From digits to numbers

a collection of problems assembled by J. Gordon

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1 Warm-up: the geometric sequences and series

1. Prove that for any number q, the identities hold:
   (a) $(q-1)(q+1) = q^2 - 1$;
   (b) $(q-1)(q^2+q+1) = q^3 - 1$;
   (c) $(q-1)(q^3+q^2+q+1) = q^4 - 1$;
   (d) $(q-1)(q^n + \cdots + q + 1) = q^{n+1} - 1$ for any natural number $n \geq 1$.
   (e) We get a very useful formula:
   $$1 + q + \cdots + q^n = \frac{q^{n+1}-1}{q-1} = \frac{1-q^{n+1}}{1-q}.$$
   (the second version of the formula is more convenient when $q < 1$).
2. Let us use our formula from the previous problem for some special values of q:
3. $1 + 10 + 100 + \ldots 1,000,000 =?$
4. $1 + 2 + \cdots + 2^n = 2^{n+1}$.

   (Here one can talk about the classical problem of one grain of rice for each square of the chess board; one can also see the sum of the geometric series with ratio 1/2 graphically by taking a 1×1-square, dividing it in half, then dividing one of the halves in half, etc. − these rectangles clearly fill up the square).
5. For $|q| < 1$, the sum of the infinite geometric series exists and equals
   $$1 + q + \cdots + q^n + \cdots = \frac{1}{1-q}.$$

2 The decimals

What do we mean when we represent numbers by digits? We go from right to left (this is probably because in Arabic, everything is written from right to left!). So, for a positive integer, the digits from right to left tell us the number of ones, tens, hundreds, etc. that our number is composed from. We will sometimes want to talk about digits of a number without knowing their actual values, so we want to introduce a notation that allows digits to be variables: $\overline{a_n \ldots a_1 a_0}$

means the number represented by the digits a_0, ... a_n from right to left, [1] e.g. $\overline{abc} = a*100+b*10+c$. The main point, of course, is: if $a_0, \ldots, a_n \in \{0, \ldots 9\}$, then

$$\overline{a_n \ldots a_1 a_0} = 10^n a_n + 10^{n-1} a_{n-1} + \cdots + 10 a_1 + a_0 = \sum_{k=0}^{n} a_k 10^k.$$

(For younger students might be enough to consider some examples, such as $879 = 8 \cdot 100 + 7 \cdot 10 + 9$ – in this example $n = 2$).

6. Recall Problem 3 above. Convince yourself that everything is consistent by checking that $1 + 10 + \cdots + 1,000,000$ is a number represented by seven 1s, which is, of course, equal to $\frac{10^7-1}{10-1} = \frac{9999999}{9}$.

Now we go into decimal fractions, which means that we are allowing *negative* powers of 10:

$$\overline{a_0.a_{-1}a_{-2}\ldots} = a_0 + a_1 \cdot \frac{1}{10} + a_2 \cdot \frac{1}{100} + \ldots.$$

The next problem aims to illustrate (for advanced students, *prove*), that **a number is rational if and only if it is represented by a periodic decimal fraction.**

7. (a) Show that $0.99999\ldots = 1$ (illustrating the annoying *non-uniqueness of decimal representation* for rational numbers whose denominator is a product of a power of 2 and a power of 5. If you want to explore this question further, consider alternate decimal representations for some of these numbers, e.g. 0.25, etc.) [2]

(b) Discuss the algorithm of long division, and express 1/3 as a decimal. Check that the result is right by summing the geometric series.
Solution:
$$3\frac{1}{10} + 3\frac{1}{10^2} + \cdots = \frac{3}{10}\frac{1}{1-\frac{1}{10}} = \frac{1}{3}.$$

(c) Express 5/26 as a decimal using long division, and again check that all is right using the geometric series.
Solution: $\frac{5}{26} = 0.19230769230769\ldots$, and this checks out:
$$0.19 + \frac{1}{100}\frac{230769}{10^6}\sum_{n=0}^{\infty}\frac{1}{10^{6n}} = \frac{5}{26}.$$

(d) (Prove that) a number can be represented by a finite decimal if and only if it is of the form $a/2^m 5^k$ for some integer a and nonnegative integers m, k.
Solution: Suppose our number c has a finite decimal expansion. Then it is a sum of an integer and finitely many fractions whose denominators are powers of 10. Bringing to common denominator, we obtain that c is a ratio of an integer and a power of 10.
Conversely, suppose we have $c = \frac{a}{2^m 5^k}$. Let $n = \max(m, k)$. Then we can write $c = \frac{a 2^{n-m} 5^{n-k}}{10^n}$, so its decimal expansion has at most n digits after the decimal point.

[1] This notation is probably difficult to digest for many school students, because of the subscripts; I will use it in this handout for convenience, but in a lesson only "short" examples with a few digits might be useful.

[2] this equality is featured prominently in the hilarious monologue "The Mathematics of Change: a comic monologue about failure at Princeton" by Josh Kornbluth.

(e) Prove that a number is rational if and only if its decimal expansion is periodic.

Solution: The fact that a rational number has periodic decimal expansion follows from the way we construct a decimal expansion: as we are doing long division (and adding zeroes to the right), eventually we will get a repeating remainder (because there are finitely many remainders), and from then on everything will repeat. (To illustrate what this means, do the long division for 5/26 as above).

Conversely, suppose we have a periodic decimal expansion. Then our number can be represented as a sum of a finite decimal (coming from everything before the period), which is rational as we have seen above, and a sum of a geometric series, which is again rational. (Again, see the example of 5/26 for an illustration).

3 Implications for divisibility

8. (a) Prove that any number, when divided by 9 (or by 3), gives the same remainder as the sum of its digits. *Hint: consider a two-digit number* $\overline{ab}$ *first, and consider the difference between this number and the sum of its digits.*

(b) Prove that any number is congruent to *alternating sum* of it digits mod 11. The alternating sum is: $a_0 - a_1 + a_2 - a_3 + a_4 - \cdots + (-1)^n a_n$, where a_0 is the last digit (the number of units). (saying that two numbers are "Congruent modulo 11" is a different way of saying that these numbers have the same remainder when divided by 11, which is equivalent to saying that their difference is divisible by 11).

Solution: Let the integer a be written with the digits $a_n, a_{n-1}, \ldots, a_0$: that is,

$$a = \overline{a_n a_{n-1} \ldots a_1 a_0}.$$

In part (a), we need to prove that $a \equiv (a_0 + a_1 + \cdots + a_n) \mod 9$. In Part (b), we need to prove $a \equiv a_0 - a_1 + a_2 - a_3 + a_4 - \cdots + (-1)^n a_n \mod 11$.

Let us start with part (a). By definition of congruence, we need to verify that

$a - (a_0 + a_1 + \cdots + a_n)$ is divisible by 9. Using the expression for a, we get:

$$a - (a_0 + \cdots + a_n) = \left(10^n a_n + 10^{n-1} a_{n-1} + \cdots + 10 a_1 + a_0\right) - (a_0 + \cdots + a_n)$$
$$= \sum_{k=0}^{n} a_k 10^k - \sum_{k=0}^{n} a_k = \sum_{k=0}^{n} (10^k - 1) a_k.$$

It remains to observe that for every $k \geq 0$, $10^k - 1$ is divisible by 9 (make sure the students know how to prove it! For example, one can prove that "congruence" is preserved by taking powers: 10 is congruent to 1 mod 9, so any power of 10 is congruent to that power of 1, i.e. to 1). Thus, every term of the sum is divisible by 9, and the whole sum is divisible by 9.

Part (b) is proved similarly. We have:

$$a - (a_0 - a_1 + a_2 - a_3 + a_4 - \cdots + (-1)^n a_n) = \sum_{k=0}^{n} a_k 10^k - \sum_{k=0}^{n} a_k (-1)^k = \sum_{k=0}^{n} a_k (10^k - (-1)^k).$$

It remains to note that for every $k \geq 0$, 10^k is congruent to $(-1)^k \mod 11$.

4 Binary numbers; other bases.

Now that we understand the relationships between the numbers and their digits, it is easy to express numbers in the binary or other systems. Here are some examples:

1. Express the number 86 in binary.

   Solution: $86 = 64 + 12 = 2^6 + 2^3 + 2^2$, so its binary representation is 1001100.

2. Convert to decimal: 120120211 in base 3.

   Solution: Reading from right to left, we get: $1 + 3 + 2 \cdot 3^2 + 0 \cdot 3^4 + 2 \cdot 3^5 + 1 \cdot 3^6 + 2 \cdot 3^7 + 3^8 =$ (still a pretty big number!).

3. What is bigger: $1,000,000$ in base 6 or $20,000,000$ in base 3?

   Solution: $1,000,000$ in base 6 is 6^6, and $2,000,000$ in base 3 is $2 \cdot 3^7 = 6 \cdot 3^6$, so clearly 6^6 is bigger.

4. Which numbers have finite "decimal" (precisely, 3-adic) expansions in base 3?

   Answer: rational numbers whose denominator is a power of 3. These are the same numbers that have more than one "decimal" expansion.

5. Suppose we are writing "decimal" (i.e. 3-adic) expansions in base 3. Is there a shorter way to write the number $0.222222222\ldots$?

   Answer:

$$0.22222\ldots = \sum_{k=1}^{\infty} \frac{2}{3^k} = \frac{2}{3} \sum_{k=0}^{\infty} \frac{1}{3^k} = \frac{2}{3} \frac{1}{1 - 1/3} = 1.$$