1 Hilbert's Hotel

You are the manager of Hilbert's Hotel, a one of a kind establishment renown for having infinitely many rooms, numbered $1, 2, 3, 4\ldots$ When taking the job, you pledged to never turn down a guest, no matter what! Luckily for business, the hotel is completely booked, guests occupying every single room.

1. One more guest turns up! Knowing your hotel's reputation, he didn't bother to make a reservation, and now he's requesting a room.

   Your guests are in a really good mood –they're on vacation after all!– and they are all willing to change rooms once, as long as you tell each of them explicitly the number of the room they should go to. In other words, you should be able to tell the guest staying in room number n, to which room number m they have to go to.

   How can you accommodate the newcomer?

2. What if 50 people arrived at once, requesting separate rooms?

3. What if k people arrived at once, for some positive integer k?

Given your huge success, you decide to expand the services of the hotel to include transportation from the nearest airport. To carry on with the tradition, you fund Hilbertline, the first line of buses with infinitely many seats! When you buy the ticket, you are assigned a seat number, from the infinite list of seats $1, 2, 3, 4\ldots$

4. Your hotel is still full to capacity, and the first of your buses arrives, also full (that means it's carrying infinitely many passengers!). How will you house your new guests, if they all want to be in separate rooms?

5. Your hotel is full once more, and just when you think you have a moment to rest, three of your buses arrive at the same time, all of them full with tourists who want a room each. What can you do now?

6. What would you do if instead of three, you had k full buses arriving at the same time?

7. *A manager's headache.* What if you had infinitely many full buses arriving at the same time, Bus 1, Bus 2, Bus 3, Bus 4... ?

This problem illustrates a very important property of infinite sets –in fact, it could be used to define what an infinite set is.

> *FACT 1:* An infinite set can be mapped to a smaller subset in such a way that no two elements of the original set map to the same element in the image (this type of maps are called *injective*).

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8. Illustrate Fact 1 with some examples of subsets and mappings, taking the set of positive integers ($\{1, 2, 3, 4, \ldots\}$) as your starting infinite set.

9. *Time to call it quits.* There's going to be a huge rock concert nearby, and you make sure the hotel is completely empty by the time the show is done, so nobody needs to move at night. However, the groupies are math fanatics, and they agreed each of them would wear a T-shirt with a decimal number between 0 and 1 on it, so that no two have the same number, and that every single number between 0 and 1 is displayed on somebody's T-shirt.

   You try to give each person one room... but you can't! Can you prove why?

2 Area and perimeter

What happens with the notions of area and perimeter if instead of the usual geometric figures, we consider one that allows for an infinite construction?

The Koch snowflake can be constructed by starting with an equilateral triangle, then recursively altering each line segment as follows:

(a) Divide the line segment into three segments of equal length.

(b) Draw an equilateral triangle that has the middle segment from step (a) as its base and points outward.

(c) Remove the line segment that is the base of the triangle from step (b).

1. Starting with an equilateral triangle, draw at least three iterations of steps (a)-(b)-(c). *You might want to start with a large triangle!*

   Of course, we can't really draw the Koch snowflake, since its construction requires an infinite number of steps!

   When you get home, you might want to look up the Koch snowflake; Wikipedia (*https://en.wikipedia.org/wiki/Koch_snowflake*) has a great animation of what happens if you zoom into the curve!

2. How many sides does the snowflake have before doing any iteration? And after only one iteration of the process? After two? After three?

3. Write a formula for the number of sides of the snowflake after n iterations.

4. Let's assume that the sides of the equilateral triangle that you started with have length 1. What is the length of each side after one iteration? After two? After three?

5. Can you write a formula for the length of each side of the snowflake after n iterations?

6. Combine your answers for questions 3 and 5 to give a formula for the perimeter of the snowflake after n iterations.

7. Using your intuition: what do you think the perimeter of the Koch snowflake will be, after fully constructed?

8. Can you think of a way to prove your answer for question 7, in a way that doesn't rely on intuition and that would convince even a skeptical mathematician?

9. *Drawing with a magnifying glass.* Suppose that now the sides of the equilateral triangle that you started with have length s. Can you give a formula for the perimeter of the snowflake after n iterations? *Hint: it might help to go over your answer for questions* 4 *and* 5.

   What would the perimeter of the fully constructed snowflake be in this case? Does it matter if we make s smaller and smaller?

10. How many triangles are *added* in the first iteration of the process? In the second one? And the third?

11. How many triangles are added in the n-th iteration?

12. Let's go back to assuming that the initial equilateral triangle had sides of length 1. What is the area of each triangle added in the first iteration of the process? In the second one? And the third?

13. What is the area of each triangle added in the n-th iteration?

14. Combine questions 11 and 13 to write a formula for the total new area added after the n-th iteration.

15. Will the area of the fully constructed Koch snowflake be infinite or finite? *Even if you can't figure it out from the formula, there is an ingenious, "non-mathematical" way to answer this question!*

16. Does your answer for question 15 change if the sides of the equilateral triangle that you started with have length s instead of 1? What if s is taken to be larger and larger?

FACT 2: When we allow for geometric figures whose construction involves infinity in some way, our notions of area and perimeter are challenged. Strange things can happen, like with the Koch snowflake, which has a finite area enclosed in an infinite perimeter! This is not a contradiction with our knowledge so far, because these notions need to be re-defined to accommodate for this infinite scenario.

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17. Can you create other "infinite" geometric figures that also have the strange property mentioned in Fact 2?

3 The leprechaun and the witch

A certain girl in a vividly coloured riding hood is walking through the woods towards her granny's house, when she encounters a leprechaun sitting on the road. The leprechaun, tired of being lonely, asks the girl to sit and keep him company for an hour. In order to convince her, he promises the girl that after 30 minutes have passed, he will give her 10 gold coins out of his magic coffer; 15 minutes later he will give her 10 more coins, 7 and a half minutes later another 10 coins, and so on.

1. How many coins will she have if she takes the leprechaun's deal?

The girl feels tempted, but her mother had told her not to stop on the road, so she tells the leprechaun she will think about it and come back. After walking for a few minutes she decides to accept his offer, and on her way to meet him she runs into a witch.

Having overheard the conversation, the witch makes an offer of her own to the girl: if she stays with her for an hour, then after 30 minutes she will give the girl 10 gold coins, numbered 1 through 10. Now, the witch doesn't have a magical coffer, but she does have the power to replicate gold, so after 15 more minutes have passed, she will take back the first coin, and give the girl 100 more coins, numbered 11 through 110. After 7 and a half more minutes, she will take back coin number 2, and give the girl 1000 gold coins, numbered 111 through 1110, and so on.

The girl is greedy, so she decides to take the witch's deal.

2. How many gold coins will the girl have after the hour has passed? And the witch?

The leprechaun, benign by nature, intercedes and tries to renegotiate with the witch on behalf of the girl. He proposes the following: after 30 minutes, the witch will give the girl 10 gold coins, numbered 1 through 10. After 15 minutes, the witch will take the 10'th coin back, and give the girl 10 more coins, numbered 11 through 20. Then, 7 and a half minutes later, she will take the 20'th coin and give the girl 10 more coins numbered 21 to 30, and so on.

3. How many coins would the girl have an hour later if she took this deal? And the witch?

The witch is not quite convinced, and thinks the leprechaun is trying to trick her. They both agree on the following offer, thinking that fate will settle the matter: after 30 minutes have passed, the witch will give 10 gold coins to the girl. 15 minutes later, she will take back a random coin from the lot, and give the girl 10 more coins. 7 and a half minutes after that, she will take back another coin at random, and give the girl 10 more coins, and so on. The girl finally accepts.

4. Can you tell how many coins the girl will have after an hour?

5. Can you make a good guess of what will happen? *Hint: with probability 1 –meaning almost all the time– the girl will have either no coins or infinitely many after an hour has passed.*

Let's use the intuition we gained from this peculiar story to try and answer the following question:

6. What do we get if we evaluate $\infty - \infty$?

FACT 3: Some operations involving infinity, for example $\infty - \infty$, are what we call *indeterminate*: it's not possible to assign a value to them, and need to be studied case by case. This example shows how the rules of arithmetic change when we are dealing with infinite quantities.

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7. Can you think of other arithmetic expressions involving ∞ that would be examples for Fact 3? And some that would not? *Hint: think of what happens with other operations other than substraction*

4 Supertasks

Have you ever wondered what would happen if we were able to do an infinite number of things, infinitely fast, in a finite period of time? Such an action is called a *supertask*, and they baffled mathematicians and philosophers alike for quite sometime. Let's take a look at some "famous" supertasks, and see the crazy things that happen!

4.1 Thomson's lamp

The first case is known as Thomson's Lamp, named after James Thomson who was the first to write about it.

Imagine we have a special kind of machine hooked up to a desk lamp, able to flick the lamp on and off. As usual, flicking the switch once turns the lamp on; another flick will turn the lamp off. Now, what is not usual is the speed with which this machine can do this: it's able to do infinitely many flicks in a finite period of time!

We start a timer, and the machine turns the lamp on. At the end of one minute, it turns it off. At the end of another half minute, it turns it on again. At the end of another quarter of a minute, it turns it off. At the next eighth of a minute, it turns it on again, and this process continues, flicking the switch each time after waiting exactly one-half the time it waited before flicking it previously.

1. Even though the machine flicks the light on and off infinitely many times, the process doesn't go on forever! Find a way to convince yourself that the machine is done after two minutes.

2. Once the two minutes are up and the machine is done, we take a look at the lamp. Is it on, or is it off? How is this possible?

4.2 Frodo and the ring

At the end of his long, perilous journey, Frodo stands 100 meters away from the crater of Mount Doom, ready to destroy the ring. Little does he know that evil Sauron has built an infinity machine and attached it to the path!

It works as follows: as soon as Frodo travels a positive distance of d meters past the beginning of the path, the machine senses and immediately puts up a force field ahead of him at a distance of $2d$ meters from the start line. So, for example, when Frodo gets to the 20 meter mark, the machine will put up a force field at the 40 meter mark, thereby preventing Frodo from ever going past 40 meters.

1. Will Frodo ever make it to the 50 meter mark? Explain.
2. Will he make it to the 20 meter mark? Explain.
3. Will Frodo even be able to move towards Mount Doom?

Sauron has achieved his evil purpose! With the infinity machine attached to the path, Frodo will never be able to start running towards the volcano.

4. Can you find the paradox in this situation?

4.3 Supertasks vs. physics

Before we start, let's recall some definitions from physics:

- The *momentum* of an object (usually denoted by the letter p) is defined as the product of its mass and its velocity, $p = m.v$
- The *kinetic energy* of an object (commonly denoted by E_k) is defined as the product $E_k = \frac{m.v^2}{2}$

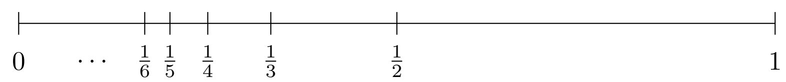
Suppose we have two identic balls of mass m lying on the floor; we leave one of them, B_1, still, and we push the other, B_2, towards B_1 in a straight line. Let's name the velocities of the balls v_1 and v_2. Eventually the two balls will collide, and we want to know the final velocities that the balls will have after the collision, which we call v_1' and v_2'.

Newtonian physics tells us that both the momentum and the kinetic energy will be preserved; this means, the total momentum (the sum of the momentum for each of the balls) before and after the collision must remain the same (and similarly for kinetic energy).

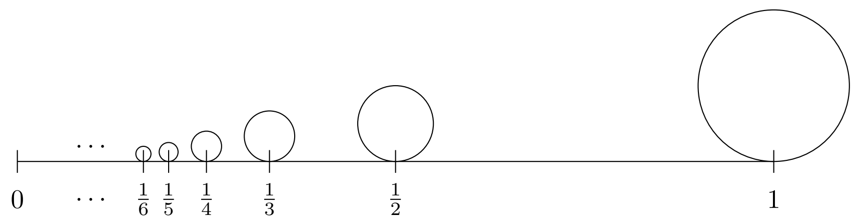
1. Express each of these two conditions as an equation, using the definitions of momentum and of kinetic energy. *Hint: the fact that B_1 was still at the beginning should help you simplify the equations that you get.*
2. Solve for v_2' in your equation for the preservation of momentum. *Hint: remember m and v_2 are fixed values, so even if you don't know them, you can treat them as if they were numbers, not variables*
3. Substitute the value of v_2' that you just got into your equation for the preservation of kinetic energy, and solve for v_2'.
4. Once you have a result for v_2', go back to your equation for the preservation of momentum, substitute v_2' by that result, and solve for v_1'.

We are now ready to discuss our next supertask!

Let's say we draw a 1 meter long line segment on the floor, and for each positive integer n, we make a mark on the spot $\frac{1}{n}$. We will have something like this:



Suppose that now we place a ball of mass m right over each mark, and that we use smaller and smaller balls, so that they don't touch one another, like this:



Having done all this, we set our timer to zero and push the first (largest) ball towards 0, giving it a speed of 1m/s. The first ball then collides with the second ball.

5. At what time do the balls collide?
6. Based on your study for questions 1-4, describe what happens to the first and second ball after they collide.

The second ball then rolls until it hits the third ball, which rolls until it hits the fourth ball, and so on.

7. What happens after 1 second has passed?
8. What is the total kinetic energy of the system when we start the timer? Is the kinetic energy preserved in the system (like we know it should) after 1 second has passed?

5 Zeno's paradox

Zeno of Elea asked a simple yet troubling question: *How is movement possible?*

Imagine that you're running a race, say the 100 meter dash. Obviously, before you reach the finish line you'll have to pass the 50 meter mark. Once you've done that, before you run the remaining 50 meters you'll have to go half that distance—you'll have to reach the 75 meter mark.

Of course, this line of reasoning can go on forever: before you go any given distance, you have to go half that, and before you go the remaining distance, you'll have to go half again that distance, and so on.

So it seems that in order to complete a simple race, you're going to have to accomplish infinitely many things along the way (first run 50 meters, then run 25 more meters, then run 12.5 more meters, and so on): an impossibility!

Let's try to understand how this daunting task can be achieved. For this purpose, suppose you're moving at a constant speed of 36 kilometers per hour.

1. What is your velocity in meters per second?
2. How long will it take you to reach the 50 meter mark? How much longer will it take you to reach the 75 meter mark? And then, how long until you get to the 87.5 meter mark? What is the pattern here?
3. Convince yourself (much in the same way as in the Thomson's lamp problem) that you will reach the end line in a finite amount of time!
4. Was the actual speed important in your reasoning?
5. Give an example where a runner runs with positive, non-constant speed and fails to complete the race.

FACT 4: A key concept is that of a *geometric sequence*: a sequence of numbers for which the ratio between any number and its predecessor is constant. For example, the sequence of distances in Zeno's paradox,

$$(50, 25, 12.5, ...)$$

is a geometric sequence since the ratio between 25 and 50 is 1/2, the ratio between 12.5 and 25 is also 1/2, and so on down the whole sequence.

If the first number in a geometric sequence is a and the common ratio is r, then we can write every number in the sequence in terms of a and r:

$$(a, ar, ar^2, ar^3, \ldots, ar^k, \ldots)$$

The (infinite) sum of all of the terms of a geometric sequence is called a *geometric series*, and we can prove that whenever the ratio satisfies $-1 < r < 1$, the sum of these infinitely many terms will be equal to

$$\frac{1}{1-r}$$

(no matter what a is!)[a]

[a] If you're feeling curious, you can find the proof of this fact on *https : //en.wikipedia.org/wiki/Geometricseries* (and many other places).