A More General Relative Künneth Formula

The relative version of the Künneth formula for pairs (X,A) and (Y,B) is a split short exact sequence

$$0 \longrightarrow \bigoplus_i \big(H_i(X,A;R)\otimes_R H_{n-i}(Y,B;R)\big) \longrightarrow H_n(X\times Y, A\times Y\cup X\times B;R) \longrightarrow \\ \bigoplus_i \operatorname{Tor}_R\big(H_i(X,A;R), H_{n-i-1}(Y,B;R)\big) \longrightarrow 0$$

where the coefficient ring R is assumed to be a principal ideal domain. In the case that (X,A) and (Y,B) are CW pairs we derived this formula from the algebraic Künneth formula in the paragraph following Corollary 3B.7. We will give an example below showing that the formula does not hold for arbitrary pairs (X,A) and (Y,B).

A version of the formula for non-CW pairs (X,A) and (Y,B) satisfying certain extra conditions can be obtained by considering CW approximations $(X',A')\to(X,A)$ and $(Y',B')\to(Y,B)$. These induce isomorphisms on relative and absolute homology groups, so in order to deduce a Künneth formula for (X,A) and (Y,B) from the corresponding formula for (X',A') and (Y',B') it suffices to find conditions under which the map $(X'\times Y', A'\times Y'\cup X'\times B')\to(X\times Y, A\times Y\cup X\times B)$ induces isomorphisms on homology. Looking at the long exact sequences of homology for the pairs $(X'\times Y', A'\times Y'\cup X'\times B')$ and $(X\times Y, A\times Y\cup X\times B)$ and using the fact that the map $X'\times Y'\to X\times Y$ is a CW approximation and so induces isomorphisms on homology, we see via the five-lemma that it suffices to find conditions for the map $A'\times Y'\cup X'\times B'\to A\times Y\cup X\times B$ to induce isomorphisms on homology.

To do this we use Mayer-Vietoris sequences. For the first union $A'\times Y'\cup X'\times B'$ we are dealing with CW complexes and subcomplexes so we certainly have a Mayer-Vietoris sequence. For the second union $A\times Y\cup X\times B$ we need some extra hypotheses on these spaces in order to have a Mayer-Vietoris sequence. It suffices to assume for example that A and B are open sets in X and Y. More generally we can assume that the union of the interiors of $A\times Y$ and $X\times B$ in $A\times Y\cup X\times B$ is all of $A\times Y\cup X\times B$.

Having Mayer-Vietoris sequences, we can now finish the argument. The maps $A'\times Y'\to A\times Y$ and $X'\times B'\to X\times B$ are CW approximations, as is the map of their intersections $A'\times B'\to A\times B$, so these maps induce isomorphisms on homology. Hence the five-lemma gives isomorphisms on the homology of the unions.

An example where the Künneth formula for pairs fails can be obtained by taking both (X,A) and (Y,B) to be the pair (I,S) where $I=[0,1]$ as usual and S is the sequence $1, 1/2, 1/3, 1/4, \cdots$ together with its limit 0. From the long exact sequence of reduced homology for the pair (I,S) and the contractibility of I we see that the only nontrivial group $H_n(I,S)$ is $H_1(I,S)\approx \widetilde{H}_0(S)=\mathbb{Z}^\infty$. This is torsionfree so the Tor terms in the Künneth formula all vanish. The formula would then say that $H_n(I\times I, S\times I\cup I\times S)$ is nonzero only for $n=2$ when it is $\mathbb{Z}^\infty\otimes\mathbb{Z}^\infty$. This is a countable group, but in fact the group $H_2(I\times I, S\times I\cup I\times S)\approx H_1(S\times I\cup I\times S)$ is uncountable as we now show. The space $Z=S\times I\cup I\times S$ is a union of horizontal and vertical line segments forming an infinite grid of rectangles together with the limiting line segments in the two coordinate axes. Let r_i be the retraction of Z onto the i^{th} diagonal square in the grid, obtained by first projecting Z onto the i^{th} horizontal strip of rectangles and then onto the i^{th} vertical strip of rectangles. The induced maps r_{i*} form the coordinates of a map $H_1(Z) \to \prod_\infty \mathbb{Z}$. This map is surjective since for every sequence of integers n_i one can construct a loop in Z that winds around the i^{th} diagonal square n_i times. There are uncountably many such sequences (one could just use sequences of zeros and ones) so we conclude that $H_1(Z)$ is uncountable, contradicting what the Künneth formula would yield.