The decimal system and a strange world of p-adic numbers

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Warm-up

A number represented by digits a, b, c in the decimal system is:

$$\overline{abc} = 100 * a + 10b + c.$$

Interestingly, we read it from right to left!
This has cool consequences:

$$\overline{abc} - (a + b + c) = 99a + 9b,$$

so we can detect remainder of a number $\mod 9$ by the sum of its digits.

$1001 = 7 \times 11 \times 13$

...and less elegant consequences: to decide whether a really big number is divisible by 7 or 13, can break it into 3-digit blocks and take the alternating sum:

$$\overline{abcdefghiklmnopqr} -$$
$$(\overline{pqr} - \overline{mno} + \overline{ikl} - \overline{fgh} + \overline{cde} - \overline{ab})$$
$$= 1001(\overline{mno} + 999\overline{ikl}$$
$$+ 999001\overline{fgh} + 999999999\overline{cde}$$
$$+ 999000999001\overline{ab})$$

The starting point

"God created the integers, all else is work of Man"
Leopold Kronecker, 1823-1891 (Germany)

Then . . .

Man made the rational numbers: $\frac{a}{b}$; $a, b \in \mathbb{Z}$,

and decimal expansions:

$$\frac{1}{3} = 0.333\dots$$

$$\frac{5}{26} = 0.19230769230769\dots$$

Why does this work:

$$3\frac{1}{10} + 3\frac{1}{10^2} + \cdots = \frac{3}{10}\frac{1}{1-\frac{1}{10}} = \frac{1}{3}.$$

$$0.19 + \frac{1}{100}\frac{230769}{10^6}\sum_{n=0}^{\infty}\frac{1}{10^{6n}} = \frac{5}{26}.$$

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Axiomatically . . .

The reals are defined by the axioms:

- Algebraic axioms ($\mathbb{R}$ is a field).
- Ordering axioms ($\mathbb{R}$ is an ordered field).
- The completeness axiom (The least upper bound axiom):

  If a set of real numbers has an upper bound, it has the least upper bound.

Using the least upper bound axiom, we can establish a bijection between the axiomatically-defined reals, and the set of decimal expansions (have to deal with minor problem of non-uniqueness).

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But do we know any real numbers?

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37510582097494459230781640628620899862803482534211
70679821480865132823066470938446095505822317253594
0812848111745028410270193852110555964462294895493
8196442881097566593344612847564823378678316527120
0914564856692346034861045432664821339360726024914
7372458700660631558817488152092096282925409171536
6789259036001133053054882046652138414695194151160
3305727036575959195309218611738193261179310511854
7446237996274956735188575272489122793818301194912
3367336244065664308602139494639522473719070217986
4370277053921717629317675238467481846766940513200
5420199561121290219608640344181598136297747713099
5187072113499999983729780499510597317328160963185
0244594553469083026425223082533446850352619311881
0100031378387528865875332083814206171776691473035
2534904287554687311595628638820654858632788659361
3818279682303019520353018529689957736225994138912
7217752834791315155748572424541506959508295331168
7278558890750983817546374649393192550604009277016
1390098488240128583616035637076601047101819429555

We have:

- approximations
- algebraic equations: $\sqrt{2}$ is defined by $x^2 = 2$.
- special properties: π, e, and a handful of others.

References and further reading

- C. Cunningham, "A report from the ambassador to Cida-2", College Mathematics Journal 39:5, 2008.
- A. Rich, "Leftist Numbers", same issue of College Math Journal. http://arxiv.org/pdf/1108.6310v1.pdf