
\cdots \longrightarrow H^{n-1}(X^{n-1}, X^{n-2}) \xrightarrow{d_{n-1}} H^n(X^n, X^{n-1}) \xrightarrow{d_n} H^{n+1}(X^{n+1}, X^n) \longrightarrow \cdots \\
j_n \searrow \quad \nearrow \delta_n \\
H^n(X^n) \\
H^n(X) \approx H^n(X^{n+1}) \nearrow \quad \searrow 0 \\
0 \nearrow
\end{array}
$$
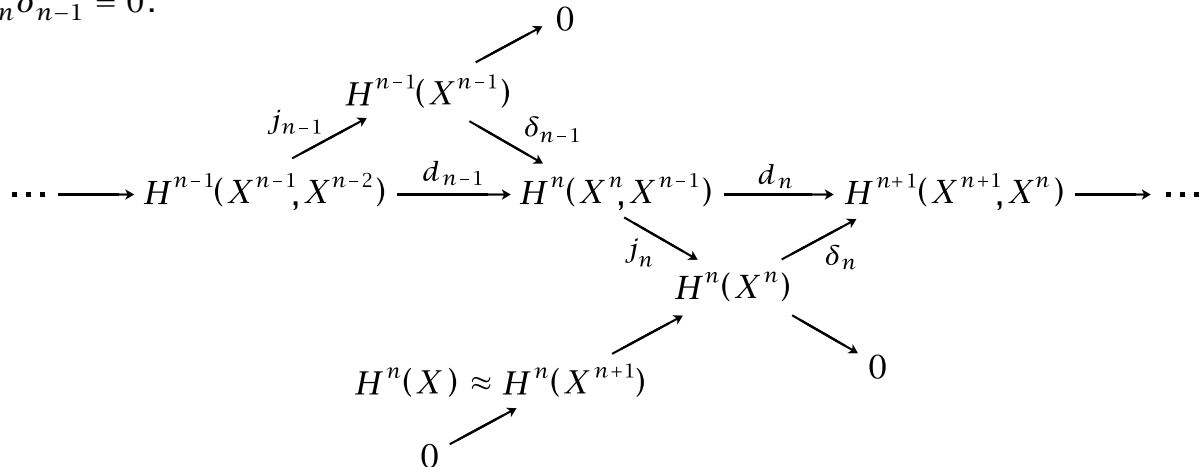
Theorem 3.5. *$H^n(X; G) \approx \operatorname{Ker} d_n / \operatorname{Im} d_{n-1}$. Furthermore, the cellular cochain complex $\{H^n(X^n, X^{n-1}; G), d_n\}$ is isomorphic to the dual of the cellular chain complex, obtained by applying $\operatorname{Hom}(-, G)$.*

Proof: The universal coefficient theorem implies that $H^k(X^n, X^{n-1}; G) = 0$ for $k \neq n$. The long exact sequence of the pair (X^n, X^{n-1}) then gives isomorphisms $H^k(X^n; G) \approx H^k(X^{n-1}; G)$ for $k \neq n,\ n-1$. Hence by induction on n we obtain $H^k(X^n; G) = 0$ if $k > n$. Thus the diagonal sequences in the preceding diagram are exact. The universal coefficient theorem also gives $H^k(X, X^{n+1}; G) = 0$ for $k \leq n+1$, so $H^n(X; G) \approx H^n(X^{n+1}; G)$. The diagram then yields isomorphisms

$$
H^n(X; G) \approx H^n(X^{n+1}; G) \approx \operatorname{Ker} \delta_n \approx \operatorname{Ker} d_n / \operatorname{Im} \delta_{n-1} \approx \operatorname{Ker} d_n / \operatorname{Im} d_{n-1}
$$

For the second statement in the theorem we have the diagram

$$
\begin{array}{ccccc}
H^k(X^k, X^{k-1}; G) & \longrightarrow & H^k(X^k; G) & \xrightarrow{\ \delta\ } & H^{k+1}(X^{k+1}, X^k; G) \\
\downarrow h & & \downarrow h & & \downarrow h \\
\operatorname{Hom}(H_k(X^k, X^{k-1}), G) & \longrightarrow & \operatorname{Hom}(H_k(X^k), G) & \xrightarrow{\ \partial^*\ } & \operatorname{Hom}(H_{k+1}(X^{k+1}, X^k), G)
\end{array}
$$

The cellular coboundary map is the composition across the top, and we want to see that this is the same as the composition across the bottom. The first and third vertical maps are isomorphisms by the universal coefficient theorem, so it suffices to show the diagram commutes. The first square commutes by naturality of h, and commutativity of the second square was shown in the discussion of the long exact sequence of cohomology groups of a pair (X, A). $\square$

Mayer–Vietoris Sequences. In the absolute case these take the form

$$
\cdots \longrightarrow H^n(X; G) \xrightarrow{\ \Psi\ } H^n(A; G) \oplus H^n(B; G) \xrightarrow{\ \Phi\ } H^n(A \cap B; G) \longrightarrow H^{n+1}(X; G) \longrightarrow \cdots
$$

where X is the union of the interiors of A and B. This is the long exact sequence associated to the short exact sequence of cochain complexes

$$
0 \longrightarrow C^n(A+B; G) \xrightarrow{\ \psi\ } C^n(A; G) \oplus C^n(B; G) \xrightarrow{\ \varphi\ } C^n(A \cap B; G) \longrightarrow 0
$$

Here $C^n(A + B; G)$ is the dual of the subgroup $C_n(A + B) \subset C_n(X)$ consisting of sums of singular n-simplices lying in A or in B. The inclusion $C_n(A + B) \subset C_n(X)$ is a chain homotopy equivalence by Proposition 2.21, so the dual restriction map $C^n(X; G) \to C^n(A + B; G)$ is also a chain homotopy equivalence, hence induces an isomorphism on cohomology as shown in the discussion of excision a couple pages back. The map ψ has coordinates the two restrictions to A and B, and φ takes the difference of the restrictions to $A \cap B$, so it is obvious that φ is onto with kernel the image of ψ.

There is a relative Mayer–Vietoris sequence

$$\cdots \longrightarrow H^n(X, Y; G) \longrightarrow H^n(A, C; G) \oplus H^n(B, D; G) \longrightarrow H^n(A \cap B, C \cap D; G) \longrightarrow \cdots$$

for a pair $(X, Y) = (A \cup B, C \cup D)$ with $C \subset A$ and $D \subset B$ such that X is the union of the interiors of A and B while Y is the union of the interiors of C and D. To derive this, consider first the map of short exact sequences of cochain complexes

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C^n(X, Y; G) & \longrightarrow & C^n(X; G) & \longrightarrow & C^n(Y; G) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^n(A + B, C + D; G) & \longrightarrow & C^n(A + B; G) & \longrightarrow & C^n(C + D; G) & \longrightarrow & 0
\end{array}
$$

Here $C^n(A + B, C + D; G)$ is defined as the kernel of $C^n(A + B; G) \to C^n(C + D; G)$, the restriction map, so the second sequence is exact. The vertical maps are restrictions. The second and third of these induce isomorphisms on cohomology, as we have seen, so by the five-lemma the first vertical map also induces isomorphisms on cohomology. The relative Mayer–Vietoris sequence is then the long exact sequence associated to the short exact sequence of cochain complexes

$$0 \to C^n(A + B, C + D; G) \xrightarrow{\psi} C^n(A, C; G) \oplus C^n(B, D; G) \xrightarrow{\varphi} C^n(A \cap B, C \cap D; G) \to 0$$

This is exact since it is the dual of the short exact sequence

$$0 \to C_n(A \cap B, C \cap D) \to C_n(A, C) \oplus C_n(B, D) \to C_n(A + B, C + D) \to 0$$

constructed in §2.2, which splits since $C_n(A + B, C + D)$ is free with basis the singular n-simplices in A or in B that do not lie in C or in D.

Exercises

1. Show that $\text{Ext}(H, G)$ is a contravariant functor of H for fixed G, and a covariant functor of G for fixed H.

2. Show that the maps $G \xrightarrow{n} G$ and $H \xrightarrow{n} H$ multiplying each element by the integer n induce multiplication by n in $\text{Ext}(H, G)$.

3. Regarding $\mathbb{Z}_2$ as a module over the ring $\mathbb{Z}_4$, construct a resolution of $\mathbb{Z}_2$ by free modules over $\mathbb{Z}_4$ and use this to show that $\text{Ext}^n_{\mathbb{Z}_4}(\mathbb{Z}_2, \mathbb{Z}_2)$ is nonzero for all n.

4. What happens if one defines homology groups $h_n(X;G)$ as the homology groups of the chain complex $\cdots \to \mathrm{Hom}(G,C_n(X)) \to \mathrm{Hom}(G,C_{n-1}(X)) \to \cdots$? More specifically, what are the groups $h_n(X;G)$ when $G = \mathbb{Z}$, $\mathbb{Z}_m$, and $\mathbb{Q}$?

5. Regarding a cochain $\varphi \in C^1(X;G)$ as a function from paths in X to G, show that if φ is a cocycle, then
 (a) $\varphi(f \cdot g) = \varphi(f) + \varphi(g)$,
 (b) φ takes the value 0 on constant paths,
 (c) $\varphi(f) = \varphi(g)$ if $f \simeq g$,
 (d) φ is a coboundary iff $\varphi(f)$ depends only on the endpoints of f, for all f.
[In particular, (a) and (c) give a map $H^1(X;G) \to \mathrm{Hom}(\pi_1(X),G)$, which the universal coefficient theorem says is an isomorphism if X is path-connected.]

6. (a) Directly from the definitions, compute the simplicial cohomology groups of $S^1 \times S^1$ with $\mathbb{Z}$ and $\mathbb{Z}_2$ coefficients, using the Δ-complex structure given in §2.1.
(b) Do the same for $\mathbb{RP}^2$ and the Klein bottle.

7. Show that the functors $h^n(X) = \mathrm{Hom}(H_n(X),\mathbb{Z})$ do not define a cohomology theory on the category of CW complexes.

8. Many basic homology arguments work just as well for cohomology even though maps go in the opposite direction. Verify this in the following cases:
(a) Compute $H^i(S^n;G)$ by induction on n in two ways: using the long exact sequence of a pair, and using the Mayer–Vietoris sequence.
(b) Show that if A is a closed subspace of X that is a deformation retract of some neighborhood, then the quotient map $X \to X/A$ induces isomorphisms $H^n(X,A;G) \approx \tilde{H}^n(X/A;G)$ for all n.
(c) Show that if A is a retract of X then $H^n(X;G) \approx H^n(A;G) \oplus H^n(X,A;G)$.

9. Show that if $f : S^n \to S^n$ has degree d then $f^* : H^n(S^n;G) \to H^n(S^n;G)$ is multiplication by d.

10. For the lens space $L_m(\ell_1, \cdots, \ell_n)$ defined in Example 2.43, compute the cohomology groups using the cellular cochain complex and taking coefficients in $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_m$, and $\mathbb{Z}_p$ for p prime. Verify that the answers agree with those given by the universal coefficient theorem.

11. Let X be a Moore space $M(\mathbb{Z}_m, n)$ obtained from S^n by attaching a cell e^{n+1} by a map of degree m.
(a) Show that the quotient map $X \to X/S^n = S^{n+1}$ induces the trivial map on $\tilde{H}_i(-;\mathbb{Z})$ for all i, but not on $H^{n+1}(-;\mathbb{Z})$. Deduce that the splitting in the universal coefficient theorem for cohomology cannot be natural.
(b) Show that the inclusion $S^n \hookrightarrow X$ induces the trivial map on $\tilde{H}^i(-;\mathbb{Z})$ for all i, but not on $H_n(-;\mathbb{Z})$.

12. Show $H^k(X,X^n;G) = 0$ if X is a CW complex and $k \le n$, by using the cohomology version of the second proof of the corresponding result for homology in Lemma 2.34.

13. Let $\langle X, Y \rangle$ denote the set of basepoint-preserving homotopy classes of basepoint-preserving maps $X \to Y$. Using Proposition 1B.9, show that if X is a connected CW complex and G is an abelian group, then the map $\langle X, K(G,1) \rangle \to H^1(X;G)$ sending a map $f : X \to K(G,1)$ to the induced homomorphism $f_* : H_1(X) \to H_1(K(G,1)) \approx G$ is a bijection, where we identify $H^1(X;G)$ with $\mathrm{Hom}(H_1(X), G)$ via the universal coefficient theorem.

3.2   Cup Product

In the introduction to this chapter we sketched a definition of cup product in terms of another product called cross product. However, to define the cross product from scratch takes some work, so we will proceed in the opposite order, first giving an elementary definition of cup product by an explicit formula with simplices, then afterwards defining cross product in terms of cup product. The other approach of defining cup product via cross product is explained at the end of §3.B.

To define the cup product we consider cohomology with coefficients in a ring R, the most common choices being $\mathbb{Z}$, $\mathbb{Z}_n$, and $\mathbb{Q}$. For cochains $\varphi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$, the **cup product** $\varphi \smile \psi \in C^{k+\ell}(X;R)$ is the cochain whose value on a singular simplex $\sigma : \Delta^{k+\ell} \to X$ is given by the formula

$$(\varphi \smile \psi)(\sigma) = \varphi(\sigma \,|\, [v_0, \cdots, v_k]) \psi(\sigma \,|\, [v_k, \cdots, v_{k+\ell}])$$

where the right-hand side is the product in R. To see that this cup product of cochains induces a cup product of cohomology classes we need a formula relating it to the coboundary map:

‖ **Lemma 3.6.** $\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^k \varphi \smile \delta\psi$ for $\varphi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$.

Proof: For $\sigma : \Delta^{k+\ell+1} \to X$ we have

$$(\delta\varphi \smile \psi)(\sigma) = \sum_{i=0}^{k+1} (-1)^i \varphi(\sigma \,|\, [v_0, \cdots, \hat{v}_i, \cdots, v_{k+1}]) \psi(\sigma \,|\, [v_{k+1}, \cdots, v_{k+\ell+1}])$$

$$(-1)^k (\varphi \smile \delta\psi)(\sigma) = \sum_{i=k}^{k+\ell+1} (-1)^i \varphi(\sigma \,|\, [v_0, \cdots, v_k]) \psi(\sigma \,|\, [v_k, \cdots, \hat{v}_i, \cdots, v_{k+\ell+1}])$$

When we add these two expressions, the last term of the first sum cancels the first term of the second sum, and the remaining terms are exactly $\delta(\varphi \smile \psi)(\sigma) = (\varphi \smile \psi)(\partial\sigma)$ since $\partial\sigma = \sum_{i=0}^{k+\ell+1} (-1)^i \sigma \,|\, [v_0, \cdots, \hat{v}_i, \cdots, v_{k+\ell+1}]$. $\qquad \square$

From the formula $\delta(\varphi \smile \psi) = \delta\varphi \smile \psi \pm \varphi \smile \delta\psi$ it is apparent that the cup product of two cocycles is again a cocycle. Also, the cup product of a cocycle and a coboundary, in either order, is a coboundary since $\varphi \smile \delta\psi = \pm\delta(\varphi \smile \psi)$ if $\delta\varphi = 0$, and $\delta\varphi \smile \psi = \delta(\varphi \smile \psi)$ if $\delta\psi = 0$. It follows that there is an induced cup product

$$H^k(X;R) \times H^\ell(X;R) \xrightarrow{\ \smile\ } H^{k+\ell}(X;R)$$

This is associative and distributive since at the level of cochains the cup product obviously has these properties. If R has an identity element, then there is an identity element for cup product, the class $1 \in H^0(X;R)$ defined by the 0-cocycle taking the value 1 on each singular 0-simplex.

A cup product for simplicial cohomology can be defined by the same formula as for singular cohomology, so the canonical isomorphism between simplicial and singular cohomology respects cup products. Here are three examples of direct calculations of cup products using simplicial cohomology.

Example 3.7. Let M be the closed orientable surface of genus $g \geq 1$ with the Δ-complex structure shown in the figure for the case $g = 2$. The cup product of interest is $H^1(M) \times H^1(M) \to H^2(M)$. Taking $\mathbb{Z}$ coefficients, a basis for $H_1(M)$ is formed by the edges a_i and b_i, as we showed in Example 2.36 when we computed the homology of M using cellular homology. We have $H^1(M) \approx \mathrm{Hom}(H_1(M), \mathbb{Z})$ by cellular cohomology or the universal coefficient theorem. A basis

for $H_1(M)$ determines a dual basis for $\mathrm{Hom}(H_1(M), \mathbb{Z})$, so dual to a_i is the cohomology class α_i assigning the value 1 to a_i and 0 to the other basis elements, and similarly we have cohomology classes β_i dual to b_i.
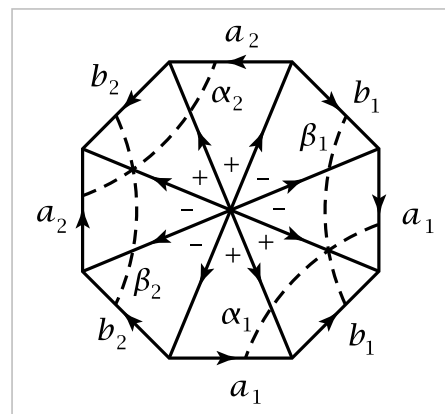
To represent α_i by a simplicial cocycle φ_i we need to choose values for φ_i on the edges radiating out from the central vertex in such a way that $\delta\varphi_i = 0$. This is the 'cocycle condition' discussed in the introduction to this chapter, where we saw that it has a geometric interpretation in terms of curves transverse to the edges of M. With this interpretation in mind, consider the arc labeled α_i in the figure, which represents a loop in M meeting a_i in one point and disjoint from all the other basis elements a_j and b_j. We define φ_i to have the value 1 on edges meeting the arc α_i and the value 0 on all other edges. Thus φ_i counts the number of intersections of each edge with the arc α_i. In similar fashion we obtain a cocycle ψ_i counting intersections with the arc β_i, and ψ_i represents the cohomology class β_i dual to b_i.

Now we can compute cup products by applying the definition. Keeping in mind that the ordering of the vertices of each 2-simplex is compatible with the indicated orientations of its edges, we see for example that $\varphi_1 \smile \psi_1$ takes the value 0 on all 2-simplices except the one with outer edge b_1 in the lower right part of the figure,

where it takes the value 1. Thus $\varphi_1 \smile \psi_1$ takes the value 1 on the 2-chain c formed by the sum of all the 2-simplices with the signs indicated in the center of the figure. It is an easy calculation that $\partial c = 0$. Since there are no 3-simplices, c is not a boundary, so it represents a nonzero element of $H_2(M)$. The fact that $(\varphi_1 \smile \psi_1)(c)$ is a generator of $\mathbb{Z}$ implies both that c represents a generator of $H_2(M) \approx \mathbb{Z}$ and that $\varphi_1 \smile \psi_1$ represents the dual generator γ of $H^2(M) \approx \mathrm{Hom}(H_2(M), \mathbb{Z}) \approx \mathbb{Z}$. Thus $\alpha_1 \smile \beta_1 = \gamma$. In similar fashion one computes:

$$\alpha_i \smile \beta_j = \begin{cases} \gamma, & i = j \\ 0, & i \neq j \end{cases} = -(\beta_i \smile \alpha_j), \qquad \alpha_i \smile \alpha_j = 0, \qquad \beta_i \smile \beta_j = 0$$

These relations determine the cup product $H^1(M) \times H^1(M) \to H^2(M)$ completely since cup product is distributive. Notice that cup product is not commutative in this example since $\alpha_i \smile \beta_i = -(\beta_i \smile \alpha_i)$. We will show in Theorem 3.11 below that this is the worst that can happen: Cup product is commutative up to a sign depending only on dimension, assuming that the coefficient ring itself is commutative.

One can see in this example that nonzero cup products of distinct classes α_i or β_j occur precisely when the corresponding loops α_i or β_j intersect. This is also true for the cup product of α_i or β_i with itself if we allow ourselves to take two copies of the corresponding loop and deform one of them to be disjoint from the other.

Example 3.8. The closed nonorientable surface N of genus g can be treated in similar fashion if we use $\mathbb{Z}_2$ coefficients. Using the Δ-complex structure shown, the edges a_i give a basis for $H_1(N; \mathbb{Z}_2)$, and the dual basis elements $\alpha_i \in H^1(N; \mathbb{Z}_2)$ can be represented by cocycles with values given by counting intersections with the arcs labeled α_i in the figure. Then one computes that $\alpha_i \smile \alpha_i$ is the nonzero element of $H^2(N; \mathbb{Z}_2) \approx \mathbb{Z}_2$ and $\alpha_i \smile \alpha_j = 0$ for $i \neq j$. In particu-

lar, when $g = 1$ we have $N = \mathbb{R}\mathrm{P}^2$, and the cup product of a generator of $H^1(\mathbb{R}\mathrm{P}^2; \mathbb{Z}_2)$ with itself is a generator of $H^2(\mathbb{R}\mathrm{P}^2; \mathbb{Z}_2)$.
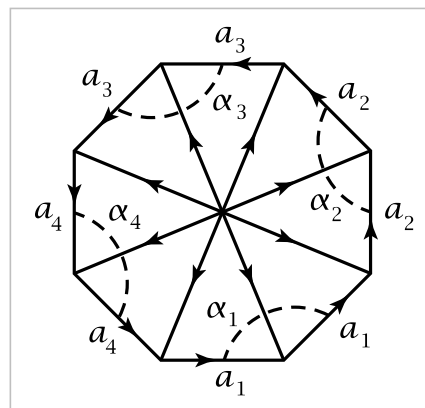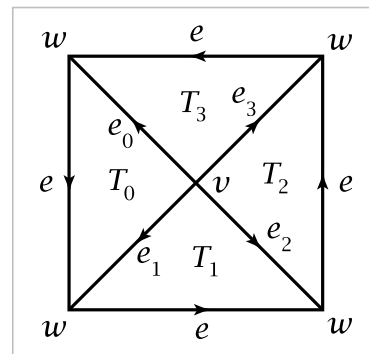
The remarks in the paragraph preceding this example apply here also, but with the following difference: When one tries to deform a second copy of the loop α_i in the present example to be disjoint from the original copy, the best one can do is make it intersect the original in one point. This reflects the fact that $\alpha_i \smile \alpha_i$ is now nonzero.

Example 3.9. Let X be the 2-dimensional CW complex obtained by attaching a 2–cell to S^1 by the degree m map $S^1 \to S^1$, $z \mapsto z^m$. Using cellular cohomology, or cellular homology and the universal coefficient theorem, we see that $H^n(X; \mathbb{Z})$ consists of a $\mathbb{Z}$ for $n = 0$ and a $\mathbb{Z}_m$ for $n = 2$, so the cup product structure with $\mathbb{Z}$ coefficients is uninteresting. However, with $\mathbb{Z}_m$ coefficients we have $H^i(X; \mathbb{Z}_m) \approx \mathbb{Z}_m$ for $i = 0, 1, 2$,

so there is the possibility that the cup product of two 1-dimensional classes can be nontrivial.

To obtain a Δ-complex structure on X, take a regular m-gon subdivided into m triangles T_i around a central vertex v, as shown in the figure for the case $m = 4$, then identify all the outer edges by rotations of the m-gon. This gives X a Δ-complex structure with 2 vertices, $m+1$ edges, and m 2-simplices. A generator α of $H^1(X;\mathbb{Z}_m)$ is represented by a cocycle φ assigning the value 1 to the edge e, which generates $H_1(X)$. The condition that φ be a cocycle means that $\varphi(e_i) + \varphi(e) = \varphi(e_{i+1})$ for all i, subscripts being taken mod m. So we may take $\varphi(e_i) = i \in \mathbb{Z}_m$. Hence $(\varphi \smile \varphi)(T_i) = \varphi(e_i)\varphi(e) = i$. The map $h : H^2(X;\mathbb{Z}_m) \to \mathrm{Hom}(H_2(X;\mathbb{Z}_m), \mathbb{Z}_m)$ is an isomorphism since $\sum_i T_i$ is a generator of $H_2(X;\mathbb{Z}_m)$ and there are 2-cocycles taking the value 1 on $\sum_i T_i$, for example the cocycle taking the value 1 on one T_i and 0 on all the others. The cocycle $\varphi \smile \varphi$ takes the value $0 + 1 + \cdots + (m-1)$ on $\sum_i T_i$, hence represents $0 + 1 + \cdots + (m-1)$ times a generator β of $H^2(X;\mathbb{Z}_m)$. In $\mathbb{Z}_m$ the sum $0 + 1 + \cdots + (m-1)$ is 0 if m is odd and k if $m = 2k$ since the terms 1 and $m-1$ cancel, 2 and $m-2$ cancel, and so on. Thus, writing α^2 for $\alpha \smile \alpha$, we have $\alpha^2 = 0$ if m is odd and $\alpha^2 = k\beta$ if $m = 2k$.

In particular, if $m = 2$, X is $\mathbb{RP}^2$ and $\alpha^2 = \beta$ in $H^2(\mathbb{RP}^2;\mathbb{Z}_2)$, as we showed already in Example 3.8.

The cup product formula $(\varphi \smile \psi)(\sigma) = \varphi(\sigma | [v_0, \cdots, v_k])\psi(\sigma | [v_k, \cdots, v_{k+\ell}])$ also gives relative cup products

$$H^k(X;R) \times H^\ell(X,A;R) \xrightarrow{\smile} H^{k+\ell}(X,A;R)$$
$$H^k(X,A;R) \times H^\ell(X;R) \xrightarrow{\smile} H^{k+\ell}(X,A;R)$$
$$H^k(X,A;R) \times H^\ell(X,A;R) \xrightarrow{\smile} H^{k+\ell}(X,A;R)$$

since if φ or ψ vanishes on chains in A then so does $\varphi \smile \psi$. There is a more general relative cup product

$$H^k(X,A;R) \times H^\ell(X,B;R) \xrightarrow{\smile} H^{k+\ell}(X,A \cup B;R)$$

when A and B are open subsets of X or subcomplexes of the CW complex X. This is obtained in the following way. The absolute cup product restricts to a cup product $C^k(X,A;R) \times C^\ell(X,B;R) \to C^{k+\ell}(X,A+B;R)$ where $C^n(X,A+B;R)$ is the subgroup of $C^n(X;R)$ consisting of cochains vanishing on sums of chains in A and chains in B. If A and B are open in X, the inclusions $C^n(X,A \cup B;R) \hookrightarrow C^n(X,A+B;R)$ induce isomorphisms on cohomology, via the five-lemma and the fact that the restriction maps $C^n(A \cup B;R) \to C^n(A+B;R)$ induce isomorphisms on cohomology as we saw in the discussion of excision in the previous section. Therefore the cup product $C^k(X,A;R) \times C^\ell(X,B;R) \to C^{k+\ell}(X,A+B;R)$ induces the desired relative cup product

$H^k(X,A;R) \times H^\ell(X,B;R) \to H^{k+\ell}(X, A \cup B;R)$. This holds also if X is a CW complex with A and B subcomplexes since here again the maps $C^n(A \cup B;R) \to C^n(A + B;R)$ induce isomorphisms on cohomology, as we saw for homology in §2.2.

Proposition 3.10. *For a map $f : X \to Y$, the induced maps $f^* : H^n(Y;R) \to H^n(X;R)$ satisfy $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$, and similarly in the relative case.*

Proof: This comes from the cochain formula $f^\sharp(\varphi) \smile f^\sharp(\psi) = f^\sharp(\varphi \smile \psi)$:

$$
\begin{aligned}
(f^\sharp \varphi \smile f^\sharp \psi)(\sigma) &= f^\sharp \varphi(\sigma | [v_0, \cdots, v_k]) f^\sharp \psi(\sigma | [v_k, \cdots, v_{k+\ell}]) \\
&= \varphi(f\sigma | [v_0, \cdots, v_k]) \psi(f\sigma | [v_k, \cdots, v_{k+\ell}]) \\
&= (\varphi \smile \psi)(f\sigma) = f^\sharp(\varphi \smile \psi)(\sigma) \qquad\qquad \square
\end{aligned}
$$

The natural question of whether the cup product is commutative is answered by the following:

Theorem 3.11. *The identity $\alpha \smile \beta = (-1)^{k\ell} \beta \smile \alpha$ holds for all $\alpha \in H^k(X,A;R)$ and $\beta \in H^\ell(X,A;R)$, when R is commutative.*

Taking $\alpha = \beta$, this implies in particular that if α is an element of $H^k(X,A;R)$ with k odd, then $2(\alpha \smile \alpha) = 0$ in $H^{2k}(X,A;R)$, or more concisely, $2\alpha^2 = 0$. Hence if $H^{2k}(X,A;R)$ has no elements of order two, then $\alpha^2 = 0$. For example, if X is the 2-complex obtained by attaching a disk to S^1 by a map of degree m as in Example 3.9 above, then we can deduce that the square of a generator of $H^1(X;\mathbb{Z}_m)$ is zero if m is odd, and is either zero or the unique element of $H^2(X;\mathbb{Z}_m) \approx \mathbb{Z}_m$ of order two if m is even. As we showed, the square is in fact nonzero when m is even.

Proof: Consider first the case $A = \varnothing$. For cochains $\varphi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$ one can see from the definition that the cup products $\varphi \smile \psi$ and $\psi \smile \varphi$ differ only by a permutation of the vertices of $\Delta^{k+\ell}$. The idea of the proof is to study a particularly nice permutation of vertices, namely the one that totally reverses their order. This has the convenient feature of also reversing the ordering of vertices in any face.

For a singular n-simplex $\sigma : [v_0, \cdots, v_n] \to X$, let $\overline{\sigma}$ be the singular n-simplex obtained by preceding σ by the linear homeomorphism of $[v_0, \cdots, v_n]$ reversing the order of the vertices. Thus $\overline{\sigma}(v_i) = \sigma(v_{n-i})$. This reversal of vertices is the product of $n + (n-1) + \cdots + 1 = n(n+1)/2$ transpositions of adjacent vertices, each of which reverses orientation of the n-simplex since it is a reflection across an $(n-1)$-dimensional hyperplane. So to take orientations into account we would expect that a sign $\varepsilon_n = (-1)^{n(n+1)/2}$ ought to be inserted. Hence we define a homomorphism $\rho : C_n(X) \to C_n(X)$ by $\rho(\sigma) = \varepsilon_n \overline{\sigma}$.

We will show that ρ is a chain map, chain homotopic to the identity, so it induces the identity on cohomology. From this the theorem quickly follows. Namely, the

formulas

$$(\rho^*\varphi \smile \rho^*\psi)(\sigma) = \varphi(\varepsilon_k\sigma|[v_k, \cdots, v_0])\psi(\varepsilon_\ell\sigma|[v_{k+\ell}, \cdots, v_k])$$

$$\rho^*(\psi \smile \varphi)(\sigma) = \varepsilon_{k+\ell}\psi(\sigma|[v_{k+\ell}, \cdots, v_k])\varphi(\sigma|[v_k, \cdots, v_0])$$

show that $\varepsilon_k\varepsilon_\ell(\rho^*\varphi \smile \rho^*\psi) = \varepsilon_{k+\ell}\rho^*(\psi \smile \varphi)$, since we assume R is commutative. A trivial calculation gives $\varepsilon_{k+\ell} = (-1)^{k\ell}\varepsilon_k\varepsilon_\ell$, hence $\rho^*\varphi \smile \rho^*\psi = (-1)^{k\ell}\rho^*(\psi \smile \varphi)$. Since ρ is chain homotopic to the identity, the ρ^*'s disappear when we pass to cohomology classes, and so we obtain the desired formula $\alpha \smile \beta = (-1)^{k\ell}\beta \smile \alpha$.

The chain map property $\partial\rho = \rho\partial$ can be verified by calculating, for a singular n-simplex σ,

$$\partial\rho(\sigma) = \varepsilon_n\sum_i(-1)^i\sigma|[v_n, \cdots, \hat{v}_{n-i}, \cdots, v_0]$$

$$\rho\partial(\sigma) = \rho\left(\sum_i(-1)^i\sigma|[v_0, \cdots, \hat{v}_i, \cdots, v_n]\right)$$

$$= \varepsilon_{n-1}\sum_i(-1)^{n-i}\sigma|[v_n, \cdots, \hat{v}_{n-i}, \cdots, v_0]$$

so the result follows from the easily checked identity $\varepsilon_n = (-1)^n\varepsilon_{n-1}$.

To define a chain homotopy between ρ and the identity we are motivated by the construction of the prism operator P in the proof that homotopic maps induce the same homomorphism on homology, in Theorem 2.10. The main ingredient in the construction of P was a subdivision of $\Delta^n \times I$ into $(n+1)$-simplices with vertices v_i in $\Delta^n \times \{0\}$ and w_i in $\Delta^n \times \{1\}$, the vertex w_i lying directly above v_i. Using the same subdivision, and letting $\pi:\Delta^n \times I \to \Delta^n$ be the projection, we now define $P:C_n(X) \to C_{n+1}(X)$ by

$$P(\sigma) = \sum_i(-1)^i\varepsilon_{n-i}(\sigma\pi)|[v_0, \cdots, v_i, w_n, \cdots, w_i]$$

Thus the w-vertices are written in reverse order, and there is a compensating sign ε_{n-i}. One can view this formula as arising from the Δ-complex structure on $\Delta^n \times I$ in which the vertices are ordered $v_0, \cdots, v_n, w_n, \cdots, w_0$ rather than the more natural ordering $v_0, \cdots, v_n, w_0, \cdots, w_n$.

To show $\partial P + P\partial = \rho - 1\!1$ we first calculate ∂P, leaving out σ's and $\sigma\pi$'s for notational simplicity:

$$\partial P = \sum_{j\leq i}(-1)^i(-1)^j\varepsilon_{n-i}[v_0, \cdots, \hat{v}_j, \cdots, v_i, w_n, \cdots, w_i]$$

$$+ \sum_{j\geq i}(-1)^i(-1)^{i+1+n-j}\varepsilon_{n-i}[v_0, \cdots, v_i, w_n, \cdots, \hat{w}_j, \cdots, w_i]$$

The $j = i$ terms in these two sums give

$$\varepsilon_n[w_n, \cdots, w_0] + \sum_{i>0}\varepsilon_{n-i}[v_0, \cdots, v_{i-1}, w_n, \cdots, w_i]$$

$$+ \sum_{i<n}(-1)^{n+i+1}\varepsilon_{n-i}[v_0, \cdots, v_i, w_n, \cdots, w_{i+1}] - [v_0, \cdots, v_n]$$

In this expression the two summation terms cancel since replacing i by $i - 1$ in the second sum produces a new sign $(-1)^{n+i}\varepsilon_{n-i+1} = -\varepsilon_{n-i}$. The remaining two terms $\varepsilon_n[w_n, \cdots, w_0]$ and $-[v_0, \cdots, v_n]$ represent $\rho(\sigma) - \sigma$. So in order to show that $\partial P + P\partial = \rho - \mathbb{1}$, it remains to check that in the formula for ∂P above, the terms with $j \neq i$ give $-P\partial$. Calculating $P\partial$ from the definitions, we have

$$P\partial = \sum_{i<j}(-1)^i(-1)^j\varepsilon_{n-i-1}[v_0, \cdots, v_i, w_n, \cdots, \widehat{w}_j, \cdots, w_i]$$

$$+ \sum_{i>j}(-1)^{i-1}(-1)^j\varepsilon_{n-i}[v_0, \cdots, \widehat{v}_j, \cdots, v_i, w_n, \cdots, w_i]$$

Since $\varepsilon_{n-i} = (-1)^{n-i}\varepsilon_{n-i-1}$, this finishes the verification that $\partial P + P\partial = \rho - \mathbb{1}$, and so the theorem is proved when $A = \varnothing$. The proof also applies when $A \neq \varnothing$ since the maps ρ and P take chains in A to chains in A, so the dual homomorphisms ρ^* and P^* act on relative cochains. $\qquad\square$

The Cohomology Ring

Since cup product is associative and distributive, it is natural to try to make it the multiplication in a ring structure on the cohomology groups of a space X. This is easy to do if we simply define $H^*(X; R)$ to be the direct sum of the groups $H^n(X; R)$. Elements of $H^*(X; R)$ are finite sums $\sum_i \alpha_i$ with $\alpha_i \in H^i(X; R)$, and the product of two such sums is defined to be $(\sum_i \alpha_i)(\sum_j \beta_j) = \sum_{i,j} \alpha_i\beta_j$. It is routine to check that this makes $H^*(X; R)$ into a ring, with identity if R has an identity. Similarly, $H^*(X, A; R)$ is a ring via the relative cup product. Taking scalar multiplication by elements of R into account, these rings can also be regarded as R-algebras.

For example, the calculations in Example 3.8 or 3.9 above show that $H^*(\mathbb{RP}^2; \mathbb{Z}_2)$ consists of the polynomials $a_0 + a_1\alpha + a_2\alpha^2$ with coefficients $a_i \in \mathbb{Z}_2$, so $H^*(\mathbb{RP}^2; \mathbb{Z}_2)$ is the quotient $\mathbb{Z}_2[\alpha]/(\alpha^3)$ of the polynomial ring $\mathbb{Z}_2[\alpha]$ by the ideal generated by α^3.

This example illustrates how $H^*(X; R)$ often has a more compact description than the sequence of individual groups $H^n(X; R)$, so there is a certain economy in the change of scale that comes from regarding all the groups $H^n(X; R)$ as part of a single object $H^*(X; R)$.

Adding cohomology classes of different dimensions to form $H^*(X; R)$ is a convenient formal device, but it has little topological significance. One always regards the cohomology ring as a **graded ring**: a ring A with a decomposition as a sum $\bigoplus_{k\geq0}A_k$ of additive subgroups A_k such that the multiplication takes $A_k \times A_\ell$ to $A_{k+\ell}$. To indicate that an element $a \in A$ lies in A_k we write $|a| = k$. This applies in particular to elements of $H^k(X; R)$. Some authors call $|a|$ the 'degree' of a, but we will use the term 'dimension' which is more geometric and avoids potential confusion with the degree of a polynomial.

A graded ring satisfying the commutativity property of Theorem 3.11, $ab = (-1)^{|a||b|}ba$, is usually called simply **commutative** in the context of algebraic topology, in spite of the potential for misunderstanding. In the older literature one finds less ambiguous terms such as *graded commutative*, *anticommutative*, or *skew commutative*.

Example 3.12: Polynomial Rings. Among the simplest graded rings are polynomial rings $R[\alpha]$ and their truncated versions $R[\alpha]/(\alpha^n)$, consisting of polynomials of degree less than n. The example we have seen is $H^*(\mathbb{RP}^2;\mathbb{Z}_2) \approx \mathbb{Z}_2[\alpha]/(\alpha^3)$. More generally we will show in Theorem 3.19 that $H^*(\mathbb{RP}^n;\mathbb{Z}_2) \approx \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$ and $H^*(\mathbb{RP}^\infty;\mathbb{Z}_2) \approx \mathbb{Z}_2[\alpha]$. In these cases $|\alpha| = 1$. We will also show that $H^*(\mathbb{CP}^n;\mathbb{Z}) \approx \mathbb{Z}[\alpha]/(\alpha^{n+1})$ and $H^*(\mathbb{CP}^\infty;\mathbb{Z}) \approx \mathbb{Z}[\alpha]$ with $|\alpha| = 2$. The analogous results for quaternionic projective spaces are also valid, with $|\alpha| = 4$. The coefficient ring $\mathbb{Z}$ in the complex and quaternionic cases could be replaced by any commutative ring R, but not for $\mathbb{RP}^n$ and $\mathbb{RP}^\infty$ since a polynomial ring $R[\alpha]$ is strictly commutative, so for this to be a commutative ring in the graded sense we must have either $|\alpha|$ even or $2 = 0$ in R.

Polynomial rings in several variables also have graded ring structures, and these graded rings can sometimes be realized as cohomology rings of spaces. For example, $\mathbb{Z}_2[\alpha_1, \cdots, \alpha_n]$ is $H^*(X;\mathbb{Z}_2)$ for X the product of n copies of $\mathbb{RP}^\infty$, with $|\alpha_i| = 1$ for each i, as we will see in Example 3.20.

Example 3.13: Exterior Algebras. Another nice example of a commutative graded ring is the **exterior algebra** $\Lambda_R[\alpha_1, \cdots, \alpha_n]$ over a commutative ring R with identity. This is the free R-module with basis the finite products $\alpha_{i_1} \cdots \alpha_{i_k}$, $i_1 < \cdots < i_k$, with associative, distributive multiplication defined by the rules $\alpha_i \alpha_j = -\alpha_j \alpha_i$ for $i \neq j$ and $\alpha_i^2 = 0$. The empty product of α_i's is allowed, and provides an identity element 1 in $\Lambda_R[\alpha_1, \cdots, \alpha_n]$. The exterior algebra becomes a commutative graded ring by specifying odd dimensions for the generators α_i.

The example we have seen is the torus $T^2 = S^1 \times S^1$, where $H^*(T^2;\mathbb{Z}) \approx \Lambda_\mathbb{Z}[\alpha, \beta]$ with $|\alpha| = |\beta| = 1$ by the calculations in Example 3.7. More generally, for the n-torus T^n, $H^*(T^n;R)$ is the exterior algebra $\Lambda_R[\alpha_1, \cdots, \alpha_n]$ as we will see in Example 3.16. The same is true for any product of odd-dimensional spheres, where $|\alpha_i|$ is the dimension of the ith sphere.

Induced homomorphisms are ring homomorphisms by Proposition 3.10. Here is an example illustrating this fact.

Example 3.14: Product Rings. The isomorphism $H^*(\coprod_\alpha X_\alpha;R) \xrightarrow{\approx} \prod_\alpha H^*(X_\alpha;R)$ whose coordinates are induced by the inclusions $i_\alpha : X_\alpha \hookrightarrow \coprod_\alpha X_\alpha$ is a ring isomorphism with respect to the usual coordinatewise multiplication in a product ring, because each coordinate function i_α^* is a ring homomorphism. Similarly for a wedge sum the isomorphism $\widetilde{H}^*(\bigvee_\alpha X_\alpha;R) \approx \prod_\alpha \widetilde{H}^*(X_\alpha;R)$ is a ring isomorphism. Here we take

reduced cohomology to be cohomology relative to a basepoint, and we use relative cup products. We should assume the basepoints $x_\alpha \in X_\alpha$ are deformation retracts of neighborhoods, to be sure that the claimed isomorphism does indeed hold.

This product ring structure for wedge sums can sometimes be used to rule out splittings of a space as a wedge sum up to homotopy equivalence. For example, consider $\mathbb{CP}^2$, which is S^2 with a cell e^4 attached by a certain map $f : S^3 \to S^2$. Using homology or just the additive structure of cohomology it is impossible to conclude that $\mathbb{CP}^2$ is not homotopy equivalent to $S^2 \vee S^4$, and hence that f is not homotopic to a constant map. However, with cup products we can distinguish these two spaces since the square of each element of $H^2(S^2 \vee S^4; \mathbb{Z})$ is zero in view of the ring isomorphism $\tilde{H}^*(S^2 \vee S^4; \mathbb{Z}) \approx \tilde{H}^*(S^2; \mathbb{Z}) \oplus \tilde{H}^*(S^4; \mathbb{Z})$, but the square of a generator of $H^2(\mathbb{CP}^2; \mathbb{Z})$ is nonzero since $H^*(\mathbb{CP}^2; \mathbb{Z}) \approx \mathbb{Z}[\alpha]/(\alpha^3)$.

More generally, cup products can be used to distinguish infinitely many different homotopy classes of maps $S^{4n-1} \to S^{2n}$ for all $n \geq 1$. This is systematized in the notion of the *Hopf invariant*, which is studied in §4.B.

Here is the evident general question raised by the preceding examples:

The Realization Problem. Which graded commutative R-algebras occur as cup product algebras $H^*(X; R)$ of spaces X?

This is a difficult problem, with the degree of difficulty depending strongly on the coefficient ring R. The most accessible case is $R = \mathbb{Q}$, where essentially every graded commutative $\mathbb{Q}$-algebra is realizable, as shown in [Quillen 1969]. Next in order of difficulty is $R = \mathbb{Z}_p$ with p prime. This is much harder than the case of $\mathbb{Q}$, and only partial results, obtained with much labor, are known. Finally there is $R = \mathbb{Z}$, about which very little is known beyond what is implied by the $\mathbb{Z}_p$ cases.

A Künneth Formula

One might guess that there should be some connection between cup product and product spaces, and indeed this is the case, as we will show in this subsection.

To begin, we define the **cross product**, or **external cup product** as it is sometimes called. This is the map

$$H^*(X; R) \times H^*(Y; R) \xrightarrow{\;\times\;} H^*(X \times Y; R)$$

given by $a \times b = p_1^*(a) \smile p_2^*(b)$ where p_1 and p_2 are the projections of $X \times Y$ onto X and Y. Since cup product is distributive, the cross product is bilinear, that is, linear in each variable separately. We might hope that the cross product map would be an isomorphism in many cases, thereby giving a nice description of the cohomology rings of these product spaces. However, a bilinear map is rarely a homomorphism, so it could hardly be an isomorphism. Fortunately there is a nice algebraic solution

to this problem, and that is to replace the direct product $H^*(X;R) \times H^*(Y;R)$ by the tensor product $H^*(X;R) \otimes_R H^*(Y;R)$.

Let us review the definition and basic properties of tensor products. For abelian groups A and B the tensor product $A \otimes B$ is defined to be the abelian group with generators $a \otimes b$ for $a \in A$, $b \in B$, and relations $(a + a') \otimes b = a \otimes b + a' \otimes b$ and $a \otimes (b + b') = a \otimes b + a \otimes b'$. The zero element of $A \otimes B$ is $0 \otimes 0 = 0 \otimes b = a \otimes 0$, and $-(a \otimes b) = -a \otimes b = a \otimes (-b)$. More generally $n(a \otimes b) = na \otimes b = a \otimes nb$ for all $n \in \mathbb{Z}$. Some readily verified elementary properties are:

(1) $A \otimes B \approx B \otimes A$.
(2) $(\bigoplus_i A_i) \otimes B \approx \bigoplus_i (A_i \otimes B)$.
(3) $(A \otimes B) \otimes C \approx A \otimes (B \otimes C)$.
(4) $\mathbb{Z} \otimes A \approx A$.
(5) $\mathbb{Z}_n \otimes A \approx A/nA$.
(6) A pair of homomorphisms $f : A \to A'$ and $g : B \to B'$ induces a homomorphism $f \otimes g : A \otimes B \to A' \otimes B'$ via $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$.
(7) A bilinear map $\varphi : A \times B \to C$ induces a homomorphism $A \otimes B \to C$ sending $a \otimes b$ to $\varphi(a, b)$.

In (1)–(5) the isomorphisms are the obvious ones, for example $a \otimes b \mapsto b \otimes a$ in (1) and $n \otimes a \mapsto na$ in (4). Properties (1), (2), (4), and (5) allow the calculation of tensor products of finitely generated abelian groups.

The generalization to tensor products of modules over a commutative ring R is easy. One defines $A \otimes_R B$ for R-modules A and B to be the quotient group of $A \otimes B$ obtained by imposing the further relations $ra \otimes b = a \otimes rb$ for $r \in R$, $a \in A$, and $b \in B$. This relation guarantees that $A \otimes_R B$ is again an R-module, with $r(a \otimes b) = ra \otimes b = a \otimes rb$. The statements (1)–(3), (6), and (7) remain valid for tensor products of R-modules. The generalization of (4) is the canonical isomorphism $R \otimes_R A \approx A$, $r \otimes a \mapsto ra$.

When R is $\mathbb{Z}$ or $\mathbb{Z}_m$ we have $A \otimes_R B = A \otimes B$, and this is also true when R is $\mathbb{Q}$ or any subring of $\mathbb{Q}$ since in $A \otimes B$ we have $\frac{p}{q} a \otimes b = \frac{1}{q} a \otimes pb = \frac{1}{q} a \otimes \frac{p}{q} qb = \frac{q}{q} a \otimes \frac{p}{q} b = a \otimes \frac{p}{q} b$. However, $A \otimes_R B$ is generally different from $A \otimes B$. For example, if $R = \mathbb{Z}[\sqrt{2}]$, which as a group is free abelian of rank 2 with basis $\{1, \sqrt{2}\}$, then $R \otimes_R R = R$ but $R \otimes R$ is free abelian of rank 4 with basis $\{1 \otimes 1, 1 \otimes \sqrt{2}, \sqrt{2} \otimes 1, \sqrt{2} \otimes \sqrt{2}\}$.

Property (7) of tensor products guarantees that the cross product as defined above gives rise to a homomorphism of R-modules

$$H^*(X;R) \otimes_R H^*(Y;R) \xrightarrow{\times} H^*(X \times Y;R), \qquad a \otimes b \mapsto a \times b$$

which we shall also call cross product. This map becomes a ring homomorphism if we define the multiplication in a tensor product of graded rings by $(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bd$ where $|x|$ denotes the dimension of x. Namely, if we denote the

cross product map by μ and we define $(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bd$, then

$$
\begin{aligned}
\mu((a \otimes b)(c \otimes d)) &= (-1)^{|b||c|} \mu(ac \otimes bd) \\
&= (-1)^{|b||c|} (a \smile c) \times (b \smile d) \\
&= (-1)^{|b||c|} p_1^*(a \smile c) \smile p_2^*(b \smile d) \\
&= (-1)^{|b||c|} p_1^*(a) \smile p_1^*(c) \smile p_2^*(b) \smile p_2^*(d) \\
&= p_1^*(a) \smile p_2^*(b) \smile p_1^*(c) \smile p_2^*(d) \\
&= (a \times b)(c \times d) = \mu(a \otimes b)\mu(c \otimes d)
\end{aligned}
$$

Theorem 3.15. *The cross product $H^*(X;R) \otimes_R H^*(Y;R) \to H^*(X \times Y;R)$ is an isomorphism of rings if X and Y are CW complexes and $H^k(Y;R)$ is a finitely generated free R-module for all k.*

Results of this type, computing homology or cohomology of a product space, are known as **Künneth formulas**. The hypothesis that X and Y are CW complexes will be shown to be unnecessary in §4.1 when we consider CW approximations to arbitrary spaces. On the other hand, the freeness hypothesis cannot always be dispensed with, as we shall see in §3.B when we obtain a completely general Künneth formula for the homology of a product space.

When the conclusion of the theorem holds, the ring structure in $H^*(X \times Y;R)$ is determined by the ring structures in $H^*(X;R)$ and $H^*(Y;R)$. Example 3E.6 shows that some hypotheses are necessary in order for this to be true.

Example 3.16. The exterior algebra $\Lambda_R[\alpha_1, \cdots, \alpha_n]$ is the graded tensor product over R of the one-variable exterior algebras $\Lambda_R[\alpha_i]$ where the α_i's have odd dimension. The Künneth formula then gives an isomorphism $H^*(S^{k_1} \times \cdots \times S^{k_n}; \mathbb{Z}) \approx \Lambda_{\mathbb{Z}}[\alpha_1, \cdots, \alpha_n]$ if the dimensions k_i are all odd. With some k_i's even, one would have the tensor product of an exterior algebra for the odd-dimensional spheres and truncated polynomial rings $\mathbb{Z}[\alpha]/(\alpha^2)$ for the even-dimensional spheres. Of course, $\Lambda_{\mathbb{Z}}[\alpha]$ and $\mathbb{Z}[\alpha]/(\alpha^2)$ are isomorphic as rings, but when one takes tensor products in the graded sense it becomes important to distinguish them as graded rings, with α odd-dimensional in $\Lambda_{\mathbb{Z}}[\alpha]$ and even-dimensional in $\mathbb{Z}[\alpha]/(\alpha^2)$. These remarks apply more generally with any coefficient ring R in place of $\mathbb{Z}$, though when $R = \mathbb{Z}_2$ there is no need to distinguish between the odd-dimensional and even-dimensional cases since signs become irrelevant.

The idea of the proof of the theorem will be to consider, for a fixed CW complex Y, the functors

$$
h^n(X, A) = \bigoplus_i \left(H^i(X, A; R) \otimes_R H^{n-i}(Y;R) \right)
$$
$$
k^n(X, A) = H^n(X \times Y, A \times Y; R)
$$

The cross product, or a relative version of it, defines a map $\mu : h^n(X, A) \to k^n(X, A)$ which we would like to show is an isomorphism when X is a CW complex and $A = \varnothing$.

We will show:

(1) h^* and k^* are cohomology theories on the category of CW pairs.

(2) μ is a natural transformation: It commutes with induced homomorphisms and with coboundary homomorphisms in long exact sequences of pairs.

It is obvious that $\mu : h^n(X) \to k^n(X)$ is an isomorphism when X is a point since it is just the scalar multiplication map $R \otimes_R H^n(Y; R) \to H^n(Y; R)$. The following general fact will then imply the theorem.

Proposition 3.17. *If a natural transformation between unreduced cohomology theories on the category of CW pairs is an isomorphism when the CW pair is $(point, \varnothing)$, then it is an isomorphism for all CW pairs.*

Proof: Let $\mu : h^*(X, A) \to k^*(X, A)$ be the natural transformation. By the five-lemma it will suffice to show that μ is an isomorphism when $A = \varnothing$.

First we do the case of finite-dimensional X by induction on dimension. The induction starts with the case that X is 0-dimensional, where the result holds by hypothesis and by the axiom for disjoint unions. For the induction step, μ gives a map between the two long exact sequences for the pair (X^n, X^{n-1}), with commuting squares since μ is a natural transformation. The five-lemma reduces the inductive step to showing that μ is an isomorphism for $(X, A) = (X^n, X^{n-1})$. Let $\Phi : \coprod_\alpha (D_\alpha^n, \partial D_\alpha^n) \to (X^n, X^{n-1})$ be a collection of characteristic maps for all the n-cells of X. By excision, Φ^* is an isomorphism for h^* and k^*, so by naturality it suffices to show that μ is an isomorphism for $(X, A) = \coprod_\alpha (D_\alpha^n, \partial D_\alpha^n)$. The axiom for disjoint unions gives a further reduction to the case of the pair $(D^n, \partial D^n)$. Finally, this case follows by applying the five-lemma to the long exact sequences of this pair, since D^n is contractible and hence is covered by the 0-dimensional case, and ∂D^n is $(n-1)$-dimensional.

The case that X is infinite-dimensional reduces to the finite-dimensional case by a telescope argument as in the proof of Lemma 2.34. We leave this for the reader since the finite-dimensional case suffices for the h^* and k^* we are considering, where the maps $h^i(X) \to h^i(X^n)$ and $k^i(X) \to k^i(X^n)$ induced by the inclusion $X^n \hookrightarrow X$ are isomorphisms when n is sufficiently large with respect to i. $\square$

Proof of 3.15: It remains to check that h^* and k^* are cohomology theories, and that μ is a natural transformation. Since we are dealing with unreduced cohomology theories there are four axioms to verify.

(1) Homotopy invariance: $f \simeq g$ implies $f^* = g^*$. This is obvious for both h^* and k^*.

(2) Excision: $h^*(X, A) \approx h^*(B, A \cap B)$ for A and B subcomplexes of the CW complex $X = A \cup B$. This is obvious, and so is the corresponding statement for k^* since $(A \times Y) \cup (B \times Y) = (A \cup B) \times Y$ and $(A \times Y) \cap (B \times Y) = (A \cap B) \times Y$.

(3) The long exact sequence of a pair. This is a triviality for k^*, but a few words of explanation are needed for h^*, where the desired exact sequence is obtained in two steps. For the first step, tensor the long exact sequence of ordinary cohomology groups for a pair (X, A) with the free R-module $H^n(Y; R)$, for a fixed n. This yields another exact sequence because $H^n(Y; R)$ is a direct sum of copies of R, so the result of tensoring an exact sequence with this direct sum is simply to produce a direct sum of copies of the exact sequence, which is again an exact sequence. The second step is to let n vary, taking a direct sum of the previously constructed exact sequences for each n, with the nth exact sequence shifted up by n dimensions.

(4) Disjoint unions. Again this axiom obviously holds for k^*, but some justification is required for h^*. What is needed is the algebraic fact that there is a canonical isomorphism $(\prod_\alpha M_\alpha) \otimes_R N \approx \prod_\alpha (M_\alpha \otimes_R N)$ for R-modules M_α and a finitely generated free R-module N. Since N is a direct product of finitely many copies R_β of R, $M_\alpha \otimes_R N$ is a direct product of corresponding copies $M_{\alpha\beta} = M_\alpha \otimes_R R_\beta$ of M_α and the desired relation becomes $\prod_\beta \prod_\alpha M_{\alpha\beta} \approx \prod_\alpha \prod_\beta M_{\alpha\beta}$, which is obviously true.

Finally there is naturality of μ to consider. Naturality with respect to maps between spaces is immediate from the naturality of cup products. Naturality with respect to coboundary maps in long exact sequences is commutativity of the following square:

$$
\begin{array}{ccc}
H^k(A; R) \times H^\ell(Y; R) & \xrightarrow{\ \delta \times \mathbb{1}\ } & H^{k+1}(X, A; R) \times H^\ell(Y; R) \\
\Big\downarrow{\scriptstyle \times} & & \Big\downarrow{\scriptstyle \times} \\
H^{k+\ell}(A \times Y; R) & \xrightarrow{\quad \delta \quad} & H^{k+\ell+1}(X \times Y, A \times Y; R)
\end{array}
$$

To check this, start with an element of the upper left product, represented by cocycles $\varphi \in C^k(A; R)$ and $\psi \in C^\ell(Y; R)$. Extend φ to a cochain $\overline{\varphi} \in C^k(X; R)$. Then the pair (φ, ψ) maps rightward to $(\delta\overline{\varphi}, \psi)$ and then downward to $p_1^\sharp(\delta\overline{\varphi}) \smile p_2^\sharp(\psi)$. Going the other way around the square, (φ, ψ) maps downward to $p_1^\sharp(\varphi) \smile p_2^\sharp(\psi)$ and then rightward to $\delta(p_1^\sharp(\overline{\varphi}) \smile p_2^\sharp(\psi))$ since $p_1^\sharp(\overline{\varphi}) \smile p_2^\sharp(\psi)$ extends $p_1^\sharp(\varphi) \smile p_2^\sharp(\psi)$ over $X \times Y$. Finally, $\delta(p_1^\sharp(\overline{\varphi}) \smile p_2^\sharp(\psi)) = p_1^\sharp(\delta\overline{\varphi}) \smile p_2^\sharp(\psi)$ since $\delta\psi = 0$. $\qquad\square$

It is sometimes important to have a relative version of the Künneth formula in Theorem 3.15. The relative cross product is

$$
H^*(X, A; R) \otimes_R H^*(Y, B; R) \xrightarrow{\ \times\ } H^*(X \times Y, A \times Y \cup X \times B; R)
$$

for CW pairs (X, A) and (Y, B), defined just as in the absolute case by $a \times b = p_1^*(a) \smile p_2^*(b)$ where $p_1^*(a) \in H^*(X \times Y, A \times Y; R)$ and $p_2^*(b) \in H^*(X \times Y, X \times B; R)$.

Theorem 3.18. *For CW pairs (X, A) and (Y, B) the cross product homomorphism $H^*(X, A; R) \otimes_R H^*(Y, B; R) \to H^*(X \times Y, A \times Y \cup X \times B; R)$ is an isomorphism of rings if $H^k(Y, B; R)$ is a finitely generated free R-module for each k.*

Proof: The case $B = \varnothing$ was covered in the course of the proof of the absolute case, so it suffices to deduce the case $B \neq \varnothing$ from the case $B = \varnothing$.

The following commutative diagram shows that collapsing B to a point reduces the proof to the case that B is a point:

$$
\begin{array}{ccc}
H^*(X,A) \otimes_R H^*(Y,B) & \xleftarrow{\;\;\approx\;\;} & H^*(X,A) \otimes_R H^*(Y/B, B/B) \\
\Big\downarrow{\scriptstyle\times} & & \Big\downarrow{\scriptstyle\times} \\
H^*(X \times Y, A \times Y \cup X \times B) & \xleftarrow{\;\approx\;} & H^*(X \times (Y/B), A \times (Y/B) \cup X \times (B/B))
\end{array}
$$

The lower map is an isomorphism since the quotient spaces $(X \times Y)/(A \times Y \cup X \times B)$ and $(X \times (Y/B))/(A \times (Y/B) \cup X \times (B/B))$ are the same.

For the case that B is a point $y_0 \in Y$ we will use the following commutative diagram:

$$
\begin{array}{ccccc}
H^*(X,A) \otimes_R H^*(Y, y_0) & \longrightarrow & H^*(X,A) \otimes_R H^*(Y) & \longrightarrow & H^*(X,A) \otimes_R H^*(y_0) \\
& & & & \Big\downarrow{\scriptstyle\times} \\
\Big\downarrow{\scriptstyle\times} & & \Big\downarrow{\scriptstyle\times} & & H^*(X \times y_0, A \times y_0) \\
& & & \nearrow & \Big\uparrow{\scriptstyle\approx} \\
H^*(X \times Y, X \times y_0 \cup A \times Y) & \longrightarrow & H^*(X \times Y, A \times Y) & \longrightarrow & H^*(X \times y_0 \cup A \times Y, A \times Y)
\end{array}
$$

Since y_0 is a retract of Y, the upper row of this diagram is a split short exact sequence. The lower row is the long exact sequence of a triple, and it too is a split short exact sequence since $(X \times y_0, A \times y_0)$ is a retract of $(X \times Y, A \times Y)$. The middle and right cross product maps are isomorphisms by the case $B = \varnothing$ since $H^k(Y; R)$ is a finitely generated free R-module if $H^k(Y, y_0; R)$ is. The five-lemma then implies that the left-hand cross product map is an isomorphism as well. $\qquad\square$

The relative cross product for pairs (X, x_0) and (Y, y_0) gives a reduced cross product

$$
\tilde{H}^*(X; R) \otimes_R \tilde{H}^*(Y; R) \xrightarrow{\;\times\;} \tilde{H}^*(X \wedge Y; R)
$$

where $X \wedge Y$ is the smash product $X \times Y/(X \times \{y_0\} \cup \{x_0\} \times Y)$. The preceding theorem implies that this reduced cross product is an isomorphism if either $\tilde{H}^*(X; R)$ or $\tilde{H}^*(Y; R)$ is a finitely generated free R-module in each dimension. For example, we have isomorphisms $\tilde{H}^n(X; R) \approx \tilde{H}^{n+k}(X \wedge S^k; R)$ via cross product with a generator of $H^k(S^k; R) \approx R$. The space $X \wedge S^k$ is the k-fold reduced suspension $\Sigma^k X$ of X, so we see that the suspension isomorphisms $\tilde{H}^n(X; R) \approx \tilde{H}^{n+k}(\Sigma^k X; R)$ derivable by elementary exact sequence arguments can also be obtained via cross product with a generator of $\tilde{H}^*(S^k; R)$.

Spaces with Polynomial Cohomology

Earlier in this section we mentioned that projective spaces provide examples of spaces whose cohomology rings are polynomial rings. Here is the precise statement:

Theorem 3.19. $H^*(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) \approx \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$ and $H^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2) \approx \mathbb{Z}_2[\alpha]$, where $|\alpha| = 1$. In the complex case, $H^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \approx \mathbb{Z}[\alpha]/(\alpha^{n+1})$ and $H^*(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \approx \mathbb{Z}[\alpha]$ where $|\alpha| = 2$.

This turns out to be a quite important result, and it can be proved in a number of different ways. The proof we give here uses the geometry of projective spaces to reduce the result to a very special case of the Künneth formula. Another proof using Poincaré duality will be given in Example 3.40. A third proof is contained in Example 4D.5 as an application of the Gysin sequence.

Proof: Let us do the case of $\mathbb{R}\mathrm{P}^n$ first. To simplify notation we abbreviate $\mathbb{R}\mathrm{P}^n$ to P^n and we let the coefficient group $\mathbb{Z}_2$ be implicit. Since the inclusion $P^{n-1} \hookrightarrow P^n$ induces an isomorphism on H^i for $i \le n-1$, it suffices by induction on n to show that the cup product of a generator of $H^{n-1}(P^n)$ with a generator of $H^1(P^n)$ is a generator of $H^n(P^n)$. It will be no more work to show more generally that the cup product of a generator of $H^i(P^n)$ with a generator of $H^{n-i}(P^n)$ is a generator of $H^n(P^n)$. As a further notational aid, we let $j = n - i$, so $i + j = n$.

The proof uses some of the geometric structure of P^n. Recall that P^n consists of nonzero vectors $(x_0, \cdots, x_n) \in \mathbb{R}^{n+1}$ modulo multiplication by nonzero scalars. Inside P^n is a copy of P^i represented by vectors whose last j coordinates $x_{i+1}, \cdots, x_n$ are zero. We also have a copy of P^j represented by points whose first i coordinates $x_0, \cdots, x_{i-1}$ are zero. The intersection $P^i \cap P^j$ is a single point p, represented by vectors whose only nonzero coordinate is x_i. Let U be the subspace of P^n represented by vectors with nonzero coordinate x_i. Each point in U may be represented by a unique vector with $x_i = 1$ and the other n coordinates arbitrary, so U is homeomorphic to $\mathbb{R}^n$, with p corresponding to 0 under this homeomorphism.
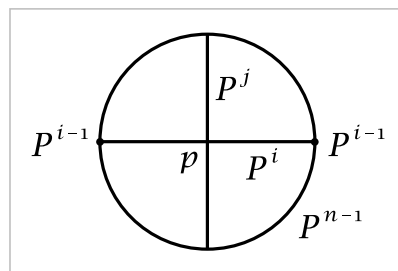
We can write this $\mathbb{R}^n$ as $\mathbb{R}^i \times \mathbb{R}^j$, with $\mathbb{R}^i$ as the coordinates $x_0, \cdots, x_{i-1}$ and $\mathbb{R}^j$ as the coordinates $x_{i+1}, \cdots, x_n$. In the figure P^n is represented as a disk with antipodal points of its boundary sphere identified to form a $P^{n-1} \subset P^n$ with $U = P^n - P^{n-1}$ the interior of the disk.

Consider the diagram

(i)
$$
\begin{array}{ccc}
H^i(P^n) \times H^j(P^n) & \xrightarrow{\smile} & H^n(P^n) \\
\uparrow & & \uparrow \\
H^i(P^n,P^n{-}P^j) \times H^j(P^n,P^n{-}P^i) & \xrightarrow{\smile} & H^n(P^n,P^n{-}\{p\}) \\
\downarrow & & \downarrow \\
H^i(\mathbb{R}^n,\mathbb{R}^n{-}\mathbb{R}^j) \times H^j(\mathbb{R}^n,\mathbb{R}^n{-}\mathbb{R}^i) & \xrightarrow{\smile} & H^n(\mathbb{R}^n,\mathbb{R}^n{-}\{0\})
\end{array}
$$

which commutes by naturality of cup product. We will show that the four vertical maps are isomorphisms and that the lower cup product map takes generator cross generator to generator. Commutativity of the diagram will then imply that the upper cup product map also takes generator cross generator to generator.

The lower map in the right column is an isomorphism by excision. For the upper map in this column, the fact that $P^n - \{p\}$ deformation retracts to a P^{n-1} gives an isomorphism $H^n(P^n, P^n - \{p\}) \approx H^n(P^n, P^{n-1})$ via the five-lemma applied to the long exact sequences for these pairs. And $H^n(P^n, P^{n-1}) \approx H^n(P^n)$ by cellular cohomology.

To see that the vertical maps in the left column of (i) are isomorphisms we will use the following commutative diagram:

(ii)
$$
\begin{array}{ccccccc}
H^i(P^n) & \longleftarrow & H^i(P^n, P^{i-1}) & \longleftarrow & H^i(P^n, P^n - P^j) & \longrightarrow & H^i(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^j) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^i(P^i) & \longleftarrow & H^i(P^i, P^{i-1}) & \longleftarrow & H^i(P^i, P^i - \{p\}) & \longrightarrow & H^i(\mathbb{R}^i, \mathbb{R}^i - \{0\})
\end{array}
$$

If we can show all these maps are isomorphisms, then the same argument will apply with i and j interchanged, and the vertical maps in the left column of (i) will be isomorphisms.

The left-hand square in (ii) consists of isomorphisms by cellular cohomology. The right-hand vertical map is obviously an isomorphism. The lower right horizontal map is an isomorphism by excision, and the map to the left of this is an isomorphism since $P^i - \{p\}$ deformation retracts onto P^{i-1}. The remaining maps will be isomorphisms if the middle map in the upper row is an isomorphism. And this map is in fact an isomorphism because $P^n - P^j$ deformation retracts onto P^{i-1} by the following argument. The subspace $P^n - P^j \subset P^n$ consists of points represented by vectors $v = (x_0, \cdots, x_n)$ with at least one of the coordinates $x_0, \cdots, x_{i-1}$ nonzero. The formula $f_t(v) = (x_0, \cdots, x_{i-1}, tx_i, \cdots, tx_n)$ for t decreasing from 1 to 0 gives a well-defined deformation retraction of $P^n - P^j$ onto P^{i-1} since $f_t(\lambda v) = \lambda f_t(v)$ for scalars $\lambda \in \mathbb{R}$.

The cup product map in the bottom row of (i) is equivalent to the cross product $H^i(I^i, \partial I^i) \times H^j(I^j, \partial I^j) \to H^n(I^n, \partial I^n)$, where the cross product of generators is a generator by the relative form of the Künneth formula in Theorem 3.18. Alternatively, if one wishes to use only the absolute Künneth formula, the cross product for cubes is equivalent to the cross product $H^i(S^i) \times H^j(S^j) \to H^n(S^i \times S^j)$ by means of the quotient maps $I^i \to S^i$ and $I^j \to S^j$ collapsing the boundaries of the cubes to points.

This finishes the proof for $\mathbb{R}\mathrm{P}^n$. The case of $\mathbb{R}\mathrm{P}^\infty$ follows from this since the inclusion $\mathbb{R}\mathrm{P}^n \hookrightarrow \mathbb{R}\mathrm{P}^\infty$ induces isomorphisms on $H^i(-; \mathbb{Z}_2)$ for $i \leq n$ by cellular cohomology.

Complex projective spaces are handled in precisely the same way, using $\mathbb{Z}$ coefficients and replacing each H^k by H^{2k} and $\mathbb{R}$ by $\mathbb{C}$.   □

There are also quaternionic projective spaces $\mathbb{H}\mathrm{P}^n$ and $\mathbb{H}\mathrm{P}^\infty$, defined exactly as in the complex case, with CW structures of the form $e^0 \cup e^4 \cup e^8 \cup \cdots$. Associativity of quaternion multiplication is needed for the identification $v \sim \lambda v$ to be an equivalence relation, so the definition does not extend to octonionic projective spaces, though there is an octonionic projective plane $\mathbb{O}\mathrm{P}^2$ defined in Example 4.47. The cup product structure in quaternionic projective spaces is just like that in complex projective spaces, except that the generator is 4–dimensional:

$$H^*(\mathbb{H}\mathrm{P}^\infty;\mathbb{Z}) \approx \mathbb{Z}[\alpha] \quad \text{and} \quad H^*(\mathbb{H}\mathrm{P}^n;\mathbb{Z}) \approx \mathbb{Z}[\alpha]/(\alpha^{n+1}), \quad \text{with } |\alpha| = 4$$

The same proof as in the real and complex cases works here as well.

The cup product structure for $\mathbb{R}\mathrm{P}^\infty$ with $\mathbb{Z}$ coefficients can easily be deduced from the cup product structure with $\mathbb{Z}_2$ coefficients, as follows. In general, a ring homomorphism $R \to S$ induces a ring homomorphism $H^*(X,A;R) \to H^*(X,A;S)$. In the case of the projection $\mathbb{Z} \to \mathbb{Z}_2$ we get for $\mathbb{R}\mathrm{P}^\infty$ an induced chain map of cellular cochain complexes with $\mathbb{Z}$ and $\mathbb{Z}_2$ coefficients:

$$
\begin{array}{ccccccccccc}
\cdots & \longleftarrow & \mathbb{Z} & \overset{2}{\longleftarrow} & \mathbb{Z} & \overset{0}{\longleftarrow} & \mathbb{Z} & \overset{2}{\longleftarrow} & \mathbb{Z} & \overset{0}{\longleftarrow} & \mathbb{Z} & \longleftarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & \mathbb{Z}_2 & \overset{0}{\longleftarrow} & \mathbb{Z}_2 & \overset{0}{\longleftarrow} & \mathbb{Z}_2 & \overset{0}{\longleftarrow} & \mathbb{Z}_2 & \overset{0}{\longleftarrow} & \mathbb{Z}_2 & \longleftarrow & 0
\end{array}
$$

From this we see that the ring homomorphism $H^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}) \to H^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2)$ is injective in positive dimensions, with image the even-dimensional part of $H^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2)$. Alternatively, this could be deduced from the universal coefficient theorem. Hence we have $H^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}) \approx \mathbb{Z}[\alpha]/(2\alpha)$ with $|\alpha| = 2$.

The cup product structure in $H^*(\mathbb{R}\mathrm{P}^n;\mathbb{Z})$ can be computed in a similar fashion, though the description is a little cumbersome:

$$H^*(\mathbb{R}\mathrm{P}^{2k};\mathbb{Z}) \approx \mathbb{Z}[\alpha]/(2\alpha, \alpha^{k+1}), \quad |\alpha| = 2$$
$$H^*(\mathbb{R}\mathrm{P}^{2k+1};\mathbb{Z}) \approx \mathbb{Z}[\alpha,\beta]/(2\alpha, \alpha^{k+1}, \beta^2, \alpha\beta), \quad |\alpha| = 2, \ |\beta| = 2k+1$$

Here β is a generator of $H^{2k+1}(\mathbb{R}\mathrm{P}^{2k+1};\mathbb{Z}) \approx \mathbb{Z}$. From this calculation we see that the rings $H^*(\mathbb{R}\mathrm{P}^{2k+1};\mathbb{Z})$ and $H^*(\mathbb{R}\mathrm{P}^{2k} \vee S^{2k+1};\mathbb{Z})$ are isomorphic, though with $\mathbb{Z}_2$ coefficients this is no longer true, as the generator $\alpha \in H^1(\mathbb{R}\mathrm{P}^{2k+1};\mathbb{Z}_2)$ has $\alpha^{2k+1} \neq 0$, while $\alpha^{2k+1} = 0$ for the generator $\alpha \in H^1(\mathbb{R}\mathrm{P}^{2k} \vee S^{2k+1};\mathbb{Z}_2)$.

Example 3.20. Combining the calculation $H^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2) \approx \mathbb{Z}_2[\alpha]$ with the Künneth formula, we see that $H^*(\mathbb{R}\mathrm{P}^\infty \times \mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2[\alpha_1] \otimes \mathbb{Z}_2[\alpha_2]$, which is just the polynomial ring $\mathbb{Z}_2[\alpha_1,\alpha_2]$. More generally it follows by induction that for a product of n copies of $\mathbb{R}\mathrm{P}^\infty$, the $\mathbb{Z}_2$–cohomology is a polynomial ring in n variables. Similar remarks apply to $\mathbb{C}\mathrm{P}^\infty$ and $\mathbb{H}\mathrm{P}^\infty$ with coefficients in $\mathbb{Z}$ or any commutative ring.

The following theorem of Hopf is a nice algebraic application of the cup product structure in $H^*(\mathbb{R}P^n \times \mathbb{R}P^n; \mathbb{Z}_2)$.

Theorem 3.21. *If $\mathbb{R}^n$ has the structure of a division algebra over the scalar field $\mathbb{R}$, then n must be a power of 2.*

Proof: For a division algebra structure on $\mathbb{R}^n$ the multiplication maps $x \mapsto ax$ and $x \mapsto xa$ are linear isomorphisms for each nonzero a, so the multiplication map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ induces a map $h: \mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1} \to \mathbb{R}P^{n-1}$ which is a homeomorphism when restricted to each subspace $\mathbb{R}P^{n-1} \times \{y\}$ and $\{x\} \times \mathbb{R}P^{n-1}$. The map h is continuous since it is a quotient of the multiplication map which is bilinear and hence continuous. The induced homomorphism h^* on $\mathbb{Z}_2$-cohomology is a ring homomorphism $\mathbb{Z}_2[\alpha]/(\alpha^n) \to \mathbb{Z}_2[\alpha_1, \alpha_2]/(\alpha_1^n, \alpha_2^n)$ determined by the element $h^*(\alpha) = k_1\alpha_1 + k_2\alpha_2$. The inclusion $\mathbb{R}P^{n-1} \hookrightarrow \mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1}$ onto the first factor sends α_1 to α and α_2 to 0, as one sees by composing with the projections of $\mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1}$ onto its two factors. The fact that h restricts to a homeomorphism on the first factor then implies that k_1 is nonzero. Similarly k_2 is nonzero, so since these coefficients lie in $\mathbb{Z}_2$ we have $h^*(\alpha) = \alpha_1 + \alpha_2$.

Since $\alpha^n = 0$ we must have $h^*(\alpha^n) = 0$, so $(\alpha_1 + \alpha_2)^n = \sum_k \binom{n}{k} \alpha_1^k \alpha_2^{n-k} = 0$. This is an equation in the ring $\mathbb{Z}_2[\alpha_1, \alpha_2]/(\alpha_1^n, \alpha_2^n)$, so the coefficient $\binom{n}{k}$ must be zero in $\mathbb{Z}_2$ for all k in the range $0 < k < n$. It is a rather easy number theory fact that this happens only when n is a power of 2. Namely, an obviously equivalent statement is that in the polynomial ring $\mathbb{Z}_2[x]$, the equality $(1+x)^n = 1+x^n$ holds only when n is a power of 2. To prove the latter statement, write n as a sum of powers of 2, $n = n_1 + \cdots + n_k$ with $n_1 < \cdots < n_k$. Then $(1+x)^n = (1+x)^{n_1} \cdots (1+x)^{n_k} = (1+x^{n_1}) \cdots (1+x^{n_k})$ since squaring is an additive homomorphism with $\mathbb{Z}_2$ coefficients. If one multiplies the product $(1+x^{n_1}) \cdots (1+x^{n_k})$ out, no terms combine or cancel since $n_i \geq 2n_{i-1}$ for each i, and so the resulting polynomial has 2^k terms. Thus if this polynomial equals $1 + x^n$ we must have $k = 1$, which means that n is a power of 2. $\qquad\square$

The same argument can be applied with $\mathbb{C}$ in place of $\mathbb{R}$, to show that if $\mathbb{C}^n$ is a division algebra over $\mathbb{C}$ then $\binom{n}{k} = 0$ for all k in the range $0 < k < n$, but now we can use $\mathbb{Z}$ rather than $\mathbb{Z}_2$ coefficients, so we deduce that $n = 1$. Thus there are no higher-dimensional division algebras over $\mathbb{C}$. This is assuming we are talking about finite-dimensional division algebras. For infinite dimensions there is for example the field of rational functions $\mathbb{C}(x)$.

We saw in Theorem 3.19 that $\mathbb{R}P^\infty$, $\mathbb{C}P^\infty$, and $\mathbb{H}P^\infty$ have cohomology rings that are polynomial algebras. We will describe now a construction for enlarging S^{2n} to a space $J(S^{2n})$ whose cohomology ring $H^*(J(S^{2n}); \mathbb{Z})$ is almost the polynomial ring $\mathbb{Z}[x]$ on a generator x of dimension $2n$. And if we change from $\mathbb{Z}$ to $\mathbb{Q}$ coefficients, then $H^*(J(S^{2n}); \mathbb{Q})$ is exactly the polynomial ring $\mathbb{Q}[x]$. This construction, known

as the **James reduced product**, is also of interest because of its connections with loopspaces described in §4.J.

For a space X, let X^k be the product of k copies of X. From the disjoint union $\coprod_{k \geq 1} X^k$, let us form a quotient space $J(X)$ by identifying $(x_1, \cdots, x_i, \cdots, x_k)$ with $(x_1, \cdots, \hat{x}_i, \cdots, x_k)$ if $x_i = e$, a chosen basepoint of X. Points of $J(X)$ can thus be thought of as k-tuples $(x_1, \cdots, x_k)$, $k \geq 0$, with no $x_i = e$. Inside $J(X)$ is the subspace $J_m(X)$ consisting of the points $(x_1, \cdots, x_k)$ with $k \leq m$. This can be viewed as a quotient space of X^m under the identifications $(x_1, \cdots, x_i, e, \cdots, x_m) \sim (x_1, \cdots, e, x_i, \cdots, x_m)$. For example, $J_1(X) = X$ and $J_2(X) = X \times X / (x, e) \sim (e, x)$. If X is a CW complex with e a 0-cell, the quotient map $X^m \to J_m(X)$ glues together the m subcomplexes of the product complex X^m where one coordinate is e. These glueings are by homeomorphisms taking cells onto cells, so $J_m(X)$ inherits a CW structure from X^m. There are natural inclusions $J_m(X) \subset J_{m+1}(X)$ as subcomplexes, and $J(X)$ is the union of these subcomplexes, hence is also a CW complex.

Proposition 3.22. *For $n > 0$, $H^*(J(S^n); \mathbb{Z})$ consists of a $\mathbb{Z}$ in each dimension a multiple of n. If n is even, the ith power of a generator of $H^n(J(S^n); \mathbb{Z})$ is $i!$ times a generator of $H^{in}(J(S^n); \mathbb{Z})$, for each $i \geq 1$. When n is odd, $H^*(J(S^n); \mathbb{Z})$ is isomorphic as a graded ring to $H^*(S^n; \mathbb{Z}) \otimes H^*(J(S^{2n}); \mathbb{Z})$.*

It follows that for n even, $H^*(J(S^n); \mathbb{Z})$ can be identified with the subring of the polynomial ring $\mathbb{Q}[x]$ additively generated by the monomials $x^i / i!$. This subring is called a **divided polynomial algebra** and is denoted $\Gamma_{\mathbb{Z}}[x]$. Thus $H^*(J(S^n); \mathbb{Z})$ is isomorphic to $\Gamma_{\mathbb{Z}}[x]$ when n is even and to $\Lambda_{\mathbb{Z}}[x] \otimes \Gamma_{\mathbb{Z}}[y]$ when n is odd.

Proof: Giving S^n its usual CW structure, the resulting CW structure on $J(S^n)$ consists of exactly one cell in each dimension a multiple of n. If $n > 1$ we deduce immediately from cellular cohomology that $H^*(J(S^n); \mathbb{Z})$ consists exactly of $\mathbb{Z}$'s in dimensions a multiple of n. For an alternative argument that works also when $n = 1$, consider the quotient map $q: (S^n)^m \to J_m(S^n)$. This carries each cell of $(S^n)^m$ homeomorphically onto a cell of $J_m(S^n)$. In particular q is a cellular map, taking k-skeleton to k-skeleton for each k, so q induces a chain map of cellular chain complexes. This chain map is surjective since each cell of $J_m(S^n)$ is the homeomorphic image of a cell of $(S^n)^m$. Hence the cellular boundary maps for $J_m(S^n)$ will be trivial if they are trivial for $(S^n)^m$, as indeed they are since $H^*((S^n)^m; \mathbb{Z})$ is free with basis in one-to-one correspondence with the cells, by Theorem 3.15.

We can compute cup products in $H^*(J_m(S^n); \mathbb{Z})$ by computing their images under q^*. Let x_k denote the generator of $H^{kn}(J_m(S^n); \mathbb{Z})$ dual to the kn-cell, represented by the cellular cocycle assigning the value 1 to the kn-cell. Since q identifies all the n-cells of $(S^n)^m$ to form the n-cell of $J_m(S^n)$, we see from cellular cohomology that $q^*(x_1)$ is the sum $\alpha_1 + \cdots + \alpha_m$ of the generators of $H^n((S^n)^m; \mathbb{Z})$ dual to the n-cells of $(S^n)^m$. By the same reasoning we have $q^*(x_k) = \sum_{i_1 < \cdots < i_k} \alpha_{i_1} \cdots \alpha_{i_k}$.

If n is even, the cup product structure in $H^*((S^n)^m;\mathbb{Z})$ is strictly commutative and $H^*((S^n)^m;\mathbb{Z}) \approx \mathbb{Z}[\alpha_1,\cdots,\alpha_m]/(\alpha_1^2,\cdots,\alpha_m^2)$. Then we have

$$q^*(x_1^m) = (\alpha_1 + \cdots + \alpha_m)^m = m!\alpha_1\cdots\alpha_m = m!q^*(x_m)$$

Since q^* is an isomorphism on H^{mn} this implies $x_1^m = m!x_m$ in $H^{mn}(J_m(S^n);\mathbb{Z})$. The inclusion $J_m(S^n) \hookrightarrow J(S^n)$ induces isomorphisms on H^i for $i \le mn$ so we have $x_1^m = m!x_m$ in $H^*(J(S^n);\mathbb{Z})$ as well, where x_1 and x_m are interpreted now as elements of $H^*(J(S^n);\mathbb{Z})$.

When n is odd we have $x_1^2 = 0$ by commutativity, and it will suffice to prove the following two formulas:

(a) $x_1 x_{2m} = x_{2m+1}$ in $H^*(J_{2m+1}(S^n);\mathbb{Z})$.

(b) $x_2 x_{2m-2} = mx_{2m}$ in $H^*(J_{2m}(S^n);\mathbb{Z})$.

For (a) we apply q^* and compute in the exterior algebra $\Lambda_{\mathbb{Z}}[\alpha_1,\cdots,\alpha_{2m+1}]$:

$$q^*(x_1 x_{2m}) = \left(\sum_i \alpha_i\right)\left(\sum_i \alpha_1 \cdots \hat{\alpha}_i \cdots \alpha_{2m+1}\right)$$
$$= \sum_i \alpha_i \alpha_1 \cdots \hat{\alpha}_i \cdots \alpha_{2m+1} = \sum_i (-1)^{i-1}\alpha_1 \cdots \alpha_{2m+1}$$

The coefficients in this last summation are $+1, -1, \cdots, +1$, so their sum is $+1$ and (a) follows. For (b) we have

$$q^*(x_2 x_{2m-2}) = \left(\sum_{i_1<i_2} \alpha_{i_1}\alpha_{i_2}\right)\left(\sum_{i_1<i_2} \alpha_1 \cdots \hat{\alpha}_{i_1} \cdots \hat{\alpha}_{i_2} \cdots \alpha_{2m}\right)$$
$$= \sum_{i_1<i_2} \alpha_{i_1}\alpha_{i_2}\alpha_1 \cdots \hat{\alpha}_{i_1} \cdots \hat{\alpha}_{i_2} \cdots \alpha_{2m} = \sum_{i_1<i_2} (-1)^{i_1-1}(-1)^{i_2-2}\alpha_1 \cdots \alpha_{2m}$$

The terms in the coefficient $\sum_{i_1<i_2}(-1)^{i_1-1}(-1)^{i_2-2}$ for a fixed i_1 have i_2 varying from i_1+1 to $2m$. These terms are $+1, -1, \cdots$ and there are $2m-i_1$ of them, so their sum is 0 if i_1 is even and 1 if i_1 is odd. Now letting i_1 vary, it takes on the odd values $1, 3, \cdots, 2m-1$, so the whole summation reduces to m 1's and we have the desired relation $x_2 x_{2m-2} = mx_{2m}$. $\square$

In $\Gamma_{\mathbb{Z}}[x] \subset \mathbb{Q}[x]$, if we let $x_i = x^i/i!$ then the multiplicative structure is given by $x_i x_j = \binom{i+j}{i}x_{i+j}$. More generally, for a commutative ring R we could define $\Gamma_R[x]$ to be the free R-module with basis $x_0 = 1, x_1, x_2, \cdots$ and multiplication defined by $x_i x_j = \binom{i+j}{i}x_{i+j}$. The preceding proposition implies that $H^*(J(S^{2n});R) \approx \Gamma_R[x]$. When $R = \mathbb{Q}$ it is clear that $\Gamma_{\mathbb{Q}}[x]$ is just $\mathbb{Q}[x]$. However, for $R = \mathbb{Z}_p$ with p prime something quite different happens: There is an isomorphism

$$\Gamma_{\mathbb{Z}_p}[x] \approx \mathbb{Z}_p[x_1, x_p, x_{p^2}, \cdots]/(x_1^p, x_p^p, x_{p^2}^p, \cdots) = \bigotimes_{i\ge 0} \mathbb{Z}_p[x_{p^i}]/(x_{p^i}^p)$$

as we show in §3.C, where we will also see that divided polynomial algebras are in a certain sense dual to polynomial algebras.

The examples of projective spaces lead naturally to the following question: Given a coefficient ring R and an integer $d > 0$, is there a space X having $H^*(X; R) \approx R[\alpha]$ with $|\alpha| = d$? Historically, it took major advances in the theory to answer this simple-looking question. Here is a table giving all the possible values of d for some of the most obvious and important choices of R, namely $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_2$, and $\mathbb{Z}_p$ with p an odd prime. As we have seen, projective

R	d
$\mathbb{Z}$	2, 4
$\mathbb{Q}$	any even number
$\mathbb{Z}_2$	1, 2, 4
$\mathbb{Z}_p$	any even divisor of $2(p-1)$

spaces give the examples for $\mathbb{Z}$ and $\mathbb{Z}_2$. Examples for $\mathbb{Q}$ are the spaces $J(S^d)$, and examples for $\mathbb{Z}_p$ are constructed in §3.G. Showing that no other d's are possible takes considerably more work. The fact that d must be even when $R \neq \mathbb{Z}_2$ is a consequence of the commutativity property of cup product. In Theorem 4L.9 and Corollary 4L.10 we will settle the case $R = \mathbb{Z}$ and show that d must be a power of 2 for $R = \mathbb{Z}_2$ and a power of p times an even divisor of $2(p-1)$ for $R = \mathbb{Z}_p$, p odd. Ruling out the remaining cases is best done using K–theory, as in [VBKT] or the classical reference [Adams & Atiyah 1966]. However there is one slightly anomalous case, $R = \mathbb{Z}_2$, $d = 8$, which must be treated by special arguments; see [Toda 1963].

It is an interesting fact that for each even d there exists a CW complex X_d which is simultaneously an example for all the admissible choices of coefficients R in the table. Moreover, X_d can be chosen to have the simplest CW structure consistent with its cohomology, namely a single cell in each dimension a multiple of d. For example, we may take $X_2 = \mathbb{C}\mathrm{P}^\infty$ and $X_4 = \mathbb{H}\mathrm{P}^\infty$. The next space X_6 would have $H^*(X_6; \mathbb{Z}_p) \approx \mathbb{Z}_p[\alpha]$ for $p = 7, 13, 19, 31, \cdots$, primes of the form $3s + 1$, the condition $6 | 2(p-1)$ being equivalent to $p = 3s + 1$. (By a famous theorem of Dirichlet there are infinitely many primes in any such arithmetic progression.) Note that, in terms of $\mathbb{Z}$ coefficients, X_d must have the property that for a generator α of $H^d(X_d; \mathbb{Z})$, each power α^i is an integer a_i times a generator of $H^{di}(X_d; \mathbb{Z})$, with $a_i \neq 0$ if $H^*(X_d; \mathbb{Q}) \approx \mathbb{Q}[\alpha]$ and a_i relatively prime to p if $H^*(X_d; \mathbb{Z}_p) \approx \mathbb{Z}_p[\alpha]$. A construction of X_d is given in [SSAT], or in the original source [Hoffman & Porter 1973].

One might also ask about realizing the truncated polynomial ring $R[\alpha]/(\alpha^{n+1})$, in view of the examples provided by $\mathbb{R}\mathrm{P}^n$, $\mathbb{C}\mathrm{P}^n$, and $\mathbb{H}\mathrm{P}^n$, leaving aside the trivial case $n = 1$ where spheres provide examples. The analysis for polynomial rings also settles which truncated polynomial rings are realizable; there are just a few more than for the full polynomial rings.

There is also the question of realizing polynomial rings $R[\alpha_1, \cdots, \alpha_n]$ with generators α_i in specified dimensions d_i. Since $R[\alpha_1, \cdots, \alpha_m] \otimes_R R[\beta_1, \cdots, \beta_n]$ is equal to $R[\alpha_1, \cdots, \alpha_m, \beta_1, \cdots, \beta_n]$, the product of two spaces with polynomial cohomology is again a space with polynomial cohomology, assuming the number of polynomial generators is finite in each dimension. For example, the n-fold product $(\mathbb{C}\mathrm{P}^\infty)^n$ has $H^*((\mathbb{C}\mathrm{P}^\infty)^n; \mathbb{Z}) \approx \mathbb{Z}[\alpha_1, \cdots, \alpha_n]$ with each α_i 2-dimensional. Similarly, products of

the spaces $J(S^{d_i})$ realize all choices of even d_i's with $\mathbb{Q}$ coefficients.

However, with $\mathbb{Z}$ and $\mathbb{Z}_p$ coefficients, products of one-variable examples do not exhaust all the possibilities. As we show in §4.D, there are three other basic examples with $\mathbb{Z}$ coefficients:

1. Generalizing the space $\mathbb{C}P^\infty$ of complex lines through the origin in $\mathbb{C}^\infty$, there is the **Grassmann manifold** $G_n(\mathbb{C}^\infty)$ of n-dimensional vector subspaces of $\mathbb{C}^\infty$, and this has $H^*(G_n(\mathbb{C}^\infty); \mathbb{Z}) \approx \mathbb{Z}[\alpha_1, \cdots, \alpha_n]$ with $|\alpha_i| = 2i$. This space is also known as $BU(n)$, the 'classifying space' of the unitary group $U(n)$. It is central to the study of vector bundles and K–theory.

2. Replacing $\mathbb{C}$ by $\mathbb{H}$, there is the quaternionic Grassmann manifold $G_n(\mathbb{H}^\infty)$, also known as $BSp(n)$, the classifying space for the symplectic group $Sp(n)$, with $H^*(G_n(\mathbb{H}^\infty); \mathbb{Z}) \approx \mathbb{Z}[\alpha_1, \cdots, \alpha_n]$ with $|\alpha_i| = 4i$.

3. There is a classifying space $BSU(n)$ for the special unitary group $SU(n)$, whose cohomology is the same as for $BU(n)$ but with the first generator α_1 omitted, so $H^*(BSU(n); \mathbb{Z}) \approx \mathbb{Z}[\alpha_2, \cdots, \alpha_n]$ with $|\alpha_i| = 2i$.

These examples and their products account for all the realizable polynomial cup product rings with $\mathbb{Z}$ coefficients, according to a theorem in [Andersen & Grodal 2008]. The situation for $\mathbb{Z}_p$ coefficients is more complicated and will be discussed in §3.G.

Polynomial algebras are examples of free graded commutative algebras, where 'free' means loosely 'having no unnecessary relations'. In general, a free graded commutative algebra is a tensor product of single-generator free graded commutative algebras. The latter are either polynomial algebras $R[\alpha]$ on even-dimension generators α or quotients $R[\alpha]/(2\alpha^2)$ with α odd-dimensional. Note that if R is a field then $R[\alpha]/(2\alpha^2)$ is either the exterior algebra $\Lambda_R[\alpha]$ if the characteristic of R is not 2, or the polynomial algebra $R[\alpha]$ otherwise. Every graded commutative algebra is a quotient of a free one, clearly.

Example 3.23: Subcomplexes of the n-Torus. To give just a small hint of the endless variety of nonfree cup product algebras that can be realized, consider subcomplexes of the n-torus T^n, the product of n copies of S^1. Here we give S^1 its standard minimal cell structure and T^n the resulting product cell structure. We know that $H^*(T^n; \mathbb{Z})$ is the exterior algebra $\Lambda_{\mathbb{Z}}[\alpha_1, \cdots, \alpha_n]$, with the monomial $\alpha_{i_1} \cdots \alpha_{i_k}$ corresponding via cellular cohomology to the k-cell $e^1_{i_1} \times \cdots \times e^1_{i_k}$. So if we pass to a subcomplex $X \subset T^n$ by omitting certain cells, then $H^*(X; \mathbb{Z})$ is the quotient of $\Lambda_{\mathbb{Z}}[\alpha_1, \cdots, \alpha_n]$ obtained by setting the monomials corresponding to the omitted cells equal to zero. Since we are dealing with rings, we are factoring out by an ideal in $\Lambda_{\mathbb{Z}}[\alpha_1, \cdots, \alpha_n]$, the ideal generated by the monomials corresponding to the 'minimal' omitted cells, those whose boundary is entirely contained in X. For example, if we take X to be the subcomplex of T^3 obtained by deleting the cells $e^1_1 \times e^1_2 \times e^1_3$ and $e^1_2 \times e^1_3$, then $H^*(X; \mathbb{Z}) \approx \Lambda_{\mathbb{Z}}[\alpha_1, \alpha_2, \alpha_3]/(\alpha_2 \alpha_3)$.

How many different subcomplexes of T^n are there? To each subcomplex $X \subset T^n$ we can associate a finite simplicial complex C_X by the following procedure. View T^n as the quotient of the n-cube $I^n = [0,1]^n \subset \mathbb{R}^n$ obtained by identifying opposite faces. If we intersect I^n with the hyperplane $x_1 + \cdots + x_n = \varepsilon$ for small $\varepsilon > 0$, we get a simplex Δ^{n-1}. Then for $q : I^n \to T^n$ the quotient map, we take C_X to be $\Delta^{n-1} \cap q^{-1}(X)$. This is a subcomplex of Δ^{n-1} whose k-simplices correspond exactly to the $(k+1)$-cells of X. In this way we get a one-to-one correspondence between subcomplexes $X \subset T^n$ and subcomplexes $C_X \subset \Delta^{n-1}$. Every simplicial complex with n vertices is a subcomplex of Δ^{n-1}, so we see that T^n has quite a large number of subcomplexes if n is not too small. The cohomology rings $H^*(X;\mathbb{Z})$ are of a type that was completely classified in [Gubeladze 1998], Theorem 3.1, and from this classification it follows that the ring $H^*(X;\mathbb{Z})$ (or even $H^*(X;\mathbb{Z}_2)$) determines the subcomplex X uniquely, up to permutation of the n circle factors of T^n.

More elaborate examples could be produced by looking at subcomplexes of the product of n copies of $\mathbb{CP}^\infty$. In this case the cohomology rings are isomorphic to polynomial rings modulo ideals generated by monomials, and it is again true that the cohomology ring determines the subcomplex up to permutation of factors. However, these cohomology rings are still a whole lot less complicated than the general case, where one takes free algebras modulo ideals generated by arbitrary polynomials having all their terms of the same dimension.

Let us conclude this section with an example of a cohomology ring that is not too far removed from a polynomial ring.

Example 3.24: Cohen–Macaulay Rings. Let X be the quotient space $\mathbb{CP}^\infty/\mathbb{CP}^{n-1}$. The quotient map $\mathbb{CP}^\infty \to X$ induces an injection $H^*(X;\mathbb{Z}) \to H^*(\mathbb{CP}^\infty;\mathbb{Z})$ embedding $H^*(X;\mathbb{Z})$ in $\mathbb{Z}[\alpha]$ as the subring generated by $1, \alpha^n, \alpha^{n+1}, \cdots$. If we view this subring as a module over $\mathbb{Z}[\alpha^n]$, it is free with basis $\{1, \alpha^{n+1}, \alpha^{n+2}, \cdots, \alpha^{2n-1}\}$. Thus $H^*(X;\mathbb{Z})$ is an example of a *Cohen–Macaulay* ring: a ring containing a polynomial subring over which it is a finitely generated free module. While polynomial cup product rings are rather rare, Cohen–Macauley cup product rings occur much more frequently.

Exercises

1. Assuming as known the cup product structure on the torus $S^1 \times S^1$, compute the cup product structure in $H^*(M_g)$ for M_g the closed orientable surface of genus g by using the quotient map from M_g to a wedge sum of g tori, shown below.
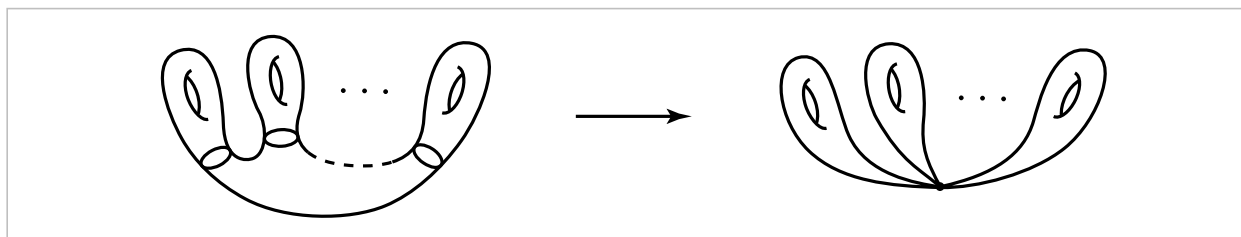
2. Using the cup product $H^k(X, A; R) \times H^\ell(X, B; R) \to H^{k+\ell}(X, A \cup B; R)$, show that if X is the union of contractible open subsets A and B, then all cup products of positive-dimensional classes in $H^*(X; R)$ are zero. This applies in particular if X is a suspension. Generalize to the situation that X is the union of n contractible open subsets, to show that all n-fold cup products of positive-dimensional classes are zero.

3. (a) Using the cup product structure, show there is no map $\mathbb{R}P^n \to \mathbb{R}P^m$ inducing a nontrivial map $H^1(\mathbb{R}P^m; \mathbb{Z}_2) \to H^1(\mathbb{R}P^n; \mathbb{Z}_2)$ if $n > m$. What is the corresponding result for maps $\mathbb{C}P^n \to \mathbb{C}P^m$?

(b) Prove the Borsuk–Ulam theorem by the following argument. Suppose on the contrary that $f : S^n \to \mathbb{R}^n$ satisfies $f(x) \neq f(-x)$ for all x. Then define $g : S^n \to S^{n-1}$ by $g(x) = (f(x) - f(-x))/|f(x) - f(-x)|$, so $g(-x) = -g(x)$ and g induces a map $\mathbb{R}P^n \to \mathbb{R}P^{n-1}$. Show that part (a) applies to this map.

4. Apply the Lefschetz fixed point theorem to show that every map $f : \mathbb{C}P^n \to \mathbb{C}P^n$ has a fixed point if n is even, using the fact that $f^* : H^*(\mathbb{C}P^n; \mathbb{Z}) \to H^*(\mathbb{C}P^n; \mathbb{Z})$ is a ring homomorphism. When n is odd show there is a fixed point unless $f^*(\alpha) = -\alpha$, for α a generator of $H^2(\mathbb{C}P^n; \mathbb{Z})$. [See Exercise 3 in §2.C for an example of a map without fixed points in this exceptional case.]

5. Show the ring $H^*(\mathbb{R}P^\infty; \mathbb{Z}_{2k})$ is isomorphic to $\mathbb{Z}_{2k}[\alpha, \beta]/(2\alpha, 2\beta, \alpha^2 - k\beta)$ where $|\alpha| = 1$ and $|\beta| = 2$. [Use the coefficient map $\mathbb{Z}_{2k} \to \mathbb{Z}_2$ and the proof of Theorem 3.19.]

6. Use cup products to compute the map $H^*(\mathbb{C}P^n; \mathbb{Z}) \to H^*(\mathbb{C}P^n; \mathbb{Z})$ induced by the map $\mathbb{C}P^n \to \mathbb{C}P^n$ that is a quotient of the map $\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ raising each coordinate to the dth power, $(z_0, \cdots, z_n) \mapsto (z_0^d, \cdots, z_n^d)$, for a fixed integer $d > 0$. [First do the case $n = 1$.]

7. Use cup products to show that $\mathbb{R}P^3$ is not homotopy equivalent to $\mathbb{R}P^2 \vee S^3$.

8. Let X be $\mathbb{C}P^2$ with a cell e^3 attached by a map $S^2 \to \mathbb{C}P^1 \subset \mathbb{C}P^2$ of degree p, and let $Y = M(\mathbb{Z}_p, 2) \vee S^4$. Thus X and Y have the same 3-skeleton but differ in the way their 4-cells are attached. Show that X and Y have isomorphic cohomology rings with $\mathbb{Z}$ coefficients but not with $\mathbb{Z}_p$ coefficients.

9. Show that if $H_n(X; \mathbb{Z})$ is free for each n, then $H^*(X; \mathbb{Z}_p)$ and $H^*(X; \mathbb{Z}) \otimes \mathbb{Z}_p$ are isomorphic as rings, so in particular the ring structure with $\mathbb{Z}$ coefficients determines the ring structure with $\mathbb{Z}_p$ coefficients.

10. Show that the cross product map $H^*(X; \mathbb{Z}) \otimes H^*(Y; \mathbb{Z}) \to H^*(X \times Y; \mathbb{Z})$ is not an isomorphism if X and Y are infinite discrete sets. [This shows the necessity of the hypothesis of finite generation in Theorem 3.15.]

11. Using cup products, show that every map $S^{k+\ell} \to S^k \times S^\ell$ induces the trivial homomorphism $H_{k+\ell}(S^{k+\ell}) \to H_{k+\ell}(S^k \times S^\ell)$, assuming $k > 0$ and $\ell > 0$.

12. Show that the spaces $(S^1 \times \mathbb{C}P^\infty)/(S^1 \times \{x_0\})$ and $S^3 \times \mathbb{C}P^\infty$ have isomorphic cohomology rings with $\mathbb{Z}$ or any other coefficients. [An exercise for §4.L is to show these two spaces are not homotopy equivalent.]

13. Describe $H^*(\mathbb{C}\mathrm{P}^\infty/\mathbb{C}\mathrm{P}^1;\mathbb{Z})$ as a ring with finitely many multiplicative generators. How does this ring compare with $H^*(S^6\times\mathbb{H}\mathrm{P}^\infty;\mathbb{Z})$?

14. Let $q:\mathbb{R}\mathrm{P}^\infty\to\mathbb{C}\mathrm{P}^\infty$ be the natural quotient map obtained by regarding both spaces as quotients of S^∞, modulo multiplication by real scalars in one case and complex scalars in the other. Show that the induced map $q^*:H^*(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})\to H^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z})$ is surjective in even dimensions by showing first by a geometric argument that the restriction $q:\mathbb{R}\mathrm{P}^2\to\mathbb{C}\mathrm{P}^1$ induces a surjection on H^2 and then appealing to cup product structures. Next, form a quotient space X of $\mathbb{R}\mathrm{P}^\infty\amalg\mathbb{C}\mathrm{P}^n$ by identifying each point $x\in\mathbb{R}\mathrm{P}^{2n}$ with $q(x)\in\mathbb{C}\mathrm{P}^n$. Show there are ring isomorphisms $H^*(X;\mathbb{Z})\approx\mathbb{Z}[\alpha]/(2\alpha^{n+1})$ and $H^*(X;\mathbb{Z}_2)\approx\mathbb{Z}_2[\alpha,\beta]/(\beta^2-\alpha^{2n+1})$, where $|\alpha|=2$ and $|\beta|=2n+1$. Make a similar construction and analysis for the quotient map $q:\mathbb{C}\mathrm{P}^\infty\to\mathbb{H}\mathrm{P}^\infty$.

15. For a fixed coefficient field F, define the **Poincaré series** of a space X to be the formal power series $p(t)=\sum_i a_i t^i$ where a_i is the dimension of $H^i(X;F)$ as a vector space over F, assuming this dimension is finite for all i. Show that $p(X\times Y)=p(X)p(Y)$. Compute the Poincaré series for S^n, $\mathbb{R}\mathrm{P}^n$, $\mathbb{R}\mathrm{P}^\infty$, $\mathbb{C}\mathrm{P}^n$, $\mathbb{C}\mathrm{P}^\infty$, and the spaces in the preceding three exercises.

16. Show that if X and Y are finite CW complexes such that $H^*(X;\mathbb{Z})$ and $H^*(Y;\mathbb{Z})$ contain no elements of order a power of a given prime p, then the same is true for $X\times Y$. [Apply Theorem 3.15 with coefficients in various fields.]

17. [This has now been incorporated into Proposition 3.22.]

18. For the closed orientable surface M of genus $g\geq 1$, show that for each nonzero $\alpha\in H^1(M;\mathbb{Z})$ there exists $\beta\in H^1(M;\mathbb{Z})$ with $\alpha\beta\neq 0$. Deduce that M is not homotopy equivalent to a wedge sum $X\vee Y$ of CW complexes with nontrivial reduced homology. Do the same for closed nonorientable surfaces using cohomology with $\mathbb{Z}_2$ coefficients.

3.3 Poincaré Duality

Algebraic topology is most often concerned with properties of spaces that depend only on homotopy type, so local topological properties do not play much of a role. Digressing somewhat from this viewpoint, we study in this section a class of spaces whose most prominent feature is their local topology, namely manifolds, which are locally homeomorphic to $\mathbb{R}^n$. It is somewhat miraculous that just this local homogeneity property, together with global compactness, is enough to impose a strong symmetry on the homology and cohomology groups of such spaces, as well as strong nontriviality of cup products. This is the Poincaré duality theorem, one of the earliest theorems in the subject. In fact, Poincaré's original work on the duality property came before homology and cohomology had even been properly defined, and it took many

years for the concepts of homology and cohomology to be refined sufficiently to put Poincaré duality on a firm footing.

Let us begin with some definitions. A **manifold of dimension** n, or more concisely an **n-manifold**, is a Hausdorff space M in which each point has an open neighborhood homeomorphic to $\mathbb{R}^n$. The dimension of M is intrinsically characterized by the fact that for $x \in M$, the local homology group $H_i(M, M - \{x\}; \mathbb{Z})$ is nonzero only for $i = n$:

$$H_i(M, M - \{x\}; \mathbb{Z}) \approx H_i(\mathbb{R}^n, \mathbb{R}^n - \{0\}; \mathbb{Z}) \qquad \text{by excision}$$
$$\approx \tilde{H}_{i-1}(\mathbb{R}^n - \{0\}; \mathbb{Z}) \qquad \text{since } \mathbb{R}^n \text{ is contractible}$$
$$\approx \tilde{H}_{i-1}(S^{n-1}; \mathbb{Z}) \qquad \text{since } \mathbb{R}^n - \{0\} \simeq S^{n-1}$$

A compact manifold is called **closed**, to distinguish it from the more general notion of a compact manifold with boundary, considered later in this section. For example S^n is a closed manifold, as are $\mathbb{R}\mathrm{P}^n$ and lens spaces since they have S^n as a covering space. Another closed manifold is $\mathbb{C}\mathrm{P}^n$. This is compact since it is a quotient space of S^{2n+1}, and the manifold property is satisfied since there is an open cover by subsets homeomorphic to $\mathbb{R}^{2n}$, the sets $U_i = \{[z_0, \cdots, z_n] \in \mathbb{C}\mathrm{P}^n \mid z_i = 1\}$. The same reasoning applies also for quaternionic projective spaces. Further examples of closed manifolds can be generated from these using the obvious fact that the product of closed manifolds of dimensions m and n is a closed manifold of dimension $m + n$.

Poincaré duality in its most primitive form asserts that for a closed orientable manifold M of dimension n, there are isomorphisms $H_k(M; \mathbb{Z}) \approx H^{n-k}(M; \mathbb{Z})$ for all k. Implicit here is the convention that homology and cohomology groups of negative dimension are zero, so the duality statement includes the fact that all the nontrivial homology and cohomology of M lies in the dimension range from 0 to n. The definition of 'orientable' will be given below. Without the orientability hypothesis there is a weaker statement that $H_k(M; \mathbb{Z}_2) \approx H^{n-k}(M; \mathbb{Z}_2)$ for all k. As we show in Corollaries A.8 and A.9 in the Appendix, the homology groups of a closed manifold are all finitely generated. So via the universal coefficient theorem, Poincaré duality for a closed orientable n-manifold M can be stated just in terms of homology: Modulo their torsion subgroups, $H_k(M; \mathbb{Z})$ and $H_{n-k}(M; \mathbb{Z})$ are isomorphic, and the torsion subgroups of $H_k(M; \mathbb{Z})$ and $H_{n-k-1}(M; \mathbb{Z})$ are isomorphic. However, the statement in terms of cohomology is really more natural.

Poincaré duality thus expresses a certain symmetry in the homology of closed orientable manifolds. For example, consider the n-dimensional torus T^n, the product of n circles. By induction on n it follows from the Künneth formula, or from the easy special case $H_i(X \times S^1; \mathbb{Z}) \approx H_i(X; \mathbb{Z}) \oplus H_{i-1}(X; \mathbb{Z})$ which was an exercise in §2.2, that $H_k(T^n; \mathbb{Z})$ is isomorphic to the direct sum of $\binom{n}{k}$ copies of $\mathbb{Z}$. So Poincaré duality is reflected in the relation $\binom{n}{k} = \binom{n}{n-k}$. The reader might also check that Poincaré duality is consistent with our calculations of the homology of projective spaces and lens spaces, which are all orientable except for $\mathbb{R}\mathrm{P}^n$ with n even.

For many manifolds there is a very nice geometric proof of Poincaré duality using the notion of *dual cell structures*. The germ of this idea can be traced back to the five regular Platonic solids: the tetrahedron, cube, octahedron, dodecahedron, and icosahedron. Each of these polyhedra has a dual polyhedron whose vertices are the center points of the faces of the given polyhedron. Thus the dual of the cube is the octahedron, and vice versa. Similarly the dodecahedron and icosahedron are dual to each other, and the tetrahedron is its own dual. One can regard each of these polyhedra as defining a cell structure C on S^2 with a dual cell structure C^* determined by the dual polyhedron. Each vertex of C lies in a dual 2-cell of C^*, each edge of C crosses a dual edge of C^*, and each 2-cell of C contains a dual vertex of C^*.

The first figure at the right shows the case of the cube and octahedron. There is no need to restrict to regular polyhedra here, and we can generalize further by replacing S^2 by any surface. A portion of a more-or-less random pair of dual cell structures is shown in the second figure. On the torus, if we lift a dual pair of cell structures to the universal cover $\mathbb{R}^2$, we get a dual pair of periodic tilings of the plane, as in the next three figures. The last two figures show that the standard CW structure on the sur-

face of genus g, obtained from a $4g$-gon by identifying edges via the product of commutators $[a_1, b_1] \cdots [a_g, b_g]$, is homeomorphic to its own dual.
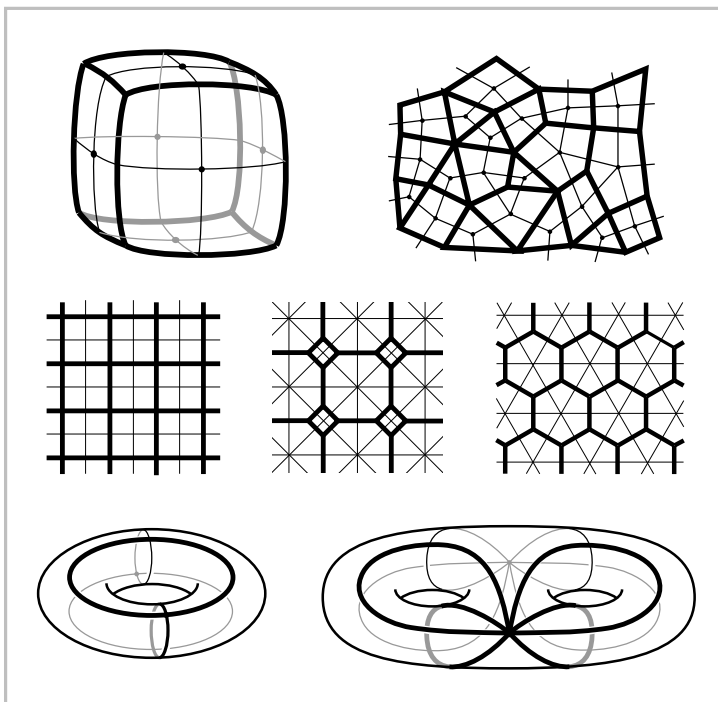
Given a pair of dual cell structures C and C^* on a closed surface M, the pairing of cells with dual cells gives identifications of cellular chain groups $C_0^* = C_2$, $C_1^* = C_1$, and $C_2^* = C_0$. If we use $\mathbb{Z}$ coefficients these identifications are not quite canonical since there is an ambiguity of sign for each cell, the choice of a generator for the corresponding $\mathbb{Z}$ summand of the cellular chain complex. We can avoid this ambiguity by considering the simpler situation of $\mathbb{Z}_2$ coefficients, where the identifications $C_i = C_{2-i}^*$ are completely canonical. The key observation now is that under these identifications, the cellular boundary map $\partial : C_i \to C_{i-1}$ becomes the cellular coboundary map $\delta : C_{2-i}^* \to C_{2-i+1}^*$ since ∂ assigns to a cell the sum of the cells which are faces of it, while δ assigns to a cell the sum of the cells of which it is a face. Thus $H_i(C; \mathbb{Z}_2) \approx H^{2-i}(C^*; \mathbb{Z}_2)$, and hence $H_i(M; \mathbb{Z}_2) \approx H^{2-i}(M; \mathbb{Z}_2)$ since C and C^* are cell structures on the same surface M.

To refine this argument to $\mathbb{Z}$ coefficients the problem of signs must be addressed. After analyzing the situation more closely, one sees that if M is orientable, it is possible to make consistent choices of orientations of all the cells of C and C^* so that the boundary maps in C agree with the coboundary maps in C^*, and therefore one gets $H_i(C;\mathbb{Z}) \approx H^{2-i}(C^*;\mathbb{Z})$, hence $H_i(M;\mathbb{Z}) \approx H^{2-i}(M;\mathbb{Z})$.

For manifolds of higher dimension the situation is entirely analogous. One would consider dual cell structures C and C^* on a closed n-manifold M, each i-cell of C being dual to a unique $(n-i)$-cell of C^* which it intersects in one point 'transversely'. For example on the 3-dimensional torus $S^1 \times S^1 \times S^1$ one could take the standard cell structure lifting to the decomposition of the universal cover $\mathbb{R}^3$ into cubes with vertices at the integer lattice points $\mathbb{Z}^3$, and then the dual cell structure is obtained by translating this by the vector $(1/2, 1/2, 1/2)$. Each edge in either cell structure then has a dual 2-cell which it pierces orthogonally, and each vertex lies in a dual 3-cell.

All the manifolds one commonly meets, for example all differentiable manifolds, have dually paired cell structures with the properties needed to carry out the proof of Poincaré duality we have just sketched. However, to construct these cell structures requires a certain amount of manifold theory. To avoid this, and to get a theorem that applies to all manifolds, we will take a completely different approach, using algebraic topology to replace the geometry of dual cell structures.

Orientations and Homology

Let us consider the question of how one might define orientability for manifolds. First there is the local question: What is an orientation of $\mathbb{R}^n$? Whatever an orientation of $\mathbb{R}^n$ is, it should have the property that it is preserved under rotations and reversed by reflections. For example, in $\mathbb{R}^2$ the notions of 'clockwise' and 'counterclockwise' certainly have this property, as do 'right-handed' and 'left-handed' in $\mathbb{R}^3$. We shall take the viewpoint that this property is what characterizes orientations, so anything satisfying the property can be regarded as an orientation.

With this in mind, we propose the following as an algebraic-topological definition: An orientation of $\mathbb{R}^n$ at a point x is a choice of generator of the infinite cyclic group $H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\})$, where the absence of a coefficient group from the notation means that we take coefficients in $\mathbb{Z}$. To verify that the characteristic property of orientations is satisfied we use the isomorphisms $H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \approx H_{n-1}(\mathbb{R}^n - \{x\}) \approx H_{n-1}(S^{n-1})$ where S^{n-1} is a sphere centered at x. Since these isomorphisms are natural, and rotations of S^{n-1} have degree 1, being homotopic to the identity, while reflections have degree -1, we see that a rotation ρ of $\mathbb{R}^n$ fixing x takes a generator α of $H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\})$ to itself, $\rho_*(\alpha) = \alpha$, while a reflection takes α to $-\alpha$.

Note that with this definition, an orientation of $\mathbb{R}^n$ at a point x determines an orientation at every other point y via the canonical isomorphisms $H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \approx H_n(\mathbb{R}^n, \mathbb{R}^n - B) \approx H_n(\mathbb{R}^n, \mathbb{R}^n - \{y\})$ where B is any ball containing both x and y.

An advantage of this definition of local orientation is that it can be applied to any n-dimensional manifold M: A **local orientation** of M at a point x is a choice of generator μ_x of the infinite cyclic group $H_n(M, M - \{x\})$.

Notational Convention. In what follows we will very often be looking at homology groups of the form $H_n(X, X - A)$. To simplify notation we will write $H_n(X, X - A)$ as $H_n(X \mid A)$, or more generally $H_n(X \mid A; G)$ if a coefficient group G needs to be specified. By excision, $H_n(X \mid A)$ depends only on a neighborhood of the closure of A in X, so it makes sense to view $H_n(X \mid A)$ as *local homology of X at A*.

Having settled what local orientations at points of a manifold are, a global orientation ought to be 'a consistent choice of local orientations at all points'. We make this precise by the following definition. An **orientation** of an n-dimensional manifold M is a function $x \mapsto \mu_x$ assigning to each $x \in M$ a local orientation $\mu_x \in H_n(M \mid x)$, satisfying the 'local consistency' condition that each $x \in M$ has a neighborhood $\mathbb{R}^n \subset M$ containing an open ball B of finite radius about x such that all the local orientations μ_y at points $y \in B$ are the images of one generator μ_B of $H_n(M \mid B) \approx H_n(\mathbb{R}^n \mid B)$ under the natural maps $H_n(M \mid B) \to H_n(M \mid y)$. If an orientation exists for M, then M is called **orientable**.

Every manifold M has an orientable two-sheeted covering space $\widetilde{M}$. For example, $\mathbb{R}\mathrm{P}^2$ is covered by S^2, and the Klein bottle has the torus as a two-sheeted covering space. The general construction goes as follows. As a set, let

$$\widetilde{M} = \{\, \mu_x \mid x \in M \text{ and } \mu_x \text{ is a local orientation of } M \text{ at } x \,\}$$

The map $\mu_x \mapsto x$ defines a two-to-one surjection $\widetilde{M} \to M$, and we wish to topologize $\widetilde{M}$ to make this a covering space projection. Given an open ball $B \subset \mathbb{R}^n \subset M$ of finite radius and a generator $\mu_B \in H_n(M \mid B)$, let $U(\mu_B)$ be the set of all $\mu_x \in \widetilde{M}$ such that $x \in B$ and μ_x is the image of μ_B under the natural map $H_n(M \mid B) \to H_n(M \mid x)$. It is easy to check that these sets $U(\mu_B)$ form a basis for a topology on $\widetilde{M}$, and that the projection $\widetilde{M} \to M$ is a covering space. The manifold $\widetilde{M}$ is orientable since each point $\mu_x \in \widetilde{M}$ has a canonical local orientation given by the element $\widetilde{\mu}_x \in H_n(\widetilde{M} \mid \mu_x)$ corresponding to μ_x under the isomorphisms $H_n(\widetilde{M} \mid \mu_x) \approx H_n(U(\mu_B) \mid \mu_x) \approx H_n(B \mid x)$, and by construction these local orientations satisfy the local consistency condition necessary to define a global orientation.

Proposition 3.25. *If M is connected, then M is orientable iff $\widetilde{M}$ has two components. In particular, M is orientable if it is simply-connected, or more generally if $\pi_1(M)$ has no subgroup of index two.*

The first statement is a formulation of the intuitive notion of nonorientability as being able to go around some closed loop and come back with the opposite orientation, since in terms of the covering space $\widetilde{M} \to M$ this corresponds to a loop in M that lifts

to a path in $\widetilde{M}$ connecting two distinct points with the same image in M. The existence of such paths is equivalent to $\widetilde{M}$ being connected.

Proof: If M is connected, $\widetilde{M}$ has either one or two components since it is a two-sheeted covering space of M. If it has two components, they are each mapped homeomorphically to M by the covering projection, so M is orientable, being homeomorphic to a component of the orientable manifold $\widetilde{M}$. Conversely, if M is orientable, it has exactly two orientations since it is connected, and each of these orientations defines a component of $\widetilde{M}$. The last statement of the proposition follows since connected two-sheeted covering spaces of M correspond to index-two subgroups of $\pi_1(M)$, by the classification of covering spaces. $\qquad\square$

The covering space $\widetilde{M} \to M$ can be embedded in a larger covering space $M_{\mathbb{Z}} \to M$ where $M_{\mathbb{Z}}$ consists of all elements $\alpha_x \in H_n(M|x)$ as x ranges over M. As before, we topologize $M_{\mathbb{Z}}$ via the basis of sets $U(\alpha_B)$ consisting of α_x's with $x \in B$ and α_x the image of an element $\alpha_B \in H_n(M|B)$ under the map $H_n(M|B) \to H_n(M|x)$. The covering space $M_{\mathbb{Z}} \to M$ is infinite-sheeted since for fixed $x \in M$, the α_x's range over the infinite cyclic group $H_n(M|x)$. Restricting α_x to be zero, we get a copy M_0 of M in $M_{\mathbb{Z}}$. The rest of $M_{\mathbb{Z}}$ consists of an infinite sequence of copies M_k of $\widetilde{M}$, $k = 1, 2, \cdots$, where M_k consists of the α_x's that are k times either generator of $H_n(M|x)$.

A continuous map $M \to M_{\mathbb{Z}}$ of the form $x \mapsto \alpha_x \in H_n(M|x)$ is called a **section** of the covering space. An orientation of M is the same thing as a section $x \mapsto \mu_x$ such that μ_x is a generator of $H_n(M|x)$ for each x.

One can generalize the definition of orientation by replacing the coefficient group $\mathbb{Z}$ by any commutative ring R with identity. Then an **R-orientation** of M assigns to each $x \in M$ a generator of $H_n(M|x;R) \approx R$, subject to the corresponding local consistency condition, where a 'generator' of R is an element u such that $Ru = R$. Since we assume R has an identity element, this is equivalent to saying that u is a unit, an invertible element of R. The definition of the covering space $M_{\mathbb{Z}}$ generalizes immediately to a covering space $M_R \to M$, and an R-orientation is a section of this covering space whose value at each $x \in M$ is a generator of $H_n(M|x;R)$.

The structure of M_R is easy to describe. In view of the canonical isomorphism $H_n(M|x;R) \approx H_n(M|x) \otimes R$, each $r \in R$ determines a subcovering space M_r of M_R consisting of the points $\pm\mu_x \otimes r \in H_n(M|x;R)$ for μ_x a generator of $H_n(M|x)$. If r has order 2 in R then $r = -r$ so M_r is just a copy of M, and otherwise M_r is isomorphic to the two-sheeted cover $\widetilde{M}$. The covering space M_R is the union of these M_r's, which are disjoint except for the equality $M_r = M_{-r}$.

In particular we see that an orientable manifold is R-orientable for all R, while a nonorientable manifold is R-orientable iff R contains a unit of order 2, which is equivalent to having $2 = 0$ in R. Thus every manifold is $\mathbb{Z}_2$-orientable. In practice this means that the two most important cases are $R = \mathbb{Z}$ and $R = \mathbb{Z}_2$. In what follows

the reader should keep these two cases foremost in mind, but we will usually state results for a general R.

The orientability of a closed manifold is reflected in the structure of its homology, according to the following result.

Theorem 3.26. *Let M be a closed connected n-manifold. Then*:
(a) *If M is R-orientable, the map $H_n(M;R) \to H_n(M|x;R) \approx R$ is an isomorphism for all $x \in M$.*
(b) *If M is not R-orientable, the map $H_n(M;R) \to H_n(M|x;R) \approx R$ is injective with image $\{r \in R \mid 2r = 0\}$ for all $x \in M$.*
(c) $H_i(M;R) = 0$ *for $i > n$.*

In particular, $H_n(M;\mathbb{Z})$ is $\mathbb{Z}$ or 0 depending on whether M is orientable or not, and in either case $H_n(M;\mathbb{Z}_2) = \mathbb{Z}_2$.

An element of $H_n(M;R)$ whose image in $H_n(M|x;R)$ is a generator for all x is called a **fundamental class** for M with coefficients in R. By the theorem, a fundamental class exists if M is closed and R-orientable. To show that the converse is also true, let $\mu \in H_n(M;R)$ be a fundamental class and let μ_x denote its image in $H_n(M|x;R)$. The function $x \mapsto \mu_x$ is then an R-orientation since the map $H_n(M;R) \to H_n(M|x;R)$ factors through $H_n(M|B;R)$ for B any open ball in M containing x. Furthermore, M must be compact since μ_x can only be nonzero for x in the image of a cycle representing μ, and this image is compact. In view of these remarks a fundamental class could also be called an **orientation class** for M.

The theorem will follow fairly easily from a more technical statement:

Lemma 3.27. *Let M be a manifold of dimension n and let $A \subset M$ be a compact subset. Then*:
(a) *If $x \mapsto \alpha_x$ is a section of the covering space $M_R \to M$, then there is a unique class $\alpha_A \in H_n(M|A;R)$ whose image in $H_n(M|x;R)$ is α_x for all $x \in A$.*
(b) $H_i(M|A;R) = 0$ *for $i > n$.*

To deduce the theorem from this, choose $A = M$, a compact set by assumption. Part (c) of the theorem is immediate from (b) of the lemma. To obtain (a) and (b) of the theorem, let $\Gamma_R(M)$ be the set of sections of $M_R \to M$. The sum of two sections is a section, and a scalar multiple of a section is a section, so $\Gamma_R(M)$ is an R-module. There is a homomorphism $H_n(M;R) \to \Gamma_R(M)$ sending a class α to the section $x \mapsto \alpha_x$, where α_x is the image of α under the map $H_n(M;R) \to H_n(M|x;R)$. Part (a) of the lemma asserts that this homomorphism is an isomorphism. If M is connected, each section is uniquely determined by its value at one point, so statements (a) and (b) of the theorem are apparent from the earlier discussion of the structure of M_R. $\quad\square$

Proof of 3.27: The coefficient ring R will play no special role in the argument so we shall omit it from the notation. We break the proof up into four steps.

(1) First we observe that if the lemma is true for compact sets A, B, and $A \cap B$, then it is true for $A \cup B$. To see this, consider the Mayer–Vietoris sequence

$$0 \longrightarrow H_n(M \mid A \cup B) \xrightarrow{\Phi} H_n(M \mid A) \oplus H_n(M \mid B) \xrightarrow{\Psi} H_n(M \mid A \cap B)$$

Here the zero on the left comes from the assumption that $H_{n+1}(M \mid A \cap B) = 0$. The map Φ is $\Phi(\alpha) = (\alpha, -\alpha)$ and Ψ is $\Psi(\alpha, \beta) = \alpha + \beta$, where we omit notation for maps on homology induced by inclusion. The terms $H_i(M \mid A \cup B)$ farther to the left in this sequence are sandwiched between groups that are zero by assumption, so $H_i(M \mid A \cup B) = 0$ for $i > n$. This gives (b). For the existence half of (a), if $x \mapsto \alpha_x$ is a section, the hypothesis gives unique classes $\alpha_A \in H_n(M \mid A)$, $\alpha_B \in H_n(M \mid B)$, and $\alpha_{A \cap B} \in H_n(M \mid A \cap B)$ having image α_x for all x in A, B, or $A \cap B$ respectively. The images of α_A and α_B in $H_n(M \mid A \cap B)$ satisfy the defining property of $\alpha_{A \cap B}$, hence must equal $\alpha_{A \cap B}$. Exactness of the sequence then implies that $(\alpha_A, -\alpha_B) = \Phi(\alpha_{A \cup B})$ for some $\alpha_{A \cup B} \in H_n(M \mid A \cup B)$. This means that $\alpha_{A \cup B}$ maps to α_A and α_B, so $\alpha_{A \cup B}$ has image α_x for all $x \in A \cup B$ since α_A and α_B have this property. To see that $\alpha_{A \cup B}$ is unique, observe that if a class $\alpha \in H_n(M \mid A \cup B)$ has image zero in $H_n(M \mid x)$ for all $x \in A \cup B$, then its images in $H_n(M \mid A)$ and $H_n(M \mid B)$ have the same property, hence are zero by hypothesis, so α itself must be zero since Φ is injective. Uniqueness of $\alpha_{A \cup B}$ follows by applying this observation to the difference between two choices for $\alpha_{A \cup B}$.

(2) A compact set $A \subset M$ can be written as the union of finitely many compact sets $A_1, \cdots, A_m$ each contained in an open $\mathbb{R}^n \subset M$. We now apply (1) to $A_1 \cup \cdots \cup A_{m-1}$ and A_m. The intersection of these two sets is $(A_1 \cap A_m) \cup \cdots \cup (A_{m-1} \cap A_m)$, a union of $m - 1$ compact sets each contained in an open $\mathbb{R}^n \subset M$. By induction on m this gives a reduction to the case $m = 1$, so $A \subset \mathbb{R}^n \subset M$.

(3) When $A \subset \mathbb{R}^n \subset M$ and A is a union of convex compact sets $A_1, \cdots, A_m$ in $\mathbb{R}^n$, an inductive argument as in (2) reduces to the case that A itself is convex. When A is convex the result is evident since the map $H_i(M \mid A) \rightarrow H_i(M \mid x)$ is equivalent to $H_i(\mathbb{R}^n \mid A) \rightarrow H_i(\mathbb{R}^n \mid x)$ by excision, and the latter map is an isomorphism for any $x \in A$, as both $\mathbb{R}^n - A$ and $\mathbb{R}^n - \{x\}$ deformation retract onto a sphere centered at x.

(4) For an arbitrary compact set $A \subset \mathbb{R}^n \subset M$ let $\alpha \in H_i(M \mid A)$ be represented by a relative cycle z in $\mathbb{R}^n$ with ∂z in $\mathbb{R}^n - A$, and let C be the union of the images of the singular simplices in ∂z. Since C is compact, it has a positive distance δ from A in $\mathbb{R}^n$. We can cover A by finitely many closed balls in $\mathbb{R}^n$ of radius less than δ centered at points of A. Let K be the union of these balls, so K is disjoint from C. The relative cycle z defines an element $\alpha_K \in H_i(M \mid K)$ mapping to the given $\alpha \in H_i(M \mid A)$. If $i > n$ then by (3) we have $H_i(M \mid K) = 0$, so $\alpha_K = 0$, which implies $\alpha = 0$ and hence $H_i(M \mid A) = 0$. For the uniqueness half of (a) when $i = n$ it suffices to show that $\alpha = 0$ if α_x is zero in $H_n(M \mid x)$ for all $x \in A$. Since K is a union of balls B meeting A and $H_n(M \mid B) \rightarrow H_n(M \mid x)$ is an isomorphism for all $x \in B$, it follows that α_K maps to

zero in $H_n(M|x)$ for each $x \in K$ since this is true when $x \in A$. Applying step (3) to the zero section of $M_R \to M$ we conclude that α_K is zero, hence also α. This finishes the uniqueness statement in (a). The existence statement is easy since we can let α_A be the image of the element α_B associated to any ball B with $A \subset B \subset \mathbb{R}^n$. $\square$

For a closed n-manifold having the structure of a Δ-complex there is a more explicit construction for a fundamental class. Consider the case of $\mathbb{Z}$ coefficients. In simplicial homology a fundamental class must be represented by some linear combination $\sum_i k_i \sigma_i$ of the n-simplices σ_i of M. The condition that the fundamental class maps to a generator of $H_n(M|x; \mathbb{Z})$ for points x in the interiors of the σ_i's means that each coefficient k_i must be ± 1. The k_i's must also be such that $\sum_i k_i \sigma_i$ is a cycle. This implies that if σ_i and σ_j share a common $(n-1)$-dimensional face, then k_i determines k_j and vice versa. Analyzing the situation more closely, one can show that a choice of signs for the k_i's making $\sum_i k_i \sigma_i$ a cycle is possible iff M is orientable, and if such a choice is possible, then the cycle $\sum_i k_i \sigma_i$ defines a fundamental class. With $\mathbb{Z}_2$ coefficients there is no issue of signs, and $\sum_i \sigma_i$ always defines a fundamental class.

Some information about $H_{n-1}(M)$ can also be squeezed out of the preceding theorem:

Corollary 3.28. *If M is a closed connected n-manifold, the torsion subgroup of $H_{n-1}(M; \mathbb{Z})$ is trivial if M is orientable and $\mathbb{Z}_2$ if M is nonorientable.*

Proof: This is an application of the universal coefficient theorem for homology, using the fact that the homology groups of M are finitely generated, from Corollaries A.8 and A.9 in the Appendix. In the orientable case, if $H_{n-1}(M; \mathbb{Z})$ contained torsion, then for some prime p, $H_n(M; \mathbb{Z}_p)$ would be larger than the $\mathbb{Z}_p$ coming from $H_n(M; \mathbb{Z})$. In the nonorientable case, $H_n(M; \mathbb{Z}_m)$ is either $\mathbb{Z}_2$ or 0 depending on whether m is even or odd. This forces the torsion subgroup of $H_{n-1}(M; \mathbb{Z})$ to be $\mathbb{Z}_2$. $\square$

The reader who is familiar with Bockstein homomorphisms, which are discussed in §3.E, will recognize that the $\mathbb{Z}_2$ in $H_{n-1}(M; \mathbb{Z})$ in the nonorientable case is the image of the Bockstein homomorphism $H_n(M; \mathbb{Z}_2) \to H_{n-1}(M; \mathbb{Z})$ coming from the short exact sequence of coefficient groups $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$.

The structure of $H_n(M; G)$ and $H_{n-1}(M; G)$ for a closed connected n-manifold M can be explained very nicely in terms of cellular homology when M has a CW structure with a single n-cell, which is the case for a large number of manifolds. Note that there can be no cells of higher dimension since a cell of maximal dimension produces nontrivial local homology in that dimension. Consider the cellular boundary map $d: C_n(M) \to C_{n-1}(M)$ with $\mathbb{Z}$ coefficients. Since M has a single n-cell we have $C_n(M) = \mathbb{Z}$. If M is orientable, d must be zero since $H_n(M; \mathbb{Z}) = \mathbb{Z}$. Then since d

is zero, $H_{n-1}(M;\mathbb{Z})$ must be free. On the other hand, if M is nonorientable then d must take a generator of $C_n(M)$ to twice a generator α of a $\mathbb{Z}$ summand of $C_{n-1}(M)$, in order for $H_n(M;\mathbb{Z}_p)$ to be zero for odd primes p and $\mathbb{Z}_2$ for $p = 2$. The cellular chain α must be a cycle since 2α is a boundary and hence a cycle. It follows that the torsion subgroup of $H_{n-1}(M;\mathbb{Z})$ must be a $\mathbb{Z}_2$ generated by α.

Concerning the homology of noncompact manifolds there is the following general statement.

Proposition 3.29. *If M is a connected noncompact n-manifold, then $H_i(M;R) = 0$ for $i \geq n$.*

Proof: Represent an element of $H_i(M;R)$ by a cycle z. This has compact image in M, so there is an open set $U \subset M$ containing the image of z and having compact closure $\overline{U} \subset M$. Let $V = M - \overline{U}$. Part of the long exact sequence of the triple $(M, U \cup V, V)$ fits into a commutative diagram

$$H_{i+1}(M, U \cup V; R) \longrightarrow H_i(U \cup V, V; R) \longrightarrow H_i(M, V; R)$$
$$\uparrow \approx \qquad\qquad\qquad \uparrow$$
$$H_i(U; R) \longrightarrow H_i(M; R)$$

When $i > n$, the two groups on either side of $H_i(U \cup V, V; R)$ are zero by Lemma 3.27 since $U \cup V$ and V are the complements of compact sets in M. Hence $H_i(U;R) = 0$, so z is a boundary in U and therefore in M, and we conclude that $H_i(M;R) = 0$.

When $i = n$, the class $[z] \in H_n(M;R)$ defines a section $x \mapsto [z]_x$ of M_R. Since M is connected, this section is determined by its value at a single point, so $[z]_x$ will be zero for all x if it is zero for some x, which it must be since z has compact image and M is noncompact. By Lemma 3.27, z then represents zero in $H_n(M, V; R)$, hence also in $H_n(U;R)$ since the first term in the upper row of the diagram above is zero when $i = n$, by Lemma 3.27 again. So $[z] = 0$ in $H_n(M;R)$, and therefore $H_n(M;R) = 0$ since $[z]$ was an arbitrary element of this group. $\square$

The Duality Theorem

The form of Poincaré duality we will prove asserts that for an R-orientable closed n-manifold, a certain naturally defined map $H^k(M;R) \to H_{n-k}(M;R)$ is an isomorphism. The definition of this map will be in terms of a more general construction called *cap product*, which has close connections with cup product.

For an arbitrary space X and coefficient ring R, define an R-bilinear cap product $\frown : C_k(X;R) \times C^\ell(X;R) \to C_{k-\ell}(X;R)$ for $k \geq \ell$ by setting

$$\sigma \frown \varphi = \varphi(\sigma | [v_0, \cdots, v_\ell]) \, \sigma | [v_\ell, \cdots, v_k]$$

for $\sigma : \Delta^k \to X$ and $\varphi \in C^\ell(X;R)$. To see that this induces a cap product in homology

and cohomology we use the formula

$$\partial(\sigma \frown \varphi) = (-1)^\ell (\partial\sigma \frown \varphi - \sigma \frown \delta\varphi)$$

which is checked by a calculation:

$$\partial\sigma \frown \varphi = \sum_{i=0}^{\ell} (-1)^i \varphi(\sigma|[v_0, \cdots, \hat{v}_i, \cdots, v_{\ell+1}])\sigma|[v_{\ell+1}, \cdots, v_k]$$

$$+ \sum_{i=\ell+1}^{k} (-1)^i \varphi(\sigma|[v_0, \cdots, v_\ell])\sigma|[v_\ell, \cdots, \hat{v}_i, \cdots, v_k]$$

$$\sigma \frown \delta\varphi = \sum_{i=0}^{\ell+1} (-1)^i \varphi(\sigma|[v_0, \cdots, \hat{v}_i, \cdots, v_{\ell+1}])\sigma|[v_{\ell+1}, \cdots, v_k]$$

$$\partial(\sigma \frown \varphi) = \sum_{i=\ell}^{k} (-1)^{i-\ell} \varphi(\sigma|[v_0, \cdots, v_\ell])\sigma|[v_\ell, \cdots, \hat{v}_i, \cdots, v_k]$$

From the relation $\partial(\sigma \frown \varphi) = \pm(\partial\sigma \frown \varphi - \sigma \frown \delta\varphi)$ it follows that the cap product of a cycle σ and a cocycle φ is a cycle. Further, if $\partial\sigma = 0$ then $\partial(\sigma \frown \varphi) = \pm(\sigma \frown \delta\varphi)$, so the cap product of a cycle and a coboundary is a boundary. And if $\delta\varphi = 0$ then $\partial(\sigma \frown \varphi) = \pm(\partial\sigma \frown \varphi)$, so the cap product of a boundary and a cocycle is a boundary. These facts imply that there is an induced cap product

$$H_k(X;R) \times H^\ell(X;R) \xrightarrow{\ \frown\ } H_{k-\ell}(X;R)$$

which is R-linear in each variable.

Using the same formulas, one checks that cap product has the relative forms

$$H_k(X, A;R) \times H^\ell(X;R) \xrightarrow{\ \frown\ } H_{k-\ell}(X, A;R)$$
$$H_k(X, A;R) \times H^\ell(X, A;R) \xrightarrow{\ \frown\ } H_{k-\ell}(X;R)$$

For example, in the second case the cap product $C_k(X;R) \times C^\ell(X;R) \to C_{k-\ell}(X;R)$ restricts to zero on the submodule $C_k(A;R) \times C^\ell(X, A;R)$, so there is an induced cap product $C_k(X, A;R) \times C^\ell(X, A;R) \to C_{k-\ell}(X;R)$. The formula for $\partial(\sigma \frown \varphi)$ still holds, so we can pass to homology and cohomology groups. There is also a more general relative cap product

$$H_k(X, A \cup B;R) \times H^\ell(X, A;R) \xrightarrow{\ \frown\ } H_{k-\ell}(X, B;R),$$

defined when A and B are open sets in X, using the fact that $H_k(X, A \cup B;R)$ can be computed using the chain groups $C_n(X, A + B;R) = C_n(X;R)/C_n(A + B;R)$, as in the derivation of relative Mayer–Vietoris sequences in §2.2.

Cap product satisfies a naturality property that is a little more awkward to state than the corresponding result for cup product since both covariant and contravariant functors are involved. Given a map $f : X \to Y$, the relevant induced maps on homology and cohomology fit into the diagram shown below. It does not quite make sense

to say this diagram commutes, but the spirit of commutativity is contained in the formula

$$f_*(\alpha) \frown \varphi = f_*(\alpha \frown f^*(\varphi))$$

$$
\begin{array}{ccc}
H_k(X) \times H^\ell(X) & \xrightarrow{\frown} & H_{k-\ell}(X) \\
\downarrow f_* \quad \uparrow f^* & & \downarrow f_* \\
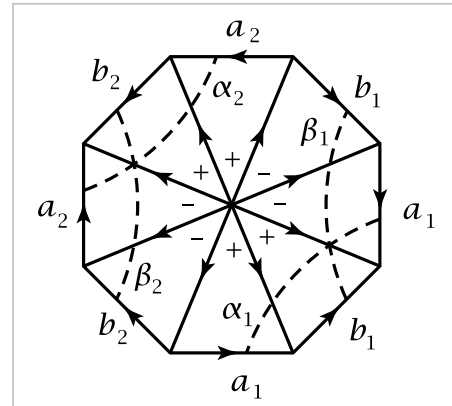H_k(Y) \times H^\ell(Y) & \xrightarrow{\frown} & H_{k-\ell}(Y)
\end{array}
$$

which is obtained by substituting $f\sigma$ for σ in the definition of cap product: $f\sigma \frown \varphi = \varphi(f\sigma | [v_0, \cdots, v_\ell]) \, f\sigma | [v_\ell, \cdots, v_k]$. There are evident relative versions as well.

Now we can state Poincaré duality for closed manifolds:

Theorem 3.30 (Poincaré Duality). *If M is a closed R-orientable n-manifold with fundamental class $[M] \in H_n(M; R)$, then the map $D : H^k(M; R) \longrightarrow H_{n-k}(M; R)$ defined by $D(\alpha) = [M] \frown \alpha$ is an isomorphism for all k.*

Recall that a fundamental class for M is an element of $H_n(M; R)$ whose image in $H_n(M | x; R)$ is a generator for each $x \in M$. The existence of such a class was shown in Theorem 3.26.

Example 3.31: Surfaces. Let M be the closed orientable surface of genus g, obtained as usual from a $4g$-gon by identifying pairs of edges according to the word $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$. A Δ-complex structure on M is obtained by coning off the $4g$-gon to its center, as indicated in the figure for the case $g = 2$. We can compute cap products using simplicial homology and cohomology since cap products are defined for simplicial homology and cohomology by exactly the same formula as for singular homology and cohomology, so the isomorphism between the simplicial and singular theories respects cap products. A fundamental class $[M]$ generating $H_2(M)$ is represented by the 2-cycle formed by the

sum of all $4g$ 2-simplices with the signs indicated. The edges a_i and b_i form a basis for $H_1(M)$. Under the isomorphism $H^1(M) \approx \mathrm{Hom}(H_1(M), \mathbb{Z})$, the cohomology class α_i corresponding to a_i assigns the value 1 to a_i and 0 to the other basis elements. This class α_i is represented by the cocycle φ_i assigning the value 1 to the 1-simplices meeting the arc labeled α_i in the figure and 0 to the other 1-simplices. Similarly we have a class β_i corresponding to b_i, represented by the cocycle ψ_i assigning the value 1 to the 1-simplices meeting the arc β_i and 0 to the other 1-simplices. Applying the definition of cap product, we have $[M] \frown \varphi_i = b_i$ and $[M] \frown \psi_i = -a_i$ since in both cases there is just one 2-simplex $[v_0, v_1, v_2]$ where φ_i or ψ_i is nonzero on the edge $[v_0, v_1]$. Thus b_i is the Poincaré dual of α_i and $-a_i$ is the Poincaré dual of β_i. If we interpret Poincaré duality entirely in terms of homology, identifying α_i with its Hom-dual a_i and β_i with b_i, then the classes a_i and b_i are Poincaré duals of each other, up to sign at least. Geometrically, Poincaré duality is reflected in the fact that the loops α_i and b_i are homotopic, as are the loops β_i and a_i.

The closed nonorientable surface N of genus g can be treated in the same way if we use $\mathbb{Z}_2$ coefficients. We view N as obtained from a $2g$-gon by identifying consecutive pairs of edges according to the word $a_1^2 \cdots a_g^2$. We have classes $\alpha_i \in H^1(N;\mathbb{Z}_2)$ represented by cocycles φ_i assigning the value 1 to the edges meeting the arc α_i. Then $[N] \frown \varphi_i = a_i$, so a_i is the Poincaré dual of α_i. In terms of homology, a_i is the Hom-dual of α_i, so a_i is its own Poincaré dual. Geometrically, the loops a_i on N are homotopic to their Poincaré dual loops α_i.
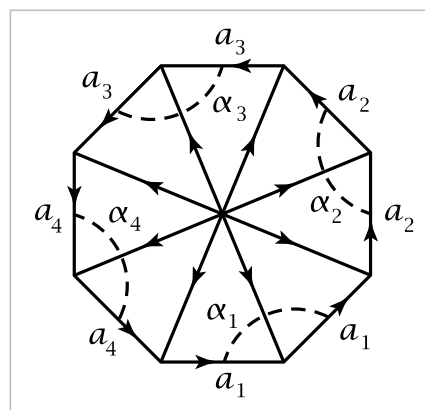
Our proof of Poincaré duality, like the construction of fundamental classes, will be by an inductive argument using Mayer–Vietoris sequences. The induction step requires a version of Poincaré duality for open subsets of M, which are noncompact and can satisfy Poincaré duality only when a different kind of cohomology called *cohomology with compact supports* is used.

Cohomology with Compact Supports

Before giving the general definition, let us look at the conceptually simpler notion of simplicial cohomology with compact supports. Here one starts with a Δ-complex X which is locally compact. This is equivalent to saying that every point has a neighborhood that meets only finitely many simplices. Consider the subgroup $\Delta_c^i(X;G)$ of the simplicial cochain group $\Delta^i(X;G)$ consisting of cochains that are compactly supported in the sense that they take nonzero values on only finitely many simplices. The coboundary of such a cochain φ can have a nonzero value only on those $(i+1)$-simplices having a face on which φ is nonzero, and there are only finitely many such simplices by the local compactness assumption, so $\delta\varphi$ lies in $\Delta_c^{i+1}(X;G)$. Thus we have a subcomplex of the simplicial cochain complex. The cohomology groups for this subcomplex will be denoted temporarily by $H_c^i(X;G)$.

Example 3.32. Let us compute these cohomology groups when $X = \mathbb{R}$ with the Δ-complex structure having vertices at the integer points. For a simplicial 0-cochain to be a cocycle it must take the same value on all vertices, but then if the cochain lies in $\Delta_c^0(X)$ it must be identically zero. Thus $H_c^0(\mathbb{R};G) = 0$. However, $H_c^1(\mathbb{R};G)$ is nonzero. Namely, consider the map $\Sigma : \Delta_c^1(\mathbb{R};G) \to G$ sending each cochain to the sum of its values on all the 1-simplices. Note that Σ is not defined on all of $\Delta^1(X)$, just on $\Delta_c^1(X)$. The map Σ vanishes on coboundaries, so it induces a map $H_c^1(\mathbb{R};G) \to G$. This is surjective since every element of $\Delta_c^1(X)$ is a cocycle. It is an easy exercise to verify that it is also injective, so $H_c^1(\mathbb{R};G) \approx G$.

Compactly supported cellular cohomology for a locally compact CW complex could be defined in a similar fashion, using cellular cochains that are nonzero on

only finitely many cells. However, what we really need is singular cohomology with compact supports for spaces without any simplicial or cellular structure. The quickest definition of this is the following. Let $C_c^i(X;G)$ be the subgroup of $C^i(X;G)$ consisting of cochains $\varphi : C_i(X) \to G$ for which there exists a compact set $K = K_\varphi \subset X$ such that φ is zero on all chains in $X - K$. Note that $\delta\varphi$ is then also zero on chains in $X - K$, so $\delta\varphi$ lies in $C_c^{i+1}(X;G)$ and the $C_c^i(X;G)$'s for varying i form a subcomplex of the singular cochain complex of X. The cohomology groups $H_c^i(X;G)$ of this subcomplex are the **cohomology groups with compact supports**.

Cochains in $C_c^i(X;G)$ have compact support in only a rather weak sense. A stronger and perhaps more natural condition would have been to require cochains to be nonzero only on singular simplices contained in some compact set, depending on the cochain. However, cochains satisfying this condition do not in general form a subcomplex of the singular cochain complex. For example, if $X = \mathbb{R}$ and φ is a 0-cochain assigning a nonzero value to one point of $\mathbb{R}$ and zero to all other points, then $\delta\varphi$ assigns a nonzero value to arbitrarily large 1-simplices.

It will be quite useful to have an alternative definition of $H_c^i(X;G)$ in terms of algebraic limits, which enter the picture in the following way. The cochain group $C_c^i(X;G)$ is the union of its subgroups $C^i(X, X - K;G)$ as K ranges over compact subsets of X. Each inclusion $K \hookrightarrow L$ induces inclusions $C^i(X, X - K;G) \hookrightarrow C^i(X, X - L;G)$ for all i, so there are induced maps $H^i(X, X - K;G) \to H^i(X, X - L;G)$. These need not be injective, but one might still hope that $H_c^i(X;G)$ is somehow describable in terms of the system of groups $H^i(X, X - K;G)$ for varying K. This is indeed the case, and it is algebraic limits that provide the description.

Suppose one has abelian groups G_α indexed by some partially ordered index set I having the property that for each pair $\alpha, \beta \in I$ there exists $\gamma \in I$ with $\alpha \leq \gamma$ and $\beta \leq \gamma$. Such an I is called a **directed set**. Suppose also that for each pair $\alpha \leq \beta$ one has a homomorphism $f_{\alpha\beta} : G_\alpha \to G_\beta$, such that $f_{\alpha\alpha} = \mathbb{1}$ for each α, and if $\alpha \leq \beta \leq \gamma$ then $f_{\alpha\gamma}$ is the composition of $f_{\alpha\beta}$ and $f_{\beta\gamma}$. Given this data, which is called a **directed system** of groups, there are two equivalent ways of defining the **direct limit** group $\varinjlim G_\alpha$. The shorter definition is that $\varinjlim G_\alpha$ is the quotient of the direct sum $\bigoplus_\alpha G_\alpha$ by the subgroup generated by all elements of the form $a - f_{\alpha\beta}(a)$ for $a \in G_\alpha$, where we are viewing each G_α as a subgroup of $\bigoplus_\alpha G_\alpha$. The other definition, which is often more convenient to work with, runs as follows. Define an equivalence relation on the set $\coprod_\alpha G_\alpha$ by $a \sim b$ if $f_{\alpha\gamma}(a) = f_{\beta\gamma}(b)$ for some γ, where $a \in G_\alpha$ and $b \in G_\beta$. This is clearly reflexive and symmetric, and transitivity follows from the directed set property. It could also be described as the equivalence relation generated by setting $a \sim f_{\alpha\beta}(a)$. Any two equivalence classes $[a]$ and $[b]$ have representatives a' and b' lying in the same G_γ, so define $[a] + [b] = [a' + b']$. One checks this is well-defined and gives an abelian group structure to the set of equivalence classes. It is easy to check further that the map sending an equivalence class $[a]$ to the coset of a

in $\varinjlim G_\alpha$ is a homomorphism, with an inverse induced by the map $\sum_i a_i \mapsto \sum_i [a_i]$ for $a_i \in G_{\alpha_i}$. Thus we can identify $\varinjlim G_\alpha$ with the group of equivalence classes $[a]$.

A useful consequence of this is that if we have a subset $J \subset I$ with the property that for each $\alpha \in I$ there exists a $\beta \in J$ with $\alpha \leq \beta$, then $\varinjlim G_\alpha$ is the same whether we compute it with α varying over I or just over J. In particular, if I has a maximal element γ, we can take $J = \{\gamma\}$ and then $\varinjlim G_\alpha = G_\gamma$.

Suppose now that we have a space X expressed as the union of a collection of subspaces X_α forming a directed set with respect to the inclusion relation. Then the groups $H_i(X_\alpha; G)$ for fixed i and G form a directed system, using the homomorphisms induced by inclusions. The natural maps $H_i(X_\alpha; G) \to H_i(X; G)$ induce a homomorphism $\varinjlim H_i(X_\alpha; G) \to H_i(X; G)$.

> **Proposition 3.33.** *If a space X is the union of a directed set of subspaces X_α with the property that each compact set in X is contained in some X_α, then the natural map $\varinjlim H_i(X_\alpha; G) \to H_i(X; G)$ is an isomorphism for all i and G.*

Proof: For surjectivity, represent a cycle in X by a finite sum of singular simplices. The union of the images of these singular simplices is compact in X, hence lies in some X_α, so the map $\varinjlim H_i(X_\alpha; G) \to H_i(X; G)$ is surjective. Injectivity is similar: If a cycle in some X_α is a boundary in X, compactness implies it is a boundary in some $X_\beta \supset X_\alpha$, hence represents zero in $\varinjlim H_i(X_\alpha; G)$. $\qquad\square$

Now we can give the alternative definition of cohomology with compact supports in terms of direct limits. For a space X, the compact subsets $K \subset X$ form a directed set under inclusion since the union of two compact sets is compact. To each compact $K \subset X$ we associate the group $H^i(X, X - K; G)$, with a fixed i and coefficient group G, and to each inclusion $K \subset L$ of compact sets we associate the natural homomorphism $H^i(X, X - K; G) \to H^i(X, X - L; G)$. The resulting limit group $\varinjlim H^i(X, X - K; G)$ is then equal to $H^i_c(X; G)$ since each element of this limit group is represented by a cocycle in $C^i(X, X - K; G)$ for some compact K, and such a cocycle is zero in $\varinjlim H^i(X, X - K; G)$ iff it is the coboundary of a cochain in $C^{i-1}(X, X - L; G)$ for some compact $L \supset K$.

Note that if X is compact, then $H^i_c(X; G) = H^i(X; G)$ since there is a unique maximal compact set $K \subset X$, namely X itself. This is also immediate from the original definition since $C^i_c(X; G) = C^i(X; G)$ if X is compact.

Example 3.34: $H^*_c(\mathbb{R}^n; G)$. To compute $\varinjlim H^i(\mathbb{R}^n, \mathbb{R}^n - K; G)$ it suffices to let K range over balls B_k of integer radius k centered at the origin since every compact set is contained in such a ball. Since $H^i(\mathbb{R}^n, \mathbb{R}^n - B_k; G)$ is nonzero only for $i = n$, when it is G, and the maps $H^n(\mathbb{R}^n, \mathbb{R}^n - B_k; G) \to H^n(\mathbb{R}^n, \mathbb{R}^n - B_{k+1}; G)$ are isomorphisms, we deduce that $H^i_c(\mathbb{R}^n; G) = 0$ for $i \neq n$ and $H^n_c(\mathbb{R}^n; G) \approx G$.

This example shows that cohomology with compact supports is not an invariant of homotopy type. This can be traced to difficulties with induced maps. For example,

the constant map from $\mathbb{R}^n$ to a point does not induce a map on cohomology with compact supports. The maps which do induce maps on H_c^* are the *proper* maps, those for which the inverse image of each compact set is compact. In the proof of Poincaré duality, however, we will need induced maps of a different sort going in the opposite direction from what is usual for cohomology, maps $H_c^i(U;G) \to H_c^i(V;G)$ associated to inclusions $U \hookrightarrow V$ of open sets in the fixed manifold M.

The group $H^i(X, X-K; G)$ for K compact depends only on a neighborhood of K in X by excision, assuming X is Hausdorff so that K is closed. As convenient shorthand notation we will write this group as $H^i(X \mid K; G)$, in analogy with the similar notation used earlier for local homology. One can think of cohomology with compact supports as the limit of these 'local cohomology groups at compact subsets'.

Duality for Noncompact Manifolds

For M an R-orientable n-manifold, possibly noncompact, we can define a duality map $D_M : H_c^k(M;R) \to H_{n-k}(M;R)$ by a limiting process in the following way. For compact sets $K \subset L \subset M$ we have a diagram

$$
\begin{array}{ccc}
H_n(M|L;R) \times H^k(M|L;R) & & \\
\downarrow i_* \qquad\qquad \uparrow i^* & \overset{\frown}{\searrow} & \\
 & & H_{n-k}(M;R) \\
H_n(M|K;R) \times H^k(M|K;R) & \overset{\frown}{\nearrow} &
\end{array}
$$

where $H_n(M \mid A; R) = H_n(M, M - A; R)$ and $H^k(M \mid A; R) = H^k(M, M - A; R)$. By Lemma 3.27 there are unique elements $\mu_K \in H_n(M \mid K; R)$ and $\mu_L \in H_n(M \mid L; R)$ restricting to a given orientation of M at each point of K and L, respectively. From the uniqueness we have $i_*(\mu_L) = \mu_K$. The naturality of cap product implies that $i_*(\mu_L) \frown x = \mu_L \frown i^*(x)$ for all $x \in H^k(M \mid K; R)$, so $\mu_K \frown x = \mu_L \frown i^*(x)$. Therefore, letting K vary over compact sets in M, the homomorphisms $H^k(M \mid K; R) \to H_{n-k}(M;R)$, $x \mapsto \mu_K \frown x$, induce in the limit a duality homomorphism $D_M : H_c^k(M;R) \to H_{n-k}(M;R)$.

Since $H_c^*(M;R) = H^*(M;R)$ if M is compact, the following theorem generalizes Poincaré duality for closed manifolds:

Theorem 3.35. *The duality map $D_M : H_c^k(M;R) \to H_{n-k}(M;R)$ is an isomorphism for all k whenever M is an R-oriented n-manifold.*

The proof will not be difficult once we establish a technical result stated in the next lemma, concerning the commutativity of a certain diagram. Commutativity statements of this sort are usually routine to prove, but this one seems to be an exception. The reader who consults other books for alternative expositions will find somewhat uneven treatments of this technical point, and the proof we give is also not as simple as one would like.

The coefficient ring R will be fixed throughout the proof, and for simplicity we will omit it from the notation for homology and cohomology.

Lemma 3.36. *If M is the union of two open sets U and V, then there is a diagram of Mayer-Vietoris sequences, commutative up to sign*:

$$\cdots \longrightarrow H_c^k(U \cap V) \longrightarrow H_c^k(U) \oplus H_c^k(V) \longrightarrow H_c^k(M) \longrightarrow H_c^{k+1}(U \cap V) \longrightarrow \cdots$$

$$\Big\downarrow D_{U \cap V} \qquad\qquad \Big\downarrow D_U \oplus -D_V \qquad\qquad \Big\downarrow D_M \qquad\qquad \Big\downarrow D_{U \cap V}$$

$$\cdots \longrightarrow H_{n-k}(U \cap V) \longrightarrow H_{n-k}(U) \oplus H_{n-k}(V) \longrightarrow H_{n-k}(M) \longrightarrow H_{n-k-1}(U \cap V) \longrightarrow \cdots$$

Proof: Compact sets $K \subset U$ and $L \subset V$ give rise to the Mayer-Vietoris sequence in the upper row of the following diagram, whose lower row is also a Mayer-Vietoris sequence:

$$\cdots \longrightarrow H^k(M|K \cap L) \longrightarrow H^k(M|K) \oplus H^k(M|L) \longrightarrow H^k(M|K \cup L) \longrightarrow \cdots$$

$$\Big\downarrow \approx \qquad\qquad\qquad \Big\downarrow \approx$$

$$H^k(U \cap V|K \cap L) \qquad H^k(U|K) \oplus H^k(V|L) \qquad\qquad \Big\downarrow \mu_{K \cup L} \frown$$

$$\Big\downarrow \mu_{K \cap L} \frown \qquad\qquad \Big\downarrow \mu_K \frown \oplus -\mu_L \frown$$

$$\cdots \longrightarrow H_{n-k}(U \cap V) \longrightarrow H_{n-k}(U) \oplus H_{n-k}(V) \longrightarrow H_{n-k}(M) \longrightarrow \cdots$$

The two maps labeled isomorphisms come from excision. Assuming this diagram commutes, consider passing to the limit over compact sets $K \subset U$ and $L \subset V$. Since each compact set in $U \cap V$ is contained in an intersection $K \cap L$ of compact sets $K \subset U$ and $L \subset V$, and similarly for $U \cup V$, the diagram induces a limit diagram having the form stated in the lemma. The first row of this limit diagram is exact since a direct limit of exact sequences is exact; this is an exercise at the end of the section, and follows easily from the definition of direct limits.

    It remains to consider the commutativity of the preceding diagram involving K and L. In the two squares shown, not involving boundary or coboundary maps, it is a triviality to check commutativity at the level of cycles and cocycles. Less trivial is the third square, which we rewrite in the following way:

$$H^k(M|K \cup L) \xrightarrow{\ \delta\ } H^{k+1}(M|K \cap L) \xrightarrow{\ \approx\ } H^{k+1}(U \cap V|K \cap L)$$

$$(*) \qquad \Big\downarrow \mu_{K \cup L} \frown \qquad\qquad\qquad\qquad\qquad \Big\downarrow \mu_{K \cap L} \frown$$

$$H_{n-k}(M) \xrightarrow{\qquad\qquad \partial \qquad\qquad} H_{n-k-1}(U \cap V)$$

Letting $A = M - K$ and $B = M - L$, the map δ is the coboundary map in the Mayer-Vietoris sequence obtained from the short exact sequence of cochain complexes

$$0 \longrightarrow C^*(M, A + B) \longrightarrow C^*(M, A) \oplus C^*(M, B) \longrightarrow C^*(M, A \cap B) \longrightarrow 0$$

where $C^*(M, A + B)$ consists of cochains on M vanishing on chains in A and chains in B. To evaluate the Mayer-Vietoris coboundary map δ on a cohomology class represented by a cocycle $\varphi \in C^*(M, A \cap B)$, the first step is to write $\varphi = \varphi_A - \varphi_B$

for $\varphi_A \in C^*(M, A)$ and $\varphi_B \in C^*(M, B)$. Then $\delta[\varphi]$ is represented by the cocycle $\delta\varphi_A = \delta\varphi_B \in C^*(M, A + B)$, where the equality $\delta\varphi_A = \delta\varphi_B$ comes from the fact that φ is a cocycle, so $\delta\varphi = \delta\varphi_A - \delta\varphi_B = 0$. Similarly, the boundary map ∂ in the homology Mayer–Vietoris sequence is obtained by representing an element of $H_i(M)$ by a cycle z that is a sum of chains $z_U \in C_i(U)$ and $z_V \in C_i(V)$, and then $\partial[z] = [\partial z_U]$.

Via barycentric subdivision, the class $\mu_{K \cup L}$ can be represented by a chain α that is a sum $\alpha_{U-L} + \alpha_{U \cap V} + \alpha_{V-K}$ of chains in $U - L$, $U \cap V$, and $V - K$, respectively, since these three open sets cover M. The chain $\alpha_{U \cap V}$ represents $\mu_{K \cap L}$ since the other two chains α_{U-L} and α_{V-K} lie in the complement of $K \cap L$, hence vanish in $H_n(M|K \cap L) \approx H_n(U \cap V|K \cap L)$. Similarly, $\alpha_{U-L} + \alpha_{U \cap V}$ represents μ_K.
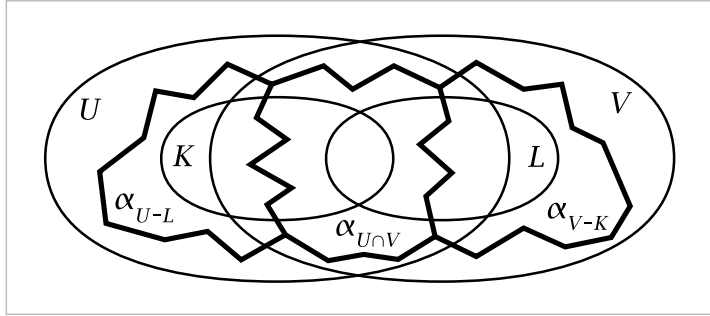
In the square $(*)$ let φ be a cocycle representing an element of $H^k(M|K \cup L)$. Under δ this maps to the cohomology class of $\delta\varphi_A$. Continuing on to $H_{n-k-1}(U \cap V)$ we obtain $\alpha_{U \cap V} \frown \delta\varphi_A$, which is in the same homology class as $\partial\alpha_{U \cap V} \frown \varphi_A$ since

$$\partial(\alpha_{U \cap V} \frown \varphi_A) = (-1)^k(\partial\alpha_{U \cap V} \frown \varphi_A - \alpha_{U \cap V} \frown \delta\varphi_A)$$

and $\alpha_{U \cap V} \frown \varphi_A$ is a chain in $U \cap V$.

Going around the square $(*)$ the other way, φ maps first to $\alpha \frown \varphi$. To apply the Mayer–Vietoris boundary map ∂ to this, we first write $\alpha \frown \varphi$ as a sum of a chain in U and a chain in V:

$$\alpha \frown \varphi = (\alpha_{U-L} \frown \varphi) + (\alpha_{U \cap V} \frown \varphi + \alpha_{V-K} \frown \varphi)$$

Then we take the boundary of the first of these two chains, obtaining the homology class $[\partial(\alpha_{U-L} \frown \varphi)] \in H_{n-k-1}(U \cap V)$. To compare this with $[\partial\alpha_{U \cap V} \frown \varphi_A]$, we have

$$\begin{aligned}
\partial(\alpha_{U-L} \frown \varphi) &= (-1)^k \partial\alpha_{U-L} \frown \varphi && \text{since } \delta\varphi = 0 \\
&= (-1)^k \partial\alpha_{U-L} \frown \varphi_A && \text{since } \partial\alpha_{U-L} \frown \varphi_B = 0, \quad \varphi_B \text{ being} \\
& && \text{zero on chains in } B = M - L \\
&= (-1)^{k+1} \partial\alpha_{U \cap V} \frown \varphi_A
\end{aligned}$$

where this last equality comes from the fact that $\partial(\alpha_{U-L} + \alpha_{U \cap V}) \frown \varphi_A = 0$ since $\partial(\alpha_{U-L} + \alpha_{U \cap V})$ is a chain in $U - K$ by the earlier observation that $\alpha_{U-L} + \alpha_{U \cap V}$ represents μ_K, and φ_A vanishes on chains in $A = M - K$.

Thus the square $(*)$ commutes up to a sign depending only on k.   □

Proof of Poincaré Duality: There are two inductive steps, finite and infinite:

(A) If M is the union of open sets U and V and if D_U, D_V, and $D_{U \cap V}$ are isomorphisms, then so is D_M. Via the five-lemma, this is immediate from the preceding lemma.

(B) If M is the union of a sequence of open sets $U_1 \subset U_2 \subset \cdots$ and each duality map $D_{U_i} : H_c^k(U_i) \to H_{n-k}(U_i)$ is an isomorphism, then so is D_M. To show this we notice first that by excision, $H_c^k(U_i)$ can be regarded as the limit of the groups $H^k(M|K)$ as K ranges over compact subsets of U_i. Then there are natural maps $H_c^k(U_i) \to H_c^k(U_{i+1})$ since the second of these groups is a limit over a larger collection of K's. Thus we can form $\varinjlim H_c^k(U_i)$ which is obviously isomorphic to $H_c^k(M)$ since the compact sets in M are just the compact sets in all the U_i's. By Proposition 3.33, $H_{n-k}(M) \approx \varinjlim H_{n-k}(U_i)$. The map D_M is thus the limit of the isomorphisms D_{U_i}, hence is an isomorphism.

Now after all these preliminaries we can prove the theorem in three easy steps:

(1) The case $M = \mathbb{R}^n$ can be proved by regarding $\mathbb{R}^n$ as the interior of Δ^n, and then the map D_M can be identified with the map $H^k(\Delta^n, \partial\Delta^n) \to H_{n-k}(\Delta^n)$ given by cap product with a unit times the generator $[\Delta^n] \in H_n(\Delta^n, \partial\Delta^n)$ defined by the identity map of Δ^n, which is a relative cycle. The only nontrivial value of k is $k = n$, when the cap product map is an isomorphism since a generator of $H^n(\Delta^n, \partial\Delta^n) \approx \mathrm{Hom}(H_n(\Delta^n, \partial\Delta^n), R)$ is represented by a cocycle φ taking the value 1 on Δ^n, so by the definition of cap product, $\Delta^n \frown \varphi$ is the last vertex of Δ^n, representing a generator of $H_0(\Delta^n)$.

(2) More generally, D_M is an isomorphism for M an arbitrary open set in $\mathbb{R}^n$. To see this, first write M as a countable union of nonempty bounded convex open sets U_i, for example open balls, and let $V_i = \bigcup_{j<i} U_j$. Both V_i and $U_i \cap V_i$ are unions of $i-1$ bounded convex open sets, so by induction on the number of such sets in a cover we may assume that D_{V_i} and $D_{U_i \cap V_i}$ are isomorphisms. By (1), D_{U_i} is an isomorphism since U_i is homeomorphic to $\mathbb{R}^n$. Hence $D_{U_i \cup V_i}$ is an isomorphism by (A). Since M is the increasing union of the V_i's and each D_{V_i} is an isomorphism, so is D_M by (B).

(3) If M is a finite or countably infinite union of open sets U_i homeomorphic to $\mathbb{R}^n$, the theorem now follows by the argument in (2), with each appearance of the words 'bounded convex open set' replaced by 'open set in $\mathbb{R}^n$'. Thus the proof is finished for closed manifolds, as well as for all the noncompact manifolds one ever encounters in actual practice.

To handle a completely general noncompact manifold M we use a Zorn's Lemma argument. Consider the collection of open sets $U \subset M$ for which the duality maps D_U are isomorphisms. This collection is partially ordered by inclusion, and the union of every totally ordered subcollection is again in the collection by the argument in (B), which did not really use the hypothesis that the collection $\{U_i\}$ was indexed by the positive integers. Zorn's Lemma then implies that there exists a maximal open set U for which the theorem holds. If $U \neq M$, choose a point $x \in M - U$ and an open neighborhood V of x homeomorphic to $\mathbb{R}^n$. The theorem holds for V and $U \cap V$ by (1) and (2), and it holds for U by assumption, so by (A) it holds for $U \cup V$, contradicting the maximality of U. $\qquad\square$

|| **Corollary 3.37.** *A closed manifold of odd dimension has Euler characteristic zero.*

Proof: Let M be a closed n-manifold. If M is orientable, we have $\operatorname{rank} H_i(M;\mathbb{Z}) = \operatorname{rank} H^{n-i}(M;\mathbb{Z})$, which equals $\operatorname{rank} H_{n-i}(M;\mathbb{Z})$ by the universal coefficient theorem. Thus if n is odd, all the terms of $\sum_i (-1)^i \operatorname{rank} H_i(M;\mathbb{Z})$ cancel in pairs.

If M is not orientable we apply the same argument using $\mathbb{Z}_2$ coefficients, with $\operatorname{rank} H_i(M;\mathbb{Z})$ replaced by $\dim H_i(M;\mathbb{Z}_2)$, the dimension as a vector space over $\mathbb{Z}_2$, to conclude that $\sum_i (-1)^i \dim H_i(M;\mathbb{Z}_2) = 0$. It remains to check that this alternating sum equals the Euler characteristic $\sum_i (-1)^i \operatorname{rank} H_i(M;\mathbb{Z})$. We can do this by using the isomorphisms $H_i(M;\mathbb{Z}_2) \approx H^i(M;\mathbb{Z}_2)$ and applying the universal coefficient theorem for cohomology. Each $\mathbb{Z}$ summand of $H_i(M;\mathbb{Z})$ gives a $\mathbb{Z}_2$ summand of $H^i(M;\mathbb{Z}_2)$. Each $\mathbb{Z}_m$ summand of $H_i(M;\mathbb{Z})$ with m even gives $\mathbb{Z}_2$ summands of $H^i(M;\mathbb{Z}_2)$ and $H^{i+1}(M,\mathbb{Z}_2)$, whose contributions to $\sum_i (-1)^i \dim H_i(M;\mathbb{Z}_2)$ cancel. And $\mathbb{Z}_m$ summands of $H_i(M;\mathbb{Z})$ with m odd contribute nothing to $H^*(M;\mathbb{Z}_2)$. $\square$

Connection with Cup Product

Cup and cap product are related by the formula

$$(\ast) \qquad \psi(\alpha \frown \varphi) = (\varphi \smile \psi)(\alpha)$$

for $\alpha \in C_{k+\ell}(X;R)$, $\varphi \in C^k(X;R)$, and $\psi \in C^\ell(X;R)$. This holds since for a singular $(k+\ell)$-simplex $\sigma : \Delta^{k+\ell} \to X$ we have

$$\psi(\sigma \frown \varphi) = \psi\big(\varphi(\sigma|[v_0,\cdots,v_k])\sigma|[v_k,\cdots,v_{k+\ell}]\big)$$
$$= \varphi(\sigma|[v_0,\cdots,v_k])\psi(\sigma|[v_k,\cdots,v_{k+\ell}]) = (\varphi \smile \psi)(\sigma)$$

The formula $(\ast)$ says that the map $\varphi \smile : C^\ell(X;R) \to C^{k+\ell}(X;R)$ is equal to the map $\operatorname{Hom}_R(C_\ell(X;R),R) \to \operatorname{Hom}_R(C_{k+\ell}(X;R),R)$ dual to $\frown \varphi$. Passing to homology and co-homology, we obtain the commutative di-agram at the right. When the maps h are isomorphisms, for example when R is a field or when $R = \mathbb{Z}$ and the homology

$$\begin{array}{ccc}
H^\ell(X;R) & \xrightarrow{\ h\ } & \operatorname{Hom}_R(H_\ell(X;R),R) \\
\downarrow{\scriptstyle\varphi\smile} & & \downarrow{\scriptstyle(\frown\varphi)^*} \\
H^{k+\ell}(X;R) & \xrightarrow{\ h\ } & \operatorname{Hom}_R(H_{k+\ell}(X;R),R)
\end{array}$$

groups of X are free, then the map $\varphi \smile$ is the dual of $\frown \varphi$. Thus in these cases cup and cap product determine each other, at least if one assumes finite generation so that cohomology determines homology as well as vice versa. However, there are examples where cap and cup products are not equivalent when $R = \mathbb{Z}$ and there is torsion in homology.

By means of the formula $(\ast)$, Poincaré duality has nontrivial implications for the cup product structure of manifolds. For a closed R-orientable n-manifold M, consider the cup product pairing

$$H^k(M;R) \times H^{n-k}(M;R) \longrightarrow R, \qquad (\varphi,\psi) \mapsto (\varphi \smile \psi)[M]$$

Such a bilinear pairing $A \times B \to R$ is said to be **nonsingular** if the maps $A \to \mathrm{Hom}_R(B, R)$ and $B \to \mathrm{Hom}_R(A, R)$, obtained by viewing the pairing as a function of each variable separately, are both isomorphisms.

Proposition 3.38. *The cup product pairing is nonsingular for closed R-orientable manifolds when R is a field, or when $R = \mathbb{Z}$ and torsion in $H^*(M; \mathbb{Z})$ is factored out.*

Proof: Consider the composition

$$H^{n-k}(M; R) \xrightarrow{\;h\;} \mathrm{Hom}_R(H_{n-k}(M; R), R) \xrightarrow{\;D^*\;} \mathrm{Hom}_R(H^k(M; R), R)$$

where h is the map appearing in the universal coefficient theorem, induced by evaluation of cochains on chains, and D^* is the Hom-dual of the Poincaré duality map $D : H^k \to H_{n-k}$. The composition $D^* h$ sends $\psi \in H^{n-k}(M; R)$ to the homomorphism $\varphi \mapsto \psi([M] \frown \varphi) = (\varphi \smile \psi)[M]$. For field coefficients or for integer coefficients with torsion factored out, h is an isomorphism. Nonsingularity of the pairing in one of its variables is then equivalent to D being an isomorphism. Nonsingularity in the other variable follows by commutativity of cup product. $\qquad\square$

Corollary 3.39. *If M is a closed connected orientable n-manifold, then an element $\alpha \in H^k(M; \mathbb{Z})$ generates an infinite cyclic summand of $H^k(M; \mathbb{Z})$ iff there exists an element $\beta \in H^{n-k}(M; \mathbb{Z})$ such that $\alpha \smile \beta$ is a generator of $H^n(M; \mathbb{Z}) \approx \mathbb{Z}$. With coefficients in a field this holds for any $\alpha \neq 0$.*

Proof: For α to generate a $\mathbb{Z}$ summand of $H^k(M; \mathbb{Z})$ is equivalent to the existence of a homomorphism $\varphi : H^k(M; \mathbb{Z}) \to \mathbb{Z}$ with $\varphi(\alpha) = \pm 1$. By the nonsingularity of the cup product pairing, φ is realized by taking cup product with an element $\beta \in H^{n-k}(M; \mathbb{Z})$ and evaluating on $[M]$, so having a β with $\alpha \smile \beta$ generating $H^n(M; \mathbb{Z})$ is equivalent to having φ with $\varphi(\alpha) = \pm 1$. The case of field coefficients is similar but easier. $\qquad\square$

Example 3.40: Projective Spaces. The cup product structure of $H^*(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$ as a truncated polynomial ring $\mathbb{Z}[\alpha]/(\alpha^{n+1})$ with $|\alpha| = 2$ can easily be deduced from this as follows. The inclusion $\mathbb{C}\mathrm{P}^{n-1} \hookrightarrow \mathbb{C}\mathrm{P}^n$ induces an isomorphism on H^i for $i \leq 2n - 2$, so by induction on n, $H^{2i}(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$ is generated by α^i for $i < n$. By the corollary, there is an integer m such that the product $\alpha \smile m\alpha^{n-1} = m\alpha^n$ generates $H^{2n}(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$. This can only happen if $m = \pm 1$, and therefore $H^*(\mathbb{C}\mathrm{P}^n; \mathbb{Z}) \approx \mathbb{Z}[\alpha]/(\alpha^{n+1})$. The same argument shows $H^*(\mathbb{H}\mathrm{P}^n; \mathbb{Z}) \approx \mathbb{Z}[\alpha]/(\alpha^{n+1})$ with $|\alpha| = 4$. For $\mathbb{R}\mathrm{P}^n$ one can use the same argument with $\mathbb{Z}_2$ coefficients to deduce that $H^*(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2) \approx \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$ with $|\alpha| = 1$. The cup product structure in infinite-dimensional projective spaces follows from the finite-dimensional case, as we saw in the proof of Theorem 3.19.

Could there be a closed manifold whose cohomology is additively isomorphic to that of $\mathbb{C}\mathrm{P}^n$ but with a different cup product structure? For $n = 2$ the answer is no since duality implies that the square of a generator of H^2 must be a generator of

H^4. For $n = 3$, duality says that the product of generators of H^2 and H^4 must be a generator of H^6, but nothing is said about the square of a generator of H^2. Indeed, for $S^2 \times S^4$, whose cohomology has the same additive structure as $\mathbb{CP}^3$, the square of the generator of $H^2(S^2 \times S^4; \mathbb{Z})$ is zero since it is the pullback of a generator of $H^2(S^2; \mathbb{Z})$ under the projection $S^2 \times S^4 \to S^2$, and in $H^*(S^2; \mathbb{Z})$ the square of the generator of H^2 is zero. More generally, an exercise for §4.D describes closed 6-manifolds having the same cohomology groups as $\mathbb{CP}^3$ but where the square of the generator of H^2 is an arbitrary multiple of a generator of H^4.

Example 3.41: Lens Spaces. Cup products in lens spaces can be computed in the same way as in projective spaces. For a lens space L^{2n+1} of dimension $2n + 1$ with fundamental group $\mathbb{Z}_m$, we computed $H_i(L^{2n+1}; \mathbb{Z})$ in Example 2.43 to be $\mathbb{Z}$ for $i = 0$ and $2n+1$, $\mathbb{Z}_m$ for odd $i < 2n + 1$, and 0 otherwise. In particular, this implies that L^{2n+1} is orientable, which can also be deduced from the fact that L^{2n+1} is the orbit space of an action of $\mathbb{Z}_m$ on S^{2n+1} by orientation-preserving homeomorphisms, using an exercise at the end of this section. By the universal coefficient theorem, $H^i(L^{2n+1}; \mathbb{Z}_m)$ is $\mathbb{Z}_m$ for each $i \le 2n+1$. Let $\alpha \in H^1(L^{2n+1}; \mathbb{Z}_m)$ and $\beta \in H^2(L^{2n+1}; \mathbb{Z}_m)$ be generators. The statement we wish to prove is:

$$H^j(L^{2n+1}; \mathbb{Z}_m) \text{ is generated by } \begin{cases} \beta^i & \text{for } j = 2i \\ \alpha\beta^i & \text{for } j = 2i + 1 \end{cases}$$

By induction on n we may assume this holds for $j \le 2n-1$ since we have a lens space $L^{2n-1} \subset L^{2n+1}$ with this inclusion inducing an isomorphism on H^j for $j \le 2n - 1$, as one sees by comparing the cellular chain complexes for L^{2n-1} and L^{2n+1}. The preceding corollary does not apply directly for $\mathbb{Z}_m$ coefficients with arbitrary m, but its proof does since the maps $h: H^i(L^{2n+1}; \mathbb{Z}_m) \to \text{Hom}(H_i(L^{2n+1}; \mathbb{Z}_m), \mathbb{Z}_m)$ are isomorphisms. We conclude that $\beta \smile k\alpha\beta^{n-1}$ generates $H^{2n+1}(L^{2n+1}; \mathbb{Z}_m)$ for some integer k. We must have k relatively prime to m, otherwise the product $\beta \smile k\alpha\beta^{n-1} = k\alpha\beta^n$ would have order less than m and so could not generate $H^{2n+1}(L^{2n+1}; \mathbb{Z}_m)$. Then since k is relatively prime to m, $\alpha\beta^n$ is also a generator of $H^{2n+1}(L^{2n+1}; \mathbb{Z}_m)$. From this it follows that β^n must generate $H^{2n}(L^{2n+1}; \mathbb{Z}_m)$, otherwise it would have order less than m and so therefore would $\alpha\beta^n$.

The rest of the cup product structure on $H^*(L^{2n+1}; \mathbb{Z}_m)$ is determined once α^2 is expressed as a multiple of β. When m is odd, the commutativity formula for cup product implies $\alpha^2 = 0$. When m is even, commutativity implies only that α^2 is either zero or the unique element of $H^2(L^{2n+1}; \mathbb{Z}_m) \approx \mathbb{Z}_m$ of order two. In fact it is the latter possibility which holds, since the 2-skeleton L^2 is the circle L^1 with a 2-cell attached by a map of degree m, and we computed the cup product structure in this 2-complex in Example 3.9. It does not seem to be possible to deduce the nontriviality of α^2 from Poincaré duality alone, except when $m = 2$.

The cup product structure for an infinite-dimensional lens space L^∞ follows from the finite-dimensional case since the restriction map $H^j(L^\infty; \mathbb{Z}_m) \to H^j(L^{2n+1}; \mathbb{Z}_m)$ is

an isomorphism for $j \leq 2n + 1$. As with $\mathbb{R}P^n$, the ring structure in $H^*(L^{2n+1}; \mathbb{Z})$ is determined by the ring structure in $H^*(L^{2n+1}; \mathbb{Z}_m)$, and likewise for L^∞, where one has the slightly simpler structure $H^*(L^\infty; \mathbb{Z}) \approx \mathbb{Z}[\alpha]/(m\alpha)$ with $|\alpha| = 2$. The case of L^{2n+1} is obtained from this by setting $\alpha^{n+1} = 0$ and adjoining the extra $\mathbb{Z} \approx H^{2n+1}(L^{2n+1}; \mathbb{Z})$.

A different derivation of the cup product structure in lens spaces is given in Example 3E.2.

Using the ad hoc notation $H^k_{free}(M)$ for $H^k(M)$ modulo its torsion subgroup, the preceding proposition implies that for a closed orientable manifold M of dimension $2n$, the middle-dimensional cup product pairing $H^n_{free}(M) \times H^n_{free}(M) \to \mathbb{Z}$ is a nonsingular bilinear form on $H^n_{free}(M)$. This form is symmetric or skew-symmetric according to whether n is even or odd. The algebra in the skew-symmetric case is rather simple: With a suitable choice of basis, the matrix of a skew-symmetric nonsingular bilinear form over $\mathbb{Z}$ can be put into the standard form consisting of 2×2 blocks $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ along the diagonal and zeros elsewhere, according to an algebra exercise at the end of the section. In particular, the rank of $H^n(M^{2n})$ must be even when n is odd. We are already familiar with these facts in the case $n = 1$ by the explicit computations of cup products for surfaces in §3.2.

The symmetric case is much more interesting algebraically. There are only finitely many isomorphism classes of symmetric nonsingular bilinear forms over $\mathbb{Z}$ of a fixed rank, but this 'finitely many' grows rather rapidly, for example it is more than 80 million for rank 32; see [Serre 1973] for an exposition of this beautiful chapter of number theory. One can ask whether all these forms actually occur as cup product pairings in closed manifolds M^{4k} for a given k. The answer is yes for $4k = 4, 8, 16$ but seems to be unknown in other dimensions. In dimensions 4, 8, and 16 one can even take M^{4k} to be simply-connected and have the bare minimum of homology: $\mathbb{Z}$'s in dimensions 0 and $4k$ and a free abelian group in dimension $2k$. In dimension 4 there are at most two nonhomeomorphic simply-connected closed 4-manifolds with the same bilinear form. Namely, there are two manifolds with the same form if the square $\alpha \smile \alpha$ of some $\alpha \in H^2(M^4)$ is an odd multiple of a generator of $H^4(M^4)$, for example for $\mathbb{C}P^2$, and otherwise the M^4 is unique, for example for S^4 or $S^2 \times S^2$; see [Freedman & Quinn 1990]. In §4.C we take the first step in this direction by proving a classical result of J. H. C. Whitehead that the homotopy type of a simply-connected closed 4-manifold is uniquely determined by its cup product structure.

Other Forms of Duality

Generalizing the definition of a manifold, an **n-manifold with boundary** is a Hausdorff space M in which each point has an open neighborhood homeomorphic either to $\mathbb{R}^n$ or to the half-space $\mathbb{R}^n_+ = \{ (x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_n \geq 0 \}$. If a point $x \in M$ corresponds under such a homeomorphism to a point $(x_1, \cdots, x_n) \in \mathbb{R}^n_+$ with

$x_n = 0$, then by excision we have $H_n(M, M - \{x\}; \mathbb{Z}) \approx H_n(\mathbb{R}^n_+, \mathbb{R}^n_+ - \{0\}; \mathbb{Z}) = 0$, whereas if x corresponds to a point $(x_1, \cdots, x_n) \in \mathbb{R}^n_+$ with $x_n > 0$ or to a point of $\mathbb{R}^n$, then $H_n(M, M - \{x\}; \mathbb{Z}) \approx H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}; \mathbb{Z}) \approx \mathbb{Z}$. Thus the points x with $H_n(M, M - \{x\}; \mathbb{Z}) = 0$ form a well-defined subspace, called the **boundary** of M and denoted ∂M. For example, $\partial \mathbb{R}^n_+ = \mathbb{R}^{n-1}$ and $\partial D^n = S^{n-1}$. It is evident that ∂M is an $(n-1)$-dimensional manifold with empty boundary.

If M is a manifold with boundary, then a **collar** neighborhood of ∂M in M is an open neighborhood homeomorphic to $\partial M \times [0, 1)$ by a homeomorphism taking ∂M to $\partial M \times \{0\}$.

Proposition 3.42. *If M is a compact manifold with boundary, then ∂M has a collar neighborhood.*

Proof: Let M' be M with an external collar attached, the quotient of the disjoint union of M and $\partial M \times [0, 1]$ in which $x \in \partial M$ is identified with $(x, 0) \in \partial M \times [0, 1]$. It will suffice to construct a homeomorphism $h : M \to M'$ since $\partial M'$ clearly has a collar neighborhood.

Since M is compact, so is the closed subspace ∂M. This implies that we can choose a finite number of continuous functions $\varphi_i : \partial M \to [0, 1]$ such that the sets $V_i = \varphi_i^{-1}(0, 1]$ form an open cover of ∂M and each V_i has closure contained in an open set $U_i \subset M$ homeomorphic to the half-space $\mathbb{R}^n_+$. After dividing each φ_i by $\sum_j \varphi_j$ we may assume $\sum_i \varphi_i = 1$.

Let $\psi_k = \varphi_1 + \cdots + \varphi_k$ and let $M_k \subset M'$ be the union of M with the points $(x, t) \in \partial M \times [0, 1]$ with $t \leq \psi_k(x)$. By definition $\psi_0 = 0$ and $M_0 = M$. We construct a homeomorphism $h_k : M_{k-1} \to M_k$ as follows. The homeomorphism $U_k \approx \mathbb{R}^n_+$ gives a collar neighborhood $\partial U_k \times [-1, 0]$ of ∂U_k in U_k, with $x \in \partial U_k$ corresponding to $(x, 0) \in \partial U_k \times [-1, 0]$. Via the external collar $\partial M \times [0, 1]$ we then have an embedding $\partial U_k \times [-1, 1] \subset M'$. We define h_k to be the identity outside this $\partial U_k \times [-1, 1]$, and for $x \in \partial U_k$ we let h_k stretch the segment $\{x\} \times [-1, \psi_{k-1}(x)]$ linearly onto $\{x\} \times [-1, \psi_k(x)]$. The composition of all the h_k's then gives a homeomorphism $M \approx M'$, finishing the proof.   □

More generally, collars can be constructed for the boundaries of paracompact manifolds in the same way.

A compact manifold M with boundary is defined to be R-orientable if $M - \partial M$ is R-orientable as a manifold without boundary. If $\partial M \times [0, 1)$ is a collar neighborhood of ∂M in M then $H_i(M, \partial M; R)$ is naturally isomorphic to $H_i(M - \partial M, \partial M \times (0, \varepsilon); R)$, so when M is R-orientable, Lemma 3.27 gives a relative fundamental class $[M]$ in $H_n(M, \partial M; R)$ restricting to a given orientation at each point of $M - \partial M$.

It will not be difficult to deduce the following generalization of Poincaré duality to manifolds with boundary from the version we have already proved for noncompact manifolds:

Theorem 3.43. *Suppose M is a compact R-orientable n-manifold whose boundary ∂M is decomposed as the union of two compact $(n-1)$-dimensional manifolds A and B with a common boundary $\partial A = \partial B = A \cap B$. Then cap product with a fundamental class $[M] \in H_n(M, \partial M; R)$ gives isomorphisms $D_M : H^k(M, A; R) \to H_{n-k}(M, B; R)$ for all k.*

The possibility that A, B, or $A \cap B$ is empty is not excluded. The cases $A = \varnothing$ and $B = \varnothing$ are sometimes called Lefschetz duality.

Proof: The cap product map $D_M : H^k(M, A; R) \to H_{n-k}(M, B; R)$ is defined since the existence of collar neighborhoods of $A \cap B$ in A and B and ∂M in M implies that A and B are deformation retracts of open neighborhoods U and V in M such that $U \cup V$ deformation retracts onto $A \cup B = \partial M$ and $U \cap V$ deformation retracts onto $A \cap B$.

The case $B = \varnothing$ is proved by applying Theorem 3.35 to $M - \partial M$. Via a collar neighborhood of ∂M we see that $H^k(M, \partial M; R) \approx H_c^k(M - \partial M; R)$, and there are obvious isomorphisms $H_{n-k}(M; R) \approx H_{n-k}(M - \partial M; R)$.

The general case reduces to the case $B = \varnothing$ by applying the five-lemma to the following diagram, where coefficients in R are implicit:

$$\cdots \longrightarrow H^k(M, \partial M) \longrightarrow H^k(M, A) \longrightarrow H^k(\partial M, A) \longrightarrow H^{k+1}(M, \partial M) \longrightarrow \cdots$$

with vertical maps $[M] \frown$, a middle column $H^k(\partial M, A) \xrightarrow{\approx} H^k(B, \partial B) \xrightarrow{[B] \frown} H_{n-k-1}(B)$, and bottom row

$$\cdots \longrightarrow H_{n-k}(M) \longrightarrow H_{n-k}(M, B) \longrightarrow H_{n-k-1}(B) \longrightarrow H_{n-k-1}(M) \longrightarrow \cdots$$

For commutativity of the middle square one needs to check that the boundary map $H_n(M, \partial M) \to H_{n-1}(\partial M)$ sends a fundamental class for M to a fundamental class for ∂M. We leave this as an exercise at the end of the section. $\square$

Here is another kind of duality which generalizes the calculation of the local homology groups $H_i(M, M - \{x\}; \mathbb{Z})$:

Theorem 3.44. *If K is a compact, locally contractible subspace of a closed orientable n-manifold M, then $H_i(M, M - K; \mathbb{Z}) \approx H^{n-i}(K; \mathbb{Z})$ for all i.*

Proof: Let U be an open neighborhood of K in M. Consider the following diagram whose rows are long exact sequences of pairs:

$$\cdots \longrightarrow H_i(M - K) \longrightarrow H_i(M) \longrightarrow H_i(M, M - K) \longrightarrow H_{i-1}(M - K) \longrightarrow \cdots$$

with $H_i(U, U - K) \approx H_i(M, M - K)$, and the diagram connecting

$$H^{n-i}(M - K, U - K), \quad H^{n-i+1}(M - K, U - K)$$

$$\cdots \longrightarrow H^{n-i}(M, U) \longrightarrow H^{n-i}(M) \longrightarrow H^{n-i}(U) \longrightarrow H^{n-i+1}(M, U) \longrightarrow \cdots$$

The second vertical map is the Poincaré duality isomorphism given by cap products with a fundamental class $[M]$. This class can be represented by a cycle which is the sum of a chain in $M - K$ and a chain in U representing elements of $H_n(M - K, U - K)$ and $H_n(U, U - K)$ respectively, and the first and third vertical maps are given by relative cap products with these classes. It is not hard to check that the diagram commutes up to sign, where for the square involving boundary and coboundary maps one uses the formula for the boundary of a cap product.

Passing to the direct limit over decreasing $U \supset K$, the first vertical arrow become the Poincaré duality isomorphism $H_i(M - K) \approx H_c^{n-i}(M - K)$. The five-lemma then gives an isomorphism $H_i(M, M - K) \approx \varinjlim H^{n-i}(U)$. We will show that the natural map from this limit to $H^{n-i}(K)$ is an isomorphism. This is easy when K has a neighborhood that is a mapping cylinder of some map $X \to K$, as in the 'letter examples' at the beginning of Chapter 0, since in this case we can compute the direct limit using neighborhoods U which are segments of the mapping cylinder that deformation retract to K.

For the general case we use Theorem A.7 and Corollary A.9 in the Appendix. The latter says that M can be embedded in some $\mathbb{R}^k$ as a retract of a neighborhood N in $\mathbb{R}^k$, and then Theorem A.7 says that K is a retract of a neighborhood in $\mathbb{R}^k$ and hence, by restriction, of a neighborhood W in M. We can compute $\varinjlim H^{n-i}(U)$ using just neighborhoods U in W, so these also retract to K and hence the map $\varinjlim H^{n-i}(U) \to H^{n-i}(K)$ is surjective. To show that it is injective, note first that the retraction $U \to K$ is homotopic to the identity $U \to U$ through maps $U \to \mathbb{R}^k$, via the standard linear homotopy. Choosing a smaller U if necessary, we may assume this homotopy is through maps $U \to N$ since K is stationary during the homotopy. Applying the retraction $N \to M$ gives a homotopy through maps $U \to M$ fixed on K. Restricting to sufficiently small $V \subset U$, we then obtain a homotopy in U from the inclusion map $V \to U$ to the retraction $V \to K$. Thus the map $H^{n-i}(U) \to H^{n-i}(V)$ factors as $H^{n-i}(U) \to H^{n-i}(K) \to H^{n-i}(V)$ where the first map is induced by inclusion and the second by the retraction. This implies that the kernel of $\varinjlim H^{n-i}(U) \to H^{n-i}(K)$ is trivial. $\square$

From this theorem we can easily deduce **Alexander duality**:

Corollary 3.45. *If K is a compact, locally contractible, nonempty, proper subspace of S^n, then $\tilde{H}_i(S^n - K; \mathbb{Z}) \approx \tilde{H}^{n-i-1}(K; \mathbb{Z})$ for all i.*

Proof: The long exact sequence of reduced homology for the pair $(S^n, S^n - K)$ gives isomorphisms $\tilde{H}_i(S^n - K; \mathbb{Z}) \approx H_{i+1}(S^n, S^n - K; \mathbb{Z})$ for most values of i. The exception is when $i = n - 1$ and we have only a short exact sequence

$$0 \longrightarrow \tilde{H}_n(S^n; \mathbb{Z}) \longrightarrow H_n(S^n, S^n - K; \mathbb{Z}) \longrightarrow \tilde{H}_{n-1}(S^n - K; \mathbb{Z}) \longrightarrow 0$$

where the initial 0 is $\tilde{H}_n(S^n - K; \mathbb{Z})$ which is zero since the components of $S^n - K$ are noncompact n-manifolds. This short exact sequence splits since we can map it to the corresponding sequence with K replaced by a point in K. Thus $\tilde{H}_{n-1}(S^n - K; \mathbb{Z})$ is $H_n(S^n, S^n - K; \mathbb{Z})$ with a $\mathbb{Z}$ summand canceled, just as $\tilde{H}^0(K; \mathbb{Z})$ is $H^0(K; \mathbb{Z})$ with a $\mathbb{Z}$ summand canceled. $\qquad\qquad\square$

The special case of Alexander duality when K is a sphere or disk was treated by more elementary means in Proposition 2B.1. As remarked there, it is interesting that the homology of $S^n - K$ does not depend on the way that K is embedded in S^n. There can be local pathologies as in the case of the Alexander horned sphere, or global complications as with knotted circles in S^3, but these have no effect on the homology of the complement. The only requirement is that K is not too bad a space itself. An example where the theorem fails without the local contractibility assumption is the 'quasi-circle', defined in an exercise for §1.3. This compact subspace $K \subset \mathbb{R}^2$ can be regarded as a subspace of S^2 by adding a point at infinity. Then we have $\tilde{H}_0(S^2 - K; \mathbb{Z}) \approx \mathbb{Z}$ since $S^2 - K$ has two path-components, but $\tilde{H}^1(K; \mathbb{Z}) = 0$ since K is simply-connected.

> **Corollary 3.46.** *If $X \subset \mathbb{R}^n$ is compact and locally contractible then $H_i(X; \mathbb{Z})$ is 0 for $i \geq n$ and torsionfree for $i = n - 1$ and $n - 2$.*

For example, a closed nonorientable n-manifold M cannot be embedded as a subspace of $\mathbb{R}^{n+1}$ since $H_{n-1}(M; \mathbb{Z})$ contains a $\mathbb{Z}_2$ subgroup, by Corollary 3.28. Thus the Klein bottle cannot be embedded in $\mathbb{R}^3$. More generally, the 2-dimensional complex $X_{m,n}$ studied in Example 1.24, the quotient spaces of $S^1 \times I$ under the identifications $(z, 0) \sim (e^{2\pi i/m} z, 0)$ and $(z, 1) \sim (e^{2\pi i/n} z, 1)$, cannot be embedded in $\mathbb{R}^3$ if m and n are not relatively prime, since $H_1(X_{m,n}; \mathbb{Z})$ is $\mathbb{Z} \times \mathbb{Z}_d$ where d is the greatest common divisor of m and n. The Klein bottle is the case $m = n = 2$.

Proof: Viewing X as a subspace of the one-point compactification S^n, Alexander duality gives isomorphisms $\tilde{H}^i(X; \mathbb{Z}) \approx \tilde{H}_{n-i-1}(S^n - X; \mathbb{Z})$. The latter group is zero for $i \geq n$ and torsionfree for $i = n - 1$, so the result follows from the universal coefficient theorem since X has finitely generated homology groups. $\qquad\square$

There is a way of extending Alexander duality and the duality in Theorem 3.44 to compact sets K that are not locally contractible, by replacing the singular cohomology of K with another kind of cohomology called Čech cohomology. This is defined in the following way. To each open cover $\mathcal{U} = \{U_\alpha\}$ of a given space X we can associate a simplicial complex $N(\mathcal{U})$ called the **nerve** of $\mathcal{U}$. This has a vertex v_α for each U_α, and a set of $k + 1$ vertices spans a k-simplex whenever the $k + 1$ corresponding U_α's have nonempty intersection. When another cover $\mathcal{V} = \{V_\beta\}$ is a refinement of $\mathcal{U}$, so each V_β is contained in some U_α, then these inclusions induce a simplicial map

$N(\mathcal{V}) \to N(\mathcal{U})$ that is well-defined up to homotopy. We can then form the direct limit $\varinjlim H^i(N(\mathcal{U}); G)$ with respect to finer and finer open covers $\mathcal{U}$. This limit group is by definition the **Čech cohomology group** $\check{H}^i(X; G)$. For a full exposition of this cohomology theory see [Eilenberg & Steenrod 1952]. With an analogous definition of relative groups, Čech cohomology turns out to satisfy the same axioms as singular cohomology. For spaces homotopy equivalent to CW complexes, Čech cohomology coincides with singular cohomology, but for spaces with local complexities it often behaves more reasonably. For example, if X is the subspace of $\mathbb{R}^3$ consisting of the spheres of radius $1/n$ and center $(1/n, 0, 0)$ for $n = 1, 2, \cdots$, then contrary to what one might expect, $H^3(X; \mathbb{Z})$ is nonzero, as shown in [Barratt & Milnor 1962]. But $\check{H}^3(X; \mathbb{Z}) = 0$ and $\check{H}^2(X; \mathbb{Z}) = \mathbb{Z}^\infty$, the direct sum of countably many copies of $\mathbb{Z}$.

Oddly enough, the corresponding Čech homology groups defined using inverse limits are not so well-behaved. This is because the exactness axiom fails due to the algebraic fact that an inverse limit of exact sequences need not be exact, as a direct limit would be; see §3.F. However, there is a way around this problem using a more refined definition. This is Steenrod homology theory, which the reader can learn about in [Milnor 1995].

Exercises

1. Show that there exist nonorientable 1-dimensional manifolds if the Hausdorff condition is dropped from the definition of a manifold.

2. Show that deleting a point from a manifold of dimension greater than 1 does not affect orientability of the manifold.

3. Show that every covering space of an orientable manifold is an orientable manifold.

4. Given a covering space action of a group G on an orientable manifold M by orientation-preserving homeomorphisms, show that M/G is also orientable.

5. Show that $M \times N$ is orientable iff M and N are both orientable.

6. Given two disjoint connected n-manifolds M_1 and M_2, a connected n-manifold $M_1 \sharp M_2$, their *connected sum*, can be constructed by deleting the interiors of closed n-balls $B_1 \subset M_1$ and $B_2 \subset M_2$ and identifying the resulting boundary spheres ∂B_1 and ∂B_2 via some homeomorphism between them. (Assume that each B_i embeds nicely in a larger ball in M_i.)
(a) Show that if M_1 and M_2 are closed then there are isomorphisms $H_i(M_1 \sharp M_2; \mathbb{Z}) \approx H_i(M_1; \mathbb{Z}) \oplus H_i(M_2; \mathbb{Z})$ for $0 < i < n$, with one exception: If both M_1 and M_2 are nonorientable, then $H_{n-1}(M_1 \sharp M_2; \mathbb{Z})$ is obtained from $H_{n-1}(M_1; \mathbb{Z}) \oplus H_{n-1}(M_2; \mathbb{Z})$ by replacing one of the two $\mathbb{Z}_2$ summands by a $\mathbb{Z}$ summand. [Euler characteristics may help in the exceptional case.]
(b) Show that $\chi(M_1 \sharp M_2) = \chi(M_1) + \chi(M_2) - \chi(S^n)$ if M_1 and M_2 are closed.

7. For a map $f : M \to N$ between connected closed orientable n-manifolds with fundamental classes $[M]$ and $[N]$, the *degree* of f is defined to be the integer d such that $f_*([M]) = d[N]$, so the sign of the degree depends on the choice of fundamental classes. Show that for any connected closed orientable n-manifold M there is a degree 1 map $M \to S^n$.

8. For a map $f : M \to N$ between connected closed orientable n-manifolds, suppose there is a ball $B \subset N$ such that $f^{-1}(B)$ is the disjoint union of balls B_i each mapped homeomorphically by f onto B. Show the degree of f is $\sum_i \varepsilon_i$ where ε_i is $+1$ or -1 according to whether $f : B_i \to B$ preserves or reverses local orientations induced from given fundamental classes $[M]$ and $[N]$.

9. Show that a p-sheeted covering space projection $M \to N$ has degree $\pm p$, when M and N are connected closed orientable manifolds.

10. Show that for a degree 1 map $f : M \to N$ of connected closed orientable manifolds, the induced map $f_* : \pi_1 M \to \pi_1 N$ is surjective, hence also $f_* : H_1(M) \to H_1(N)$. [Lift f to the covering space $\tilde{N} \to N$ corresponding to the subgroup $\mathrm{Im}\, f_* \subset \pi_1 N$, then consider the two cases that this covering is finite-sheeted or infinite-sheeted.]

11. If M_g denotes the closed orientable surface of genus g, show that degree 1 maps $M_g \to M_h$ exist iff $g \geq h$.

12. As an algebraic application of the preceding problem, show that in a free group F with basis $x_1, \cdots, x_{2k}$, the product of commutators $[x_1, x_2] \cdots [x_{2k-1}, x_{2k}]$ is not equal to a product of fewer than k commutators $[v_i, w_i]$ of elements $v_i, w_i \in F$. [Recall that the 2-cell of M_k is attached by the product $[x_1, x_2] \cdots [x_{2k-1}, x_{2k}]$. From a relation $[x_1, x_2] \cdots [x_{2k-1}, x_{2k}] = [v_1, w_1] \cdots [v_j, w_j]$ in F, construct a degree 1 map $M_j \to M_k$.]

13. Let $M_h' \subset M_g$ be a compact subsurface of genus h with one boundary circle, so M_h' is homeomorphic to M_h with an open disk removed. Show there is no retraction $M_g \to M_h'$ if $h > g/2$. [Apply the previous problem, using the fact that $M_g - M_h'$ has genus $g - h$.]

14. Let X be the shrinking wedge of circles in Example 1.25, the subspace of $\mathbb{R}^2$ consisting of the circles of radius $1/n$ and center $(1/n, 0)$ for $n = 1, 2, \cdots$.

(a) If $f_n : I \to X$ is the loop based at the origin winding once around the nth circle, show that the infinite product of commutators $[f_1, f_2][f_3, f_4] \cdots$ defines a loop in X that is nontrivial in $H_1(X)$. [Use Exercise 12.]

(b) If we view X as the wedge sum of the subspaces A and B consisting of the odd-numbered and even-numbered circles, respectively, use the same loop to show that the map $H_1(X) \to H_1(A) \oplus H_1(B)$ induced by the retractions of X onto A and B is not an isomorphism.

15. For an n-manifold M and a compact subspace $A \subset M$, show that $H_n(M, M-A; R)$ is isomorphic to the group $\Gamma_R(A)$ of sections of the covering space $M_R \to M$ over A, that is, maps $A \to M_R$ whose composition with $M_R \to M$ is the identity.

16. Show that $(\alpha \frown \varphi) \frown \psi = \alpha \frown (\varphi \smile \psi)$ for all $\alpha \in C_k(X; R)$, $\varphi \in C^\ell(X; R)$, and $\psi \in C^m(X; R)$. Deduce that cap product makes $H_*(X; R)$ a right $H^*(X; R)$-module.

17. Show that a direct limit of exact sequences is exact. More generally, show that homology commutes with direct limits: If $\{C_\alpha, f_{\alpha\beta}\}$ is a directed system of chain complexes, with the maps $f_{\alpha\beta} : C_\alpha \to C_\beta$ chain maps, then $H_n(\varinjlim C_\alpha) = \varinjlim H_n(C_\alpha)$.

18. Show that a direct limit $\varinjlim G_\alpha$ of torsionfree abelian groups G_α is torsionfree. More generally, show that any finitely generated subgroup of $\varinjlim G_\alpha$ is realized as a subgroup of some G_α.

19. Show that a direct limit of countable abelian groups over a countable indexing set is countable. Apply this to show that if X is an open set in $\mathbb{R}^n$ then $H_i(X; \mathbb{Z})$ is countable for all i.

20. Show that $H_c^0(X; G) = 0$ if X is path-connected and noncompact.

21. For a space X, let X^+ be the one-point compactification. If the added point, denoted ∞, has a neighborhood in X^+ that is a cone with ∞ the cone point, show that the evident map $H_c^n(X; G) \to H^n(X^+, \infty; G)$ is an isomorphism for all n. [Question: Does this result hold when $X = \mathbb{Z} \times \mathbb{R}$?]

22. Show that $H_c^n(X \times \mathbb{R}; G) \approx H_c^{n-1}(X; G)$ for all n.

23. Show that for a locally compact Δ-complex X the simplicial and singular cohomology groups $H_c^i(X; G)$ are isomorphic. This can be done by showing that $\Delta_c^i(X; G)$ is the union of its subgroups $\Delta^i(X, A; G)$ as A ranges over subcomplexes of X that contain all but finitely many simplices, and likewise $C_c^i(X; G)$ is the union of its subgroups $C^i(X, A; G)$ for the same family of subcomplexes A.

24. Let M be a closed connected 3-manifold, and write $H_1(M; \mathbb{Z})$ as $\mathbb{Z}^r \oplus F$, the direct sum of a free abelian group of rank r and a finite group F. Show that $H_2(M; \mathbb{Z})$ is $\mathbb{Z}^r$ if M is orientable and $\mathbb{Z}^{r-1} \oplus \mathbb{Z}_2$ if M is nonorientable. In particular, $r \geq 1$ when M is nonorientable. Using Exercise 6, construct examples showing there are no other restrictions on the homology groups of closed 3-manifolds. [In the nonorientable case consider the manifold N obtained from $S^2 \times I$ by identifying $S^2 \times \{0\}$ with $S^2 \times \{1\}$ via a reflection of S^2.]

25. Show that if a closed orientable manifold M of dimension $2k$ has $H_{k-1}(M; \mathbb{Z})$ torsionfree, then $H_k(M; \mathbb{Z})$ is also torsionfree.

26. Compute the cup product structure in $H^*(S^2 \times S^8 \sharp S^4 \times S^6; \mathbb{Z})$, and in particular show that the only nontrivial cup products are those dictated by Poincaré duality. [See Exercise 6. The result has an evident generalization to connected sums of $S^i \times S^{n-i}$'s for fixed n and varying i.]

27. Show that after a suitable change of basis, a skew-symmetric nonsingular bilinear form over $\mathbb{Z}$ can be represented by a matrix consisting of 2×2 blocks $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ along the diagonal and zeros elsewhere. [For the matrix of a bilinear form, the following operation can be realized by a change of basis: Add an integer multiple of the ith row to the jth row and add the same integer multiple of the ith column to the jth column. Use this to fix up each column in turn. Note that a skew-symmetric matrix must have zeros on the diagonal.]

28. Show that a nonsingular symmetric or skew-symmetric bilinear pairing over a field F, of the form $F^n \times F^n \to F$, cannot be identically zero when restricted to all pairs of vectors v, w in a k-dimensional subspace $V \subset F^n$ if $k > n/2$.

29. Use the preceding problem to show that if the closed orientable surface M_g of genus g retracts onto a graph $X \subset M_g$, then $H_1(X)$ has rank at most g. Deduce an alternative proof of Exercise 13 from this, and construct a retraction of M_g onto a wedge sum of k circles for each $k \le g$.

30. Show that the boundary of an R-orientable manifold is also R-orientable.

31. Show that if M is a compact R-orientable n-manifold, then the boundary map $H_n(M, \partial M; R) \to H_{n-1}(\partial M; R)$ sends a fundamental class for $(M, \partial M)$ to a fundamental class for ∂M.

32. Show that a compact manifold does not retract onto its boundary.

33. Show that if M is a compact contractible n-manifold then ∂M is a homology $(n-1)$-sphere, that is, $H_i(\partial M; \mathbb{Z}) \approx H_i(S^{n-1}; \mathbb{Z})$ for all i.

34. For a compact manifold M verify that the following diagram relating Poincaré duality for M and ∂M is commutative, up to sign at least:

$$
\begin{array}{ccccccc}
H^{k-1}(\partial M; R) & \longrightarrow & H^k(M, \partial M; R) & \longrightarrow & H^k(M; R) & \longrightarrow & H^k(\partial M; R) \\
\downarrow {\scriptstyle [\partial M]\frown} & & \downarrow {\scriptstyle [M]\frown} & & \downarrow {\scriptstyle [M]\frown} & & \downarrow {\scriptstyle [\partial M]\frown} \\
H_{n-k}(\partial M; R) & \longrightarrow & H_{n-k}(M; R) & \longrightarrow & H_{n-k}(M, \partial M; R) & \longrightarrow & H_{n-k-1}(\partial M; R)
\end{array}
$$

35. If M is a noncompact R-orientable n-manifold with boundary ∂M having a collar neighborhood in M, show that there are Poincaré duality isomorphisms $H_c^k(M; R) \approx H_{n-k}(M, \partial M; R)$ for all k, using the five-lemma and the following diagram:

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & H_c^{k-1}(\partial M; R) & \longrightarrow & H_c^k(M, \partial M; R) & \longrightarrow & H_c^k(M; R) & \longrightarrow & H_c^k(\partial M; R) & \longrightarrow \cdots \\
& \downarrow {\scriptstyle D_{\partial M}} & & \downarrow {\scriptstyle D_M} & & \downarrow {\scriptstyle D_M} & & \downarrow {\scriptstyle D_{\partial M}} & \\
\cdots \longrightarrow & H_{n-k}(\partial M; R) & \longrightarrow & H_{n-k}(M; R) & \longrightarrow & H_{n-k}(M, \partial M; R) & \longrightarrow & H_{n-k-1}(\partial M; R) & \longrightarrow \cdots
\end{array}
$$