What is dimension?

An investigation by Laura Escobar
Math Explorer's Club

The goal of this activity is to introduce you to the notion of dimension. The movie Flatland is also a great way to learn about dimension, I recommend it.

Dimension can be thought of as ways in which you can move in a space (or a world). For now, let's think that the **dimension** of a space counts in how many independent directions you can move in the space.

1. Suppose you live in a world in which you can only move in one direction. Can you imagine such a world? Can you draw it?

2. In a 1-dimensional space you can go backward as well, does this change your drawing?

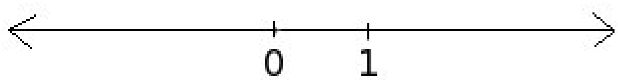
3. Suppose now you can move (forward and backward) in 2 directions. Draw this world.

4. How many dimensions does our world have? Why?

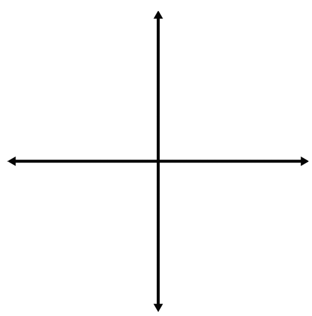
5. Are there any 0-dimensional spaces? If you think there are none, explain why. If you think there are some, describe them.

A more mathematical version of dimension

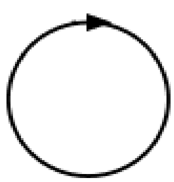
The previous handout gives a nice way to think about dimension, however it is not very precise. Mathematicians think about dimension in the following way:
Definition: The dimension of a space is how many numbers I need to specify a point in the space.
For example in a 1-dimensional space, you only need one number to specify a point. Therefore, a point corresponds to a number.



1. Why do we need 2 numbers to specify a point in the space below?



2. What about in a 3-dimensional space, why do we need here 3 numbers to specify a point?
3. What is the dimension of a hollow circle?



4. Consider now the example of the hexagon, what is the dimension of the hollow hexagon? What is the dimension of the filled hexagon?

Imagination allows us to think using 4, 5, 6, etc. numbers to define a point. However, think doesn't mean visualize; 4-dimensional spaces are very difficult to think about for experienced mathematicians. We will see some examples of this throughout the course. Just as a partial answer, we can think of time as being an independent dimension.

5. Suppose the vertical line in a 2-dimensional space represents time passing on a 1-dimensional space. Draw in the 2-dimensional plane, how Frankenstein looks like as he walks forward.

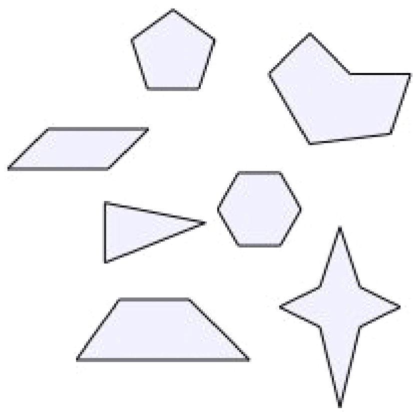
Similarly, we can think of time passing in a 2-dimensional space as the up direction in a 3-dimensional space. Then we actually live in a 4-dimensional space because we experience time. Notice a curious fact, even though we can move forward and backward, we cannot go back in time.

Polygons

An investigation by Laura Escobar

Math Explorer's Club

In this worksheet we will learn about different ways mathematicians think about polygons. We just saw some examples of polygons, let me include some more examples.

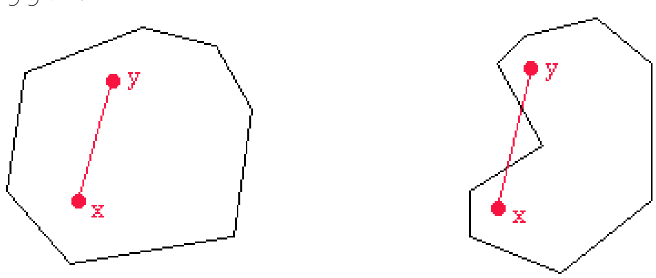


1. Try to classify the polygons on the left by stating the differences and similarities some may have.

Notice that polygons have certain pieces: points, lines and regions. We will later see a nice way to obtain such pieces. There are many ways to construct polygons, but in this course we will only consider polygons constructed in the following way.

2. **Construction:** Draw a circle and 7 points on the circle. For each point you drew, connect it with the 2 closest points with a line segment. What do you get?
3. Not all of the polygons we have encountered can be constructed like this. Which examples of the polygons can be constructed in this way?

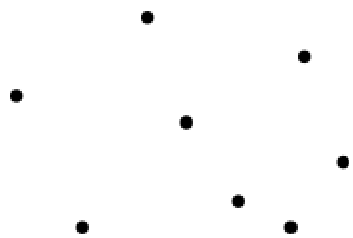
The polygons we can obtain in these way are **convex**, this means that if we take two points on the polygon and connect them using a line segment we do not go out of the polygon.



A convex polygon A non−convex polygon

Let us now learn the way some mathematicians learn about polygons.

4. We start with finitely many points in the plane, drawn below.

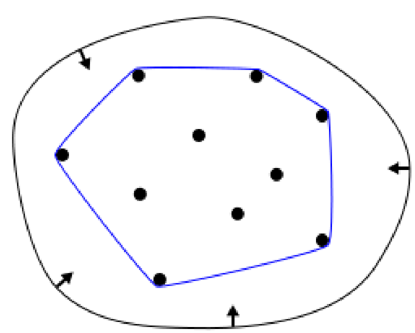


   a. First draw all the line segments connecting any two points.
   b. Now draw all the line segments connecting any two points on the lines you drew in part a.
   c. Repeat a similar process, but now drawing all line segments connecting any two points of the drawing obtained after part b. Do you get anything different?

   We then obtain an object consisting of all the lines you drew, what is it?

This process you just made is called obtaining the **convex hull** of the points you drew. You can do this process for any points you draw on the plane. The outermost

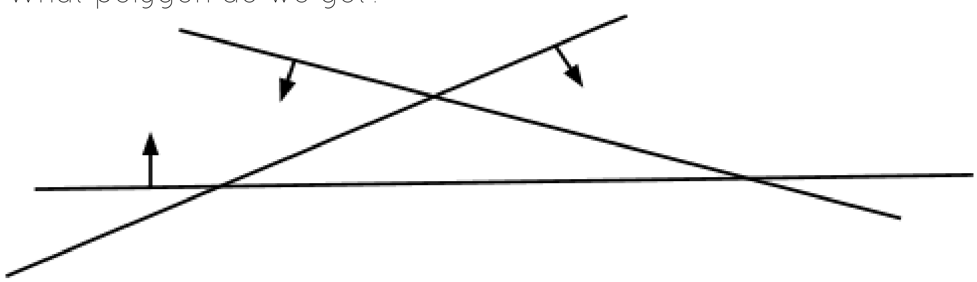
5. Explain why the object you obtain is a convex polygon.
6. Suppose you hammer some nails into a wall and then take a rubber band and wrap it around the nails as in the drawing on the right. Convince yourself that this construction is the same as the one you completed in part 4.



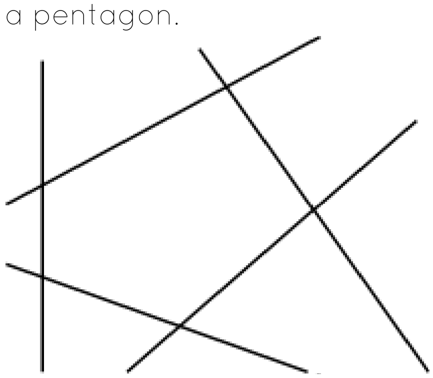
Polygons can be constructed using points or lines. We will now see how to construct a polytope using lines. A draw on the plane splits it into two regions, just like cutting a sandwich gives you two pieces.

To construct polygons, we will draw lines and pick a favorite side for each line. The polygon will then be the region that is on all our favorite sides.

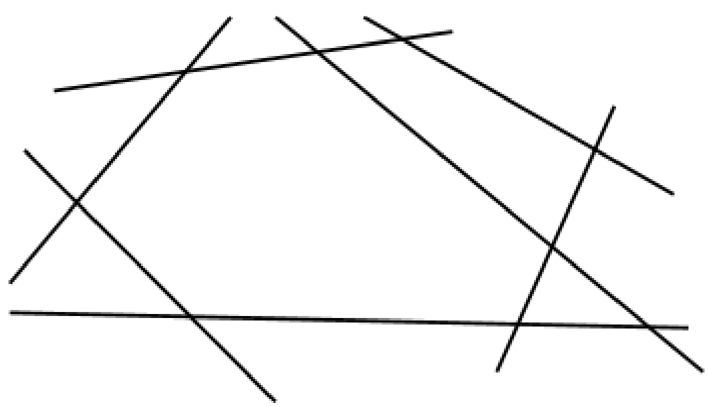
7. Consider the lines below. The little arrow indicate which is our favorite side. What polygon do we get?



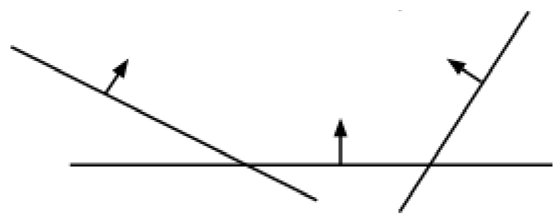
8. Now consider the following arrangement of lines. Pick the sides so that we get a pentagon.



9. Now pick the sides of the following arrangement of lines so that we get a quadrilateral. Repeat the process so that we get a hexagon. What other polygons can we get?



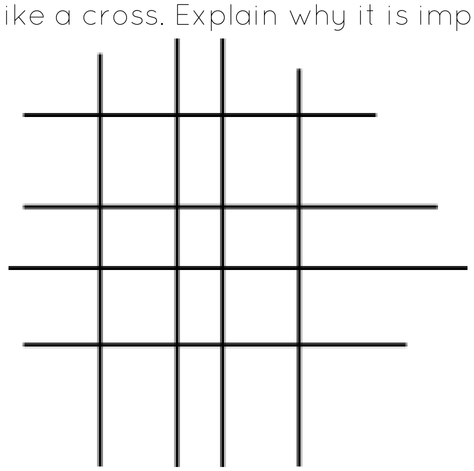
10. Consider the following arrangement of lines and regions. This one is different from the examples we have seen, how? (remember that the lines go forever in each direction.



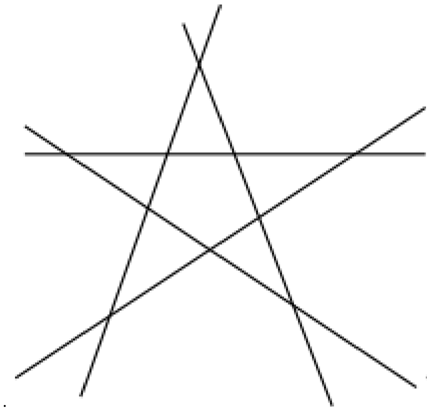
The difference between 10 and what we encountered before is that it goes in one direction forever, we call this property **unbounded**. In this course we will only work with **bounded** polygons. You can always draw a circle around a bounded polygon, this allows us to recognize them. However, an easier way to recognize bounded polygons is by walking along the perimeter of the polygon: if you loop around, then the polygon is bounded.

The process of drawing lines and picking favorite sides only gives us convex objects.

11. Consider the arrangement of lines below and suppose it gives a polygon that looks like a cross. Explain why it is impossible.



12. Consider the arrangement of lines below and suppose it gives the star polygon. Explain why this is impossible.



Challenge: Mathematicians are constantly thinking about what is true and how can they convince people that it is true. The challenge is that the proof should be general, and not just an example that it works. Try to come up with an argument to convince yourself and other people that convex polygons are the only polygons we can obtain using the construction in 2. Hint: use drawings.

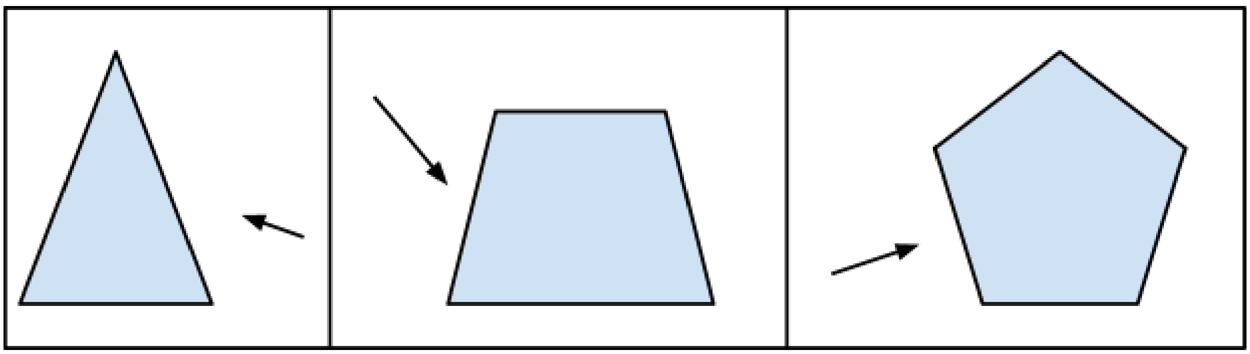
Optimization on polygons

An investigation by Laura Escobar

Math Explorer's Club, Oct. 27, 2012

The goal of this worksheet is to become familiar with the connection between certain optimization problems and polygons. We will start with how to write polygons in terms of equations, move to inequalities and then to solving optimization problems. During the last session we saw how the problem of burning a hexagon can be connected to intersecting polygons and halfspaces. Let us practice doing this visually.

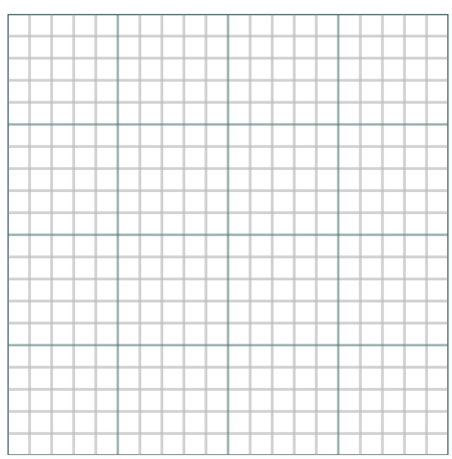
1. Look at the following drawings of polygons and directions in the plane. Highlight the face (vertex or edge) corresponding to this direction.



2. It is very difficult to follow this method and obtain an edge, most of the times if you pick a direction you will get a vertex. Try to explain why this is the case.

Last time some of you already started thinking about how to use equations to define a polygon. Let's now practice doing this. The good news is that we only need to be able to graph lines.

1. Do you remember how to graph lines? Try practicing a bit on the graph paper. For example, try graphing the lines:
   a. $y=6$
   b. $x=-2$
   c. $y=x+3$
   d. $y=1-2x$

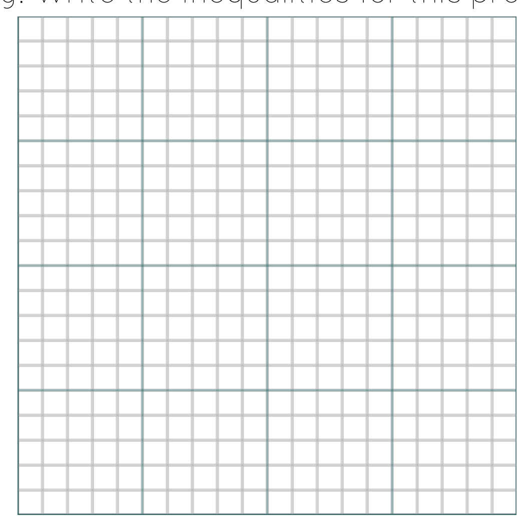


3. Since we will be working with regions, we have to graph inequalities. Graph the following inequalities:

    a. $y \leq 6$
    b. $-2 \leq x$
    c. $x+3 \leq y$
    d. $1 \leq 2x+y$
5. What object do you get after completing problem 3?
6. Now consider the direction going from the point (0,0) to the point (1,0), what face does this correspond to? Repeat the process with the direction from (0,0) to (1,2).

We are now finally ready to solve some optimization problems. We will now encounter some word problems that ask us to maximize the amount of money we can make. This problems look intimidating unless we break them up into pieces. First, try to figure out the equations we will need to solve this. It is important to identify the direction we will want to optimize. Then graph them and find the face of the polytope in the maximizing direction.

6. A carpentry shop makes night and coffee tables. Each week the shop must complete at least 9 night tables and 13 coffee tables. The shop can produce at most 30 tables each week. If the shop sells night tables for $120 and coffee tables for $150, how many of each table should be produced for a maximum weekly income?
    a. What are the variables of this problem?
    b. What quantity is to be maximized and how do I express that quantity in terms of your unknowns? This will be the direction we will use to find a face of the polygon.
    c. The constraints of the problem are some inequalities that the variables satisfy. Write the inequalities for this problem and graph them.



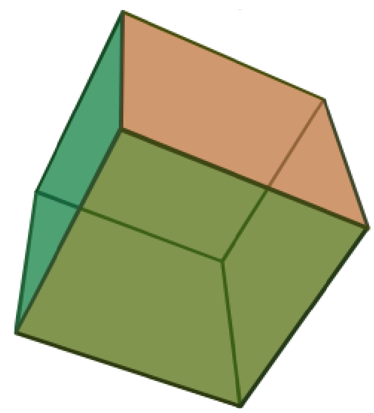
    d. Translate your equation from b. to a direction and find the corresponding face. Congratulations! you just solved this optimization problem :).

Regular polyhedra

An investigation by Laura Escobar
Math Explorer's Club, Nov. 3, 2012

The goal of this worksheet is to understand regular polyhedra. According to wikipedia, a **regular polyhedron** is a polyhedron whose faces are congruent regular polygons which are assembled in the same way around each vertex. Can you think of examples? A cube is a great example. One could naively expect that if you pick any regular polygon, we can build a polyhedron having this polygon as faces. For example, all the faces of the cube are squares. We will see that this is not the case, actually we will classify all the regular polyhedra. The way we will use to verify this is a backward construction of the one we did at the beginning of the module.

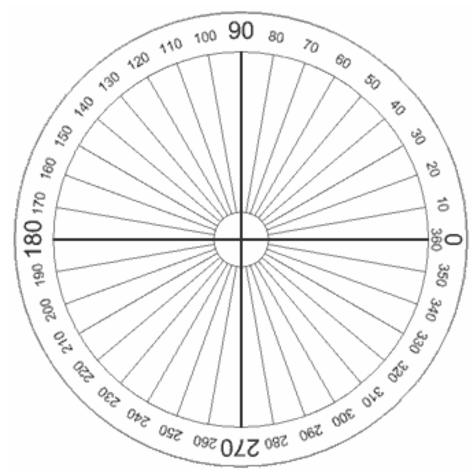
1. Let us start with a construction on (or better deconstruction of) the cube. Pick your favorite vertex on the cube and ignore everything except all the faces that contain the cube. Now think of cutting along your least favorite edge containing this vertex and open this corner to make it flat. Try drawing a cartoon of what you get.



2. This construction can be made with any polyhedron. Suppose now that we have a polyhedron made up of regular triangles with 3 triangles around each vertex. Suppose that we follow a similar process to the one on the cube, try drawing a cartoon of what you get.
3. How would you describe the cartoons you are getting?

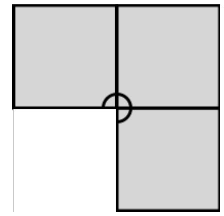
To me, they look like a number of regular polygons glued together, lying on the plane, and sharing a common vertex. Moreover, they do not fill all the space around the vertex because this space is the space we need to fold our paper model and glue it.

4. Remember angle measurements? Recall that if you go once around a circle, you travel a total of 360 . The drawing on the left has the angle measurements as you go around the circle. Consider the drawings you made in 1, 2; what can you say about the sum of the angles around the vertex?

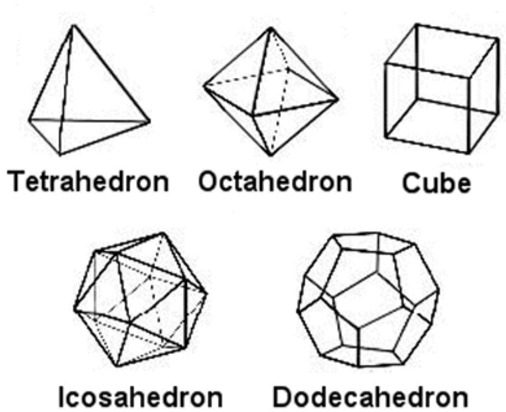


One of the many observations we can make is that the sum of the angles is less than 360 . If it were more than 360 we would not have space to fold.

5. Let's write a formula for the actual sum of the angles.
   a. Since the cube has 3 squares around each vertex, and the angle measurement of a square is 90 , write the sum of the angles around the vertex.



   b. What is the angle inside a regular triangle? Repeat exercise a. with 3 squares around each vertex.
   c. Now suppose that the number of regular n-gons around each vertex is k and the angle inside this n-gon measures A . What is the sum of the angles around such vertex?
   d. This sum must be <360 . Using trial and error figure out how large k, the number of regular n-gons, can be. (Hint: think about what happens with the angles inside an n-gon as the number of sides increases.)
   e. How small can k be? (Hint: remember we want to fold and get a 3D object.) Conclude what values k can have.
   f. What is the angle inside a regular n-gon? Use this and the inequality from d. to conclude how many sides your polygon can have.
   g. The last part would be to actually construct a polyhedron for each possible k and n. The drawing below illustrates the outcomes. Pair each regular polyhedron with the drawing.

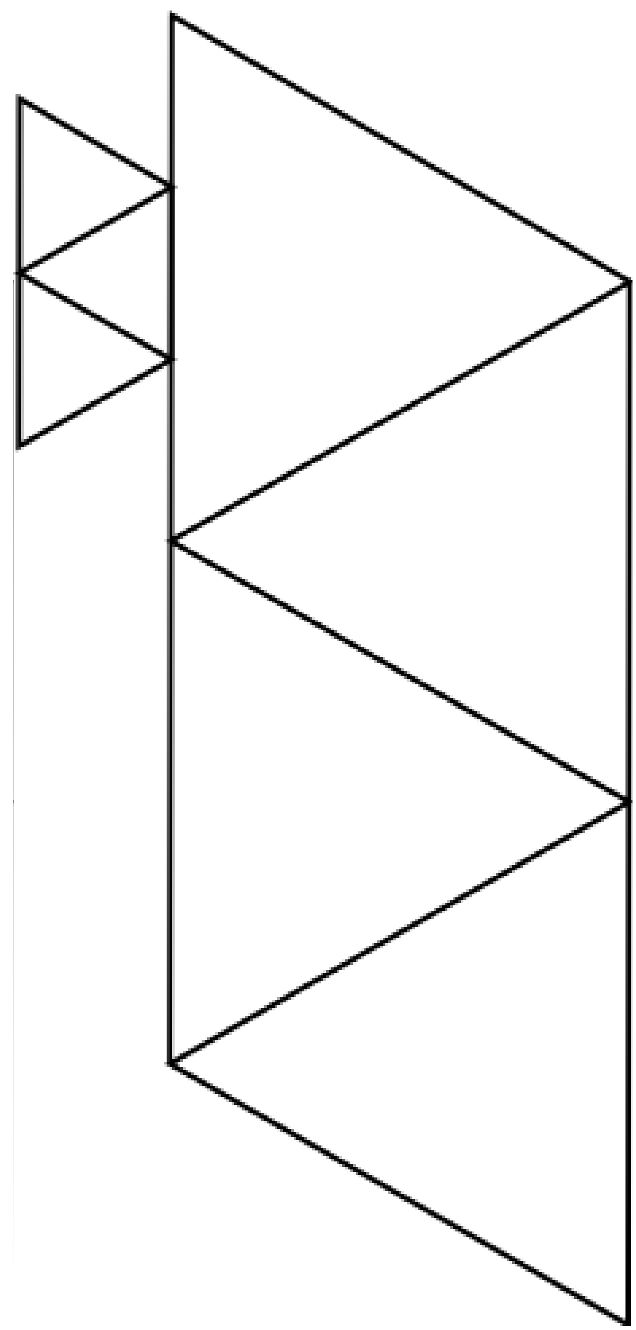


A non-convex polyhedron

An investigation by Laura Escobar

Math Explorer's Club, Nov. 3, 2012

Cut the figure belong along the perimeter, fold it and glue the edges to construct a tetrahedron with a baby tetrahedron glued to it. Use this model to explain why this object can't be constructed as the intersection of sides of planes.



4D Polytopes

An investigation by Laura Escobar
Math Explorer's Club, Nov. 10, 2012

We have seen some generalities about how to construct 4 dimensional polytopes. We will now try to draw them in a piece of paper. Let's start with the one having the least number of vertices. Before we dive into the mysterious world of 4 dimensions, let's try to find some patterns in 2 and 3 dimensions.

1. What is the polygon with the least number of vertices? How many vertices does it have? How many edges?
2. What is the polyhedron with the least number of vertices? How many vertices does it have? How many edges does it have? How many faces? (Hint: think about what would happen if it only had 2 vertices, or only 3, or only 4.)
3. How many vertices do you expect the 4D polytope with the least number of vertices to have? We will call this polytope a 4-simplex.
4. Notice that the triangle is made up of 3 edges and the tetrahedron (the 3D polyhedron with the least number of vertices) is made up of 4 triangles. The 4-simplex is made up of some polyhedra, which ones? How many?
5. The skeleton of a polytope a drawing that includes only the vertices and edges of the polytope. The skeleton of a polygon is just the hollow polygon. Draw the 1-skeleton of the cube.
6. Draw the 1-skeleton of the triangle and the tetrahedron. How many edges contain a particular vertex? Which vertices are connected?
7. Use the conclusions from 6 to draw the 1-skeleton of the 4-simplex.
8. In general, the n-dimensional simplest polytope is called the n-simplex. Do you think you can draw the skeleton of the 5-simplex? What about the 6-simplex?

We will now study the hypercube.

9. The 0-dimensional hypercube is a point, draw it. Now you obtain the 1-dimensional hypercube by moving this point one unit to the left and considering what it sweeps. Draw the 1-dimensional hypercube.
10. Now start with the 1-dimensional hypercube , pick a direction perpendicular to it and sweep it one unit in this direction. What polygon is the 2-dimensional hypercube?
11. By now you can probably guess what the 3-dimensional hypercube is. Just in case: to construct it, we start with the 2-dimensional hypercube and move it one unit in the direction perpendicular to the paper. What polyhedron is it? Draw its skeleton in a piece of paper (so we are drawing a 3-dimensional object in 2-dimensions).
12. The 4-dimensional hypercube is constructed starting with a cube, picking one direction in the 4th space and sweeping it. A cartoon of that situation is obtained by drawing two cubes and connecting similar vertices, try it out.

13. How many vertices does the 4D hypercube have? How many facets (=cubes)? Challenge: how many squares and edges?