STUDENT'S HANDOUT-ADDITIONAL PROBLEMS

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These are a set of problems for Math Explorer's club (Fall 2018). The topic is Mathematical Induction.

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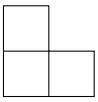
1. The 21 Game

This is a game between two players. We fix an ordering beforehand. Say "A" is the first player. "A" starts the game by saying "1" and second player increases the number by 1, 2 or 3, but may not exceed 21; the player forced to say "21" gets eliminated.

Think about a strategy which you can always adopt to win.

2. Tiling Chessboard

Setup : We have a $2^n \times 2^n$ chessboard minus a corner square.
Aim : We want to tile it using triominoes. A triomino looks like this.



"L" triomino

Problem : Is it always possible to tile our chessboard (without corner square) using triominoes?

a. Draw a 2×2 chessboard, remove a corner square. What do you find? Can you tile it using triomino?

   If yes, you are done with base case.

b. Assume that you can tile a $2^n \times 2^n$ chessboard (minus a corner square) using triominoes. How will you use this tiling to get a tiling for $2^{n+1} \times 2^{n+1}$ chessboard (minus a corner square).

3. Spot the pattern!

The following sequence of numbers is given to you:
1007, 10017, 100117,....

a. Spot the pattern in this sequence and write next 3 terms of sequence.
b. Is it true that every term is divisible by 53? How would you check this?

4. Sum of interior angles of a polygon

You must have learned in geometry that sum of angles of a triangle is 180 degrees. Do you know that there is a formula for sum of angles of any polygon?

a. What is the sum of the interior angles of a quadrilateral, a shape with four sides?
b. What is the sum of the interior angles of a pentagon? Do you have a guess for the sum of angles in an N-gon, a shape with N sides? Does it work for N = 3?
c. Let's say that your guess works for any shape with 7 sides. Does that imply it works for any shape with 8 sides?
d. Let's say your guess works for any shape with n sides. Does that imply it works for any shape with n + 1 sides? Does this mean your equation is always true?

5. Dilemma of Professors

University B. once boasted 17 tenured professors of mathematics. Tradition prescribed that at their weekly luncheon meeting, faithfully attended by all 17, any members who had discovered an error in their published work should make an announcement of this fact, and promptly resign. Such an announcement had never actually been made, because no professor was aware of any errors in her or his work. This is not to say that no errors existed, however. In fact, over the years, in the work of every member of the department at least one error had been found, by some other member of the department. This error had been mentioned to all other members of the department, but the actual author of the error had been kept ignorant of the fact, to forestall any resignations.

One fateful year, the department was augmented by a visitor from another university, one Prof. X, who had come with hopes of being offered a permanent position at the end of the academic year. Naturally, he was apprised, by various members of the department, of the published errors which had been discovered. When the hoped-for appointment failed to materialize, Prof. X obtained his revenge at the last luncheon of the year. "I have enjoyed my visit here very much," he said, "but I feel that there is one thing that I have to tell you. At least one of you has published an incorrect result, which has been discovered by others in the department." What happened the next year?

Note that in this puzzle 17 can be replaced with any number n.

a. What happens when n=1, n=2?
b. What do you think will happen when there are 3 professors?
c. Provide an inductive proof (starting at n = 2) that answers the original question: all 17 professors will end up resigning.

6. Sum of 5's and 7's

a. Is it true that 24 can be written as sum of 5's and 7's? What about 25, 26, 27?
b. Do you think that all numbers after 24 can be obtained from 7's and 5's. How would you go about proving (or disproving) this statement. Think of using induction in your argument.

7. Prime numbers

A number is called prime if it is not divisible by any other number except 1 and itself.

a. Write down first 5 prime numbers.
b. How many prime numbers are there? How do you know this?
c. Out of the following functions which on do you think will always return a prime number?
   1. $9n + 2$
   2. $n^2 - n + 41$
   3. $2n^2 + 11$

   Here is a list of first 40 primes for your reference. { 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 173, 179}

8. Even or odd

A number is called even if it is divisible by 2, otherwise it is called odd.

1. What is the remainder when an even number is divided by 2? What is the remainder when an odd number is divided by 2?
2. Is it true that every number is either even or odd? How do you know this? Use technique of induction to verify this.

   Bonus : How the problem should be formulated when we replace 2 by 3?

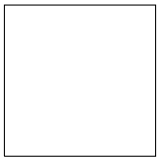
9. Graphs!

- On a piece of paper, draw some dots. These dots will be called vertices.
- Then, connect the dots using straight lines. Remember, that a line must connect two different vertices. There should be at most one line between two vertices.

Your picture is an example of a "simple graph".

Let's introduce some letters. V is number of dots on your paper, E is number of lines in your picture, F is the number of faces (regions bounded by lines, including the outer, infinitely large region).

For example, if your picture is a square



then, V=4, E=4, F=2.

Calculate V-E+F for your picture. In general, when do you think V-E+F will be 2? (Sometimes/never/always)

References

[1] http://www.math.cornell.edu/ mec/2008-2009/ABjorndahl/index.html

[2] http://pi.math.cornell.edu/files/outreach/mec2016$_i$nduction/$InductionNotes.pdf$

[3] Engel, A; Problem Solving strategies, Springer, 1999

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