Using Arithmetic of Real Numbers to Explore Limits and Continuity

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Problem 1

Let $a = .898989...$ and $b = .010011000111...$

(a) Find $a + b$.

(b) Use your ideas about how to add a and b above to describe a procedure for adding any two real numbers that are given as infinite decimals.

(c) Use the sum of .8989 and .0100 to estimate $a + b$. Explain why $a + b$ and $.8989 + .0100$ differ by less than $\frac{2}{10^4}$.

(d) Use the sum of 700.8989 and 100.0100 to estimate the sum of 700.89898989... and 100.010011000111.... Do these sums differ by more, less or the same amount as the sums in part c) above?

(e) Use the product of .8989 and .0100 to estimate a times b. Explain why a times b and (.8989)(.0100) differ by less than $\frac{1}{10^4}$

(f) Use the product of 700.8989 and 100.0100 to estimate 700.89898989... times 100.010011000111.... Do you expect these products to differ by more, or less, or the same amount as answer to part e) above?

Problem 2

(a) Describe how to find the square of $c = 9.373377333777....$

(b) Let's think about squaring any number between 0 and 50. What's the smallest number of decimal digits we can use in approximating the input and assure that the answer is within $\frac{1}{10^5}$ of the true value? In other words, let c_N be c truncated after N decimal digits. For example, for c in part a) above, $c_2 = 9.37$ and $c_4 = 9.3733$. What's the smallest number, N,that will assure that for any number c, $0 < c < 50$, c_N^2 is within $\frac{1}{10^5}$ of c^2?

Problem 3

(a) For what values of x does the sequence $x, x^2, x^3, ..., x^n, ...$ converge? Let $f(x)$ be the function defined by $\lim_{n\to\infty} f_n(x) = f(x)$. What is the domain of f? Is f continuous on its domain?

(b) Let $g_n(x) = 1 + x + ... + x^{n-1} = \frac{1-x^n}{1-x}$. For what values of x does $g_n(x)$ converge, and find its limiting function, $g(x)$. Given any tolerance ϵ, is there a place N in the sequence of functions so that for $n > N$ the error in using $g_n(x)$ as an approximation for $g(x)$ is less than ϵ for all x in I?

Solution to Problem1

Let $a = .898989...$ and $b = .010011000111...$

(a) Find $a + b$.

[Discussion] Students quickly realize one of these numbers is irrational and so it is not possible to express the sum as a sum of two rational numbers. What to do next? Most students have learned about irrational numbers but have not thought about how to add two of them, or how to add an irrational to a rational number. Students may need encouragement to "try something" and make some approximations. You might need to suggest that they compute a few approximate sums, say truncating a and b after two decimal places and taking their sum, or after ten, or after 20 decimal places. They will be hopeful that a pattern emerges. But after a number of examples students see that there is no predictable pattern for the decimal digits of $a + b$. I have often found that it helps if I ask them, "Suppose I'm your boss, and your job during the next 10 minutes is to add these two numbers. What would you do?" . Often students respond with something like, "If you were my boss, and I had to add these numbers, I'd ask you how close you need the answer, and then I'd figure out how many decimal places I needed to go out and chop them off and add them. " That of course is exactly how you add the two numbers.

(b) Use your ideas about how to add a and b to describe a procedure for adding any two real numbers that are given as infinite decimals.

[Discussion] The process that students go through taking longer decimal approximations to a and b and adding them is a process that defines $a+b$ to within any tolerance we have for error. Let a_N and b_N be the finite decimal numbers we obtain by truncating a and b after the Nth decimal place. The difference between a and a_N is less than $\frac{1}{10^N}$, and similarly the difference between b and b_N is less than $\frac{1}{10^N}$. Therefore, the difference between $a + b$ and $a_N + b_N$ is less than $2\frac{1}{10^N}$. We have described a process that determines $a+b$ with arbitrary accuracy. The process traps $a + b$ in progressively narrower closed intervals that are nested one inside the next. This question of how to add two real numbers leads to the topic of interval arithmetic, not only addition but multiplication and division of infinite decimal numbers which are more complex. Key tools that are used in expressing these ideas more formally are absolute value inequalities and the triangle inequality. We know that $|a - a_N| < \frac{1}{10^N}$ and $|b - b_N| < \frac{1}{10^N}$. By the triangle inequality ($|c + d| \leq |c| + |d|$) we have $|a + b - (a_N + b_N)| \leq |a - a_N| + |b - b_N| < \frac{1}{10^N} + \frac{1}{10^N} = \frac{2}{10^N}$. Use enough decimal places, N, so that $\frac{2}{10^N}$ is less than your tolerance for error. Because many real numbers are only known to us through a convergent sequence of rational numbers (think infinite decimals as a sequence of finite decimal numbers) we add real numbers by adding the convergent sequences that represent them.

(c) Use the sum of .8989 and .0100 to estimate $a + b$. Explain why $a + b$ and $.8989 + .0100$ differ by less than $\frac{2}{10^4}$.

[Discussion] If you choose not to use absolute value notation, here is how the discussion might go: We know that $0 < a - a_4 < \frac{1}{10^4}$ *and* $0 < b - b_4 < \frac{1}{10^4}$. *Therefore* $a + b - (a_4 + b_4) < \frac{1}{10^4} + \frac{1}{10^4} = \frac{2}{10^4}$.

(d) Use the sum of 700.8989 and 100.0100 to estimate the sum of 700.89898989... and 100.010011000111.... Do these sums differ by more, less or the same amount as the sums in part c) above?

[Discussion] Use the same argument used for part b) letting $a = 700.89898989...$ *and* $b = 100.010011000111...$ *We know that* $0 < a - a_4 < \frac{1}{10^4}$ *and* $0 < b - b_4 < \frac{1}{10^4}$. *Therefore* $a + b - (a_4 + b_4) < \frac{1}{10^4} + \frac{1}{10^4} = \frac{2}{10^4}$.

(e) Use the product of .8989 and .0100 to estimate a times b. Explain why a times b and $(.8989)(.0100)$ differ by less than $\frac{1}{10^4}$

[Discussion] Let $a = .89898989...$ *and* $b = .010011000111...$ *We know that* $0 < a < a_4 + \frac{1}{10^4}$ *and* $0 < b < b_4 + \frac{1}{10^4}$. *Multiplying the inequalities we get* $ab < a_4b_4 + a_4\frac{1}{10^4} + b_4\frac{1}{10^4} + \frac{1}{10^4}\frac{1}{10^4}$. *Subtracting* a_4b_4 *from both sides and factoring out the common* $\frac{1}{10^4}$ *on the right side we get* $ab - (.8989)(.0100) < \frac{1}{10^4}(.8989 + .0100 + .0001) < \frac{1}{10^4}$.

(f) Use the product of 700.8989 and 100.0100 to estimate 700.89898989... times 100.010011000111.... Do you expect these products to differ by more, or less, or the same amount as answer to part e) above?

[Discussion] If we think the case is analogous to the case of addition, we would be mistaken. Let $a = 700.89898989...$ *and* $b = 100.010011000111...$ *then* $a_4 = 700.8989$ *and* $b_4 = 100.0100$. *As we can see from the work done in the solution to part d) above* $ab < a_4b_4 + a_4\frac{1}{10^4} + b_4\frac{1}{10^4} + \frac{1}{10^4}\frac{1}{10^4}$.. *We get* $ab - a_4b_4 < (700.8989 + 100.0100 + .0001)\frac{1}{10^4} = .08009090$. *The size of the factors effects the error in using the product of rounded off numbers.*

Solution to Problem2

(a) Describe how to find the square of $c = 9.373377333777....$

[Discussion] Students will likely suggest that you can get an answer as close as you like to c^2, by squaring a long enough decimal approximation to c. This is the essence of the concept of continuity at a point c. It's important because many numbers, in fact most numbers, have to be rounded or approximated before we apply the function. If f is continuous at c then given any tolerance we have for error, $\epsilon > 0$ there is a precision for the input δ so that whenever x is within δ of c, $f(x)$ is within ϵ of $f(c)$. That's what it means when we write $\lim_{x \to c} f(x) = f(c)$.

(b) Let's think about squaring any number between 0 and 50. What's the smallest number of decimal digits we can use in approximating the input and assure that the answer is within $\frac{1}{10^5}$ of the true value? In other words, let c_N be c truncated after N decimal digits. For example, for c in part a) above, $c_2 = 9.37$ and $c_4 = 9.3733$. What's the smallest number, N, that will assure that for any number c, $0 < c < 50$, c_N^2 is within $\frac{1}{10^5}$ of c^2?

[Discussion] We want to know how close c_N and c need to be so that $|c_N^2 - c^2| \le \frac{1}{10^5}$. Factoring the difference of squares we see that this is equivalent to $|c_N - c| \le \frac{1}{10^5}\frac{1}{c_N + c}$. Observe that $0 < c_N + c < 50 + 50 = 10^2$. We see that when $|c_N - c| \le \frac{1}{10^7}$, $|c_N - c||c_N + c| < |c_N - c|100 \le \frac{1}{10^7}(100) = \frac{1}{10^5}$. So if we use 7 decimal digits, c_7, the error we make in squaring c_7 as opposed to squaring c does not exceed $\frac{1}{10^5}$. [Further Discussion] This problem raises a very practical matter. If we are going to square truncated numbers in between 0 and 50, we'd like to use the same decimal precision for all of the inputs and know that the error in the output is within our tolerance for error. If a function f has this property on an interval I, we say f is uniformly continuous on I. So more generally we are asking, is $f(x) = x^2$ uniformly continuous on $(0, 50)$? As you can see, our argument above can be used for any tolerance for error, not just $\frac{1}{10^5}$. So the answer is yes. [Even Further Discussion] Controlling error in the output that arises from rounding off inputs is related to a function's sensitivity to change. That is, how does a change in input $x - c$ compare to the change in output $f(x) - f(c)$. This is measured by the slope of the secant line between the points on the graph, and by the derivative when the change in input is small. Let's see how to generalize the approach above for a function that has a continuous derivative on a closed interval $[a, b]$ and how to use knowledge about the derivative to determine levels of precision for input on an interval. That is, given any tolerance for error ϵ, and points any points x and c in $[a, b]$, how close must x be to c to assure that $|f(x) - f(c)| < \epsilon$? Writing $\frac{|f(x)-f(c)|}{|x-c|}|x - c| < \epsilon$, and using the mean value theorem for derivatives we know that for some number d between x and c, $\frac{|f(x)-f(c)|}{|x-c|} = |f'(d)|$. Since we assumed that f' is continuous on $[a, b]$, $|f'|$ is continuous as well. By the extreme value theorem $|f'|$ has a maximum M on $[a, b]$. As long as f is not a constant function, M is not 0 and we can assure that $|f(x) - f(c)| < \epsilon$ by taking $|x - c| < \frac{\epsilon}{M}$. (Constant functions are uniformly continuous on any interval because the change in the output $f(x) - f(c) = 0 < \epsilon$ whatever x and c are.)

Solution to Problem 3

(a) For what values of x does the sequence $x, x^2, x^3, ..., x^n, ...$ converge? Let $f(x)$ be the function defined by $\lim_{n\to\infty} f_n(x) = f(x)$. What is the domain of f? Is f continuous on its domain? Why do you think this example is interesting or important?

[Discussion] For any x such that $|x| < 1$ we have that $\lim_{n\to\infty} x^n = 0$ and $\lim_{n\to\infty} 1^n = 1$. So the sequence of functions f_n converge to the function $f(x) = 0$ for $|x| < 0$ and $f(1) = 1$. f is not continuous on its domain, $-1 < x \leq 1$. The surprising thing about this example is that a sequence of functions that are each continuous on the interval $-1 < x \leq 1$ can converge to a function that is NOT continuous on that interval. We would like to avoid this kind of outcome. For this reason we are interested in having a stronger definition of convergence of a sequence of continuous functions on an interval. We don't just want the limit to exist at each point, clearly that is not enough. If we insist that the sequence of continuous functions converges uniformly on an interval, then it is possible to prove that its limit function is continuous on the interval. We say that f_n converges uniformly on an interval I, if given any tolerance ϵ, there is an N so for $n > N$ $|f(x) - f_n(x)| < \epsilon$ for all x in I.

(b) Let $g_n(x) = 1 + x + ... + x^{n-1} = \frac{1-x^n}{1-x}$. For what values of x does $g_n(x)$ converge, and find its limiting function, $g(x)$. Given any tolerance ϵ, is there a place N in the sequence of functions so that for $n > N$ the error in using $g_n(x)$ as an approximation for $g(x)$ is less than ϵ for all x in I?

[Discussion] Let $0 < c < 1$. We will show that $g_n(x)$ converges uniformly to $g(x) = \frac{1}{1-x}$ on the interval $[-c, c]$. Let ϵ be any tolerance for error in approximating g by g_n. The error in using $g_n(x)$ to approximate $g(x)$ is $|\frac{1-x^n}{1-x} - \frac{1}{1-x}| = |\frac{x^n}{1-x}|$. Since x is between $-c$ and c which are between -1 and 1, we see that $|\frac{x^n}{1-x}| < |\frac{c^n}{1-c}|$. Since c is a fixed number between 0 and 1 once we are given ϵ we can find a large enough N so that $\frac{c^N}{1-c} < \epsilon$. Thus we have shown that for $n > N$, $|g_n(x) - g(x)| = |\frac{1-x^n}{1-x} - \frac{1}{1-x}| = |\frac{x^n}{1-x}| < \frac{c^n}{1-c} < \epsilon$ for all x in $[-c, c]$. The sequence of functions $g_n(x) = 1 + x + ... + x^{n-1}$ converges uniformly to $\frac{1}{1-x}$ on $[-c, c]$. Uniform convergence of a sequence of functions is the kind of convergence you imagine a sequence of functions should have–that if you go out far enough in the sequence and sketch the graphs of the g_n the graphs are all within a narrow band of each other. This result means that on the interval $[-.99, .99]$ for example, for n big enough the graphs of the polynomial functions $g_n(x) = 1 + x + ... + x^{n-1}$ become indistinguishable from the graphs of each other and the graph of $\frac{1}{1-x}$. Why do we care about sequences of functions converging uniformly to a function? It's because in order to evaluate a function like $\sin x$ or e^x we'd like to be able to replace the function by a convergent sequence of approximating polynomials, and we'd like to evaluate the approximating polynomial at numbers that are all rounded off to the same level of precision. And we want to know that the result of evaluating the wrong function at the wrong value gives the very close to right answer!