# AUTOMORPHISMS OF THE DISCRETE HEISENBERG GROUP

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## 1. Introduction

The discrete Heisenberg group $\mathcal{H}$ may be described as the set $\mathbb{Z}^3$ of all integer triples endowed with the following multiplication:

$$(x, y, z) \cdot (u, v, w) = (x + u + yw, y + v, z + w). \tag{1}$$

This note describes some computational results that were obtained while studying the associated group of automorphisms $Aut(\mathcal{H})$, [2]. Specifically, we give an explicit description of a splitting of this group as an extension of $GL(2, \mathbb{Z})$ by $\mathbb{Z}^2$. That such a splitting exists could, in principle, be shown by computing a 2-cocycle for the extension and showing that it represents 0 in the cohomology group $H^2(GL(2, \mathbb{Z}), \mathbb{Z}^2) \approx \mathbb{Z}_2$, where, here, $\mathbb{Z}^2$ has the standard (left)-$GL(2, \mathbb{Z})$-module structure. However, this computation is no easier than the direct construction—indeed, it may be substantially harder— and does not appear to be as useful. We shall also include a proof of the isomorphism $H^2(GL(2, \mathbb{Z}), \mathbb{Z}^2) \approx \mathbb{Z}_2$, which is presumably well known, but the main point of this note is to provide explicit computations in $Aut(\mathcal{H})$, such as that of the mentioned splitting that may be of use to someone working with this automorphism group. A useful reference for the cohomology computations is [1].

## 2. Facts about $\mathcal{H}$

The following computational facts about $\mathcal{H}$ can be easily derived from the definition of multiplication in (1).

1. **Proposition.** *Let $x, y, z, u, v, w, n$ be any integers. Then the multiplication in $\mathcal{H}$ satisfies the following equations:*

(a) $(x, y, z)^{-1} = (-x + yz, -y, -z)$.
(b) $(x, y, z) \cdot (u, v, w) \cdot (x, y, z)^{-1} = (u + yw - zv, v, w)$.
(c) $[(x, y, z), (u, v, w)] = (yw - zv, 0, 0)$.
(d) *In particular,* $[(0, 1, 0), (0, 0, 1)] = (1, 0, 0)$.
(e) (i) $(x, 0, 0) \cdot (0, y, z) = (x, y, z)$.
(ii) $(0, y, 0) \cdot (0, 0, z) = (yz, y, z)$.
(iii) $(0, 0, z) \cdot (0, y, 0) = (0, y, z)$.
(f) (i) $(1, 0, 0)^n = (n, 0, 0)$.
(ii) $(0, 1, 0)^n = (0, n, 0)$.
(iii) $(0, 0, 1)^n = (0, 0, n)$. $\square$

It follows immediately from (b) that the center $\mathcal{Z}[\mathcal{H}]$ coincides with $\mathbb{Z} \times 0 \times 0 \subseteq \mathbb{Z}^3 = \mathcal{H}$ and from (c) that

$$[\mathcal{H}, \mathcal{H}] = \mathcal{Z}[\mathcal{H}]. \tag{2}$$

Therefore, $\mathcal{H}$ is nilpotent of nilpotency class two, and the canonical exact sequence

$$[\mathcal{H},\mathcal{H}] \rightarrowtail \mathcal{H} \twoheadrightarrow \mathcal{H}_{ab} \tag{3}$$

presents $\mathcal{H}$ as a central extension of $\mathbb{Z}^2$ by $\mathbb{Z}$.

Formulae (d)-(f) show that $(0,1,0)$ and $(0,0,1)$ generate $\mathcal{H}$. Specifically,

$$(x,y,z) = [(0,1,0),(0,0,1)]^x \cdot (0,0,1)^z \cdot (0,1,0)^y, \tag{4}$$

for all $(x,y,z) \in \mathcal{H}$.

For the next result, we use the non-standard notation $n^{(2)}$ to stand for $n(n-1)/2$, for any integer $n$.

2. **Proposition.** *For any $(x,y,z) \in \mathcal{H}$ and any $n \in \mathbb{Z}$, we have*

$$(x,y,z)^n = (nx + n^{(2)}yz, ny, nz). \tag{5}$$

*Proof.* For $n \geq 0$, the proof is an easy induction, which we leave to the reader. The case $n = -1$ is the statement in Proposition 1 (a). Combining this with the result for positive $n$ immediately gives the result for negative $n$. $\square$

The following fact was pointed out to me by K. Brown:

3. **Proposition.** *$\mathcal{H}$ may be presented as*

$$< \alpha, \beta : [\alpha,[\alpha,\beta]] = 1 = [\beta,[\alpha,\beta]] >, \tag{6}$$

*with $\alpha$ (resp., $\beta$) corresponding to the generator* $(0,1,0)$ *(resp.,* $(0,0,1)$*).*

*Proof.* Let $F$ be the free group on $\alpha$ and $\beta$, let $R$ be the relator subgroup presented in (6), and let $G$ be the quotient $F/R$. For $w \in F$, let $\overline{w}$ denote its image in $G$.

It follows immediately from the presentation that the canonical exact sequence

$$[G,G] \rightarrowtail G \twoheadrightarrow G_{ab} \tag{7}$$

is a central extension, that $[G,G]$ is cyclic with generator $[\overline{\alpha},\overline{\beta}]$, and that the abelian group $G_{ab}$ is generated by $\overline{\alpha}$ and $\overline{\beta}$.

By Proposition 1 (c) and equation (2), every pair of elements $\sigma$ and $\tau$ in $\mathcal{H}$ satisfy the relations of the presentation (6), so that there exists a unique homomorphism $f : G \to H$ such that $f(\overline{\alpha}) = \sigma$ and $f(\overline{\beta}) = \tau$. Let us choose $\sigma$ and $\tau$ to be the generators $(0,1,0)$ and $(0,0,1)$, respectively. Then, $f$ is onto. Clearly, $f$ induces a map of exact sequences (7)$\mapsto$ (3) which is surjective on every term. It follows that $[G,G]$ is infinite cyclic and $G_{ab} \approx \mathbb{Z}^2$. Therefore, the surjections $[G,G] \twoheadrightarrow [\mathcal{H},\mathcal{H}]$ and $G_{ab} \twoheadrightarrow \mathcal{H}_{ab}$ induced by $f$ are bijections, implying, by the 5-lemma, that $f$ is a bijection. $\square$

The following fact is an obvious corollary.

4. **Corollary.** *Let $L$ be any group, and let $\sigma$ and $\tau$ be any elements of $L$ satisfying the two relations in (6). Then, there is a unique homomorphism $h : \mathcal{H} \to L$ such that $h(0,1,0) = \sigma$ and $h(0,0,1) = \tau$.* $\square$

5. **Corollary.** *Let $\sigma$ and $\tau$ be any elements of $\mathcal{H}$. There exists a unique endomorphism $h$ of $\mathcal{H}$ such that $h(0,1,0) = \sigma$ and $h(0,0,1) = \tau$.*

*Proof.* As noted in the proof of Proposition 3, $\sigma$ and $\tau$ satisfy the relations described in (6), so the result follows from Corollary 4 with $L = \mathcal{H}$. $\square$

**6. Proposition.** *Let $h : \mathcal{H} \to \mathcal{H}$ be an endomorphism. Then $h$ is an automorphism if and only if $h_{ab}$ is.*

*Proof.* Suppose that $h_{ab}$ is an automorphism, and let $h'$ be the endomorphism of $[\mathcal{H}, \mathcal{H}]$ induced by $h$. Since $h_{ab}$ is surjective, there exist $\sigma$ and $\tau$ in $\mathcal{H}$ such that $h(\sigma) = (a, 1, 0)$ and $h(\tau) = (b, 0, 1)$, for some $a, b \in \mathbb{Z}$. By Proposition 1 (d), $h'([\sigma, \tau]) = (1, 0, 0)$, so $h'$ is surjective, hence bijective. The 5-lemma now implies that $h$ is an automorphism. The converse is immediate. □

*Remark.* If $h : \mathcal{H} \to \mathcal{H}$ is an endomorphism and $h(0, 1, 0) = (a, b, c)$, $h(0, 0, 1) = (d, e, f)$, then the endomorphism $h_{ab}$ of $\mathcal{H}_{ab}$ has determinant $bf - ce$. Indeed, using the obvious identification of $\mathcal{H}_{ab}$ with $\mathbb{Z}^2$, the standard matrix of $h_{ab}$ is just $\begin{pmatrix} b & e \\ c & f \end{pmatrix}$.

## 3. The multiplication in End($\mathcal{H}$)

To give a reasonably compact formula for the multiplication in $End(\mathcal{H})$, we need to introduce additional notation. For any square matrix $M$, let $|M|$ denote its determinant and $M_{ij}$ its entry in the $i, j^{th}$ position. If $A$ and $B$ are $2 \times 2$ integer matrices, define $P(A, B)$ to be the ordered pair of integers given by the matrix product

$$\begin{pmatrix} A_{11}A_{21}, & A_{12}A_{22}, & A_{12}A_{21} \end{pmatrix} \cdot \begin{pmatrix} {B_{11}}^{(2)} & {B_{12}}^{(2)} \\ {B_{21}}^{(2)} & {B_{22}}^{(2)} \\ B_{11}B_{21} & B_{12}B_{22} \end{pmatrix}. \tag{8}$$

Now, according to Corollary 5, any two elements $(a, b, c)$ and $(d, e, f)$ of $\mathcal{H}$ uniquely determine an endomorphism of $\mathcal{H}$ by the rule $h(0, 1, 0) = (a, b, c)$ and $h(0, 0, 1) = (d, e, f)$. Therefore, $h$ may be uniquely represented by the pair $((a, d), \left(\begin{smallmatrix} b & e \\ c & f \end{smallmatrix}\right))$, which we may abbreviate as $(u, U)$. (Note that we want to make use of $u$ as a row vector, i.e., a $1 \times 2$ matrix.) Let $k$ be another endomorphism, with corresponding pair $(v, V)$. We can now state the desired multiplication formula:

**7. Proposition.** *The composition $h \circ k$ in $End(\mathcal{H})$ is given by*

$$(u, U) \circ (v, V) = (|U|v + uV + P(U, V), UV), \tag{9}$$

*where $|U|v$ is the usual scalar product of the vector $v$, and $uV$ and $UV$ are given by the usual matrix multiplication.*

*Proof.* Suppose that $k(0, 1, 0) = (r, s, t)$ and $k(0, 0, 1) = (x, y, z)$, so that

$$(v, V) = ((r, x), \begin{pmatrix} s & y \\ t & z \end{pmatrix}).$$

To compute $(u, U) \circ (v, V)$, we use equation (4) to calculate $h \circ k(0, 1, 0)$ and $h \circ k(0, 0, 1)$:

$$\begin{aligned} h \circ k(0, 1, 0) &= h(r, s, t) \\ &= h([(0, 1, 0), (0, 0, 1)]^r (0, 1, 0)^t (0, 0, 1)^s) \\ &= [(a, b, c), (d, e, f)]^r (d, e, f)^t (a, b, c)^s \\ &= (r(bf - ce), 0, 0)(d, e, f)^t (a, b, c)^s. \end{aligned}$$

We now apply Proposition 2, obtaining

$$\begin{aligned} h \circ k(0,1,0) &= (r(bf-ce),0,0)\cdot(td+t^{(2)}ef, te, tf)\cdot(sa+s^{(2)}bc, sb, sc),\\ &= (r(bf-ce),0,0)\cdot(sa+td+s^{(2)}bc+t^{(2)}ef+stec, sb+te, sc+tf),\\ &= (r|U|+sa+td+s^{(2)}bc+t^{(2)}ef+stec, sb+te, sc+tf). \end{aligned}$$

A similar computation applies to $h\circ k(0,0,1)$, and, together, these yield the desired formula. $\square$

It is worth listing the following two straightforward computational consequences of Proposition 7:

8. **Corollary.** *For any* $(u,U),(v,V)$ *in* $Aut(\mathcal{H})$,

$$(u,U)^{-1} = (-|U|(uU^{-1}+P(U,U^{-1})), U^{-1}),$$

*and*

$$(u,U)\circ(v,V)\circ(u,U)^{-1} =$$
$$(-|V|(uU^{-1}+P(U,U^{-1}))+|U|vU^{-1}+uVU^{-1}+P(U,V)U^{-1}+P(UV,U^{-1}), UVU^{-1}).$$

$\square$

9. **Corollary.** *The center* $\mathcal{Z}[Aut(\mathcal{H})]$ *is trivial.*

*Proof.* Suppose that $(v,V)\in\mathcal{Z}[Aut(\mathcal{H})]$. Then Corollary 8 implies that $V\in\mathcal{Z}[GL(2,\mathbb{Z})]$, hence, as is well known, $V=\pm I$. The possibility $V=-I$ can be ruled out by a simple computation using Corollary 8. For example, suppose $V=-I$ and $U=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$. Let us call this matrix $J$ for short. We first compute $P(J,J^{-1})=P(-J,J^{-1})=P(J,-I)=(0,0)$, so we get $(v,V)=(v,-I)=(u,J)\circ(v,-I)\circ(u,J)^{-1}=(-2uJ^{-1}+vJ^{-1},-I)$, implying that, for any choice of $v$, $v-vJ=2u$, for all $u$, which is clearly impossible.

Therefore, $V=I$. By a computation similar to the one just done, we can conclude that $v\neq 0$ is impossible. $\square$

10. **Corollary.** *The surjection* $Aut(\mathcal{H})\to GL(2,\mathbb{Z})$ *given by* $h\mapsto h_{ab}$ *has kernel isomorphic to* $\mathbb{Z}^2$. *The natural left-*$GL(2,\mathbb{Z})$*-module structure on this kernel induced by inner automorphisms of* $Aut(\mathcal{H})$ *is isomorphic to the canonical structure, i.e., the one induced by left matrix multiplication on column* 2*-vectors.*

*Proof.* In terms of the notation we introduced, the map $h\mapsto h_{ab}$ may be identified with the map $(u,U)\mapsto U$. Therefore, the kernel of this map equals $\{(u,I): u\in\mathbb{Z}^2\}$, which identifies in the obvious way with $\mathbb{Z}^2$ as a set. Using equation (9), it is easy to see that

$$(u,I)\circ(v,I) = (u+v, I),$$

so that the kernel identifies with $\mathbb{Z}^2$ as an abelian group.

To compute the action of $GL(2,\mathbb{Z})$ on $\mathbb{Z}^2$ corresponding to this, we choose any $U\in GL(2,\mathbb{Z})$ and lift it to $(0,U)\in Aut(\mathcal{H})$. Using Corollary 8, we get, for any $v\in\mathbb{Z}^2$,

$$(0,U)\circ(v,I)\circ(0,U)^{-1} = (|U|vU^{-1}, I).$$

It will now be convenient to think of $v$ as ranging over $M_{1,2}(\mathbb{Z})$, the additive group of $1\times 2$ integer matrices. We define the group isomorphism $\tau: M_{1,2}(\mathbb{Z})\to M_{2,1}(\mathbb{Z})$ by

$$\tau(x,y) = \begin{pmatrix}-y\\ x\end{pmatrix}.$$

Then, one easily verifies that

$$\tau(|U|vU^{-1}) = U \cdot \tau(v),$$

for every $U \in GL(2,\mathbb{Z})$ and $v \in M_{1,2}(\mathbb{Z})$. Thus, $\tau$ yields the desired left-$GL(2,\mathbb{Z})$-module isomorphism. □

## 4. A splitting of $Aut(\mathcal{H}) \twoheadrightarrow GL(2,\mathbb{Z})$

Equations (8) and (9) suggest that a closed formula for a splitting of the map $(u, U) \mapsto U$ is difficult to define. We shall, instead, proceed by using a presentation of $GL(2,\mathbb{Z})$ and define a splitting by specifying its values on the generators of the presentation. The task will then be to verify that these values respect the relations in the presentation.

In [3], Milnor gives the following presentation for the group $SL(2,\mathbb{Z})$:

$$< x, y : xy^{-1}x = y^{-1}xy^{-1}, (xy^{-1}x)^4 = 1 >, \tag{10}$$

where $x$ and $y$ may be taken to correspond to the elementary matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

respectively.

Using this, it is easy to check that $GL(2,\mathbb{Z})$ can be given by the presentation

$$< x, y, z : xy^{-1}x = y^{-1}xy^{-1}, (xy^{-1}x)^4 = 1,\ zxz^{-1} = x^{-1},\ zyz^{-1} = y^{-1} >, \tag{11}$$

where $x$ and $y$ may be taken to correspond to the elementary matrices already described and $z$ taken to correspond to

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let us call these three matrices $e_{12}, e_{21}$, and $r$, respectively. We shall find elements $E_{12}, E_{21}$, and $R$ in $Aut(\mathcal{H})$ satisfying the relations satisfied by $x, y$, and $z$ in (11), respectively, such that

$$E_{12} \mapsto e_{12}$$
$$E_{21} \mapsto e_{21}$$
$$R \mapsto r$$

under the surjection in question. Of course, this last means that $E_{12}, E_{21}$, and $R$ have the form $(u, e_{12}), (v, e_{21})$, and $(w, r)$, respectively, so it remains to describe $u, v, w$. We shall set $u = w = (0, 1)$ and $v = (1, 0)$. The rest is now a computational exercise to verify the relations in (11). We shall give a few details for the reader.

First, set $(1, 0) = e_1$ and use Corollary 8 to compute

$$E_{21}^{-1} = (-e_1, e_{21}^{-1}).$$

Then, use this and equation (1) twice to compute

$$E_{12}E_{21}^{-1}E_{12} = (-e_1,\ e_{12}e_{21}^{-1}e_{12}) = (-e_1,\ e_{21}^{-1}e_{12}e_{21}^{-1}) = E_{21}^{-1}E_{12}E_{21}^{-1}.$$

The second equality follows from the Milnor presentation (10). Therefore, $E_{12}$ and $E_{21}$ satisfy the first relation of the Milnor presentation. One verifies easily that $e_{21}^{-1}e_{12}e_{21}^{-1} = -J$, where $J$ is as defined the proof of Corollary 9 above, so that the common value in the above equation can be written as $(-e_1,\ -J)$. Apply equation (9) twice to this value to verify the second relation in the Milnor presentation. It remains to verify that our choices satisfy the two remaining relations in (11), and we leave these to the reader.

It follows that the subgroup of $Aut(\mathcal{H})$ generated by $E_{12}, E_{21}$, and $R$ projects isomorphically onto $GL(2,\mathbb{Z})$ via the canonical map. The inverse of this isomorphism is the desired splitting.

## 5. Computing $H^2(GL(2,\mathbb{Z}),\mathbb{Z}^2)$

Consider the presentation

$$< x, y : x^4 = 1, y^3 = x^2 >$$

of the amalgamated free product $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$. This is also known ([4], p. 81)to be a presentation of $SL(2,\mathbb{Z})$,where the generators $x$ and $y$ are realized by the $2 \times 2$ integer matrices

$$\alpha = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{respectively.}$$

It is convenient to note that the matrix $\gamma = \alpha^2 = \beta^3$ is just

$$\gamma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We now consider $\mathbb{Z}_2$, $\mathbb{Z}_4$, and $\mathbb{Z}_6$ to be subgroups of $SL(2,\mathbb{Z})$, with generators $\gamma, \alpha$, and $\beta$, respectively. We let $M$ denote $\mathbb{Z}^2$, endowed with the canonical left-$SL(2,\mathbb{Z})$ module structure. Thus $M$ inherits the structure of a left-module over $\mathbb{Z}_2$, $\mathbb{Z}_4$, and $\mathbb{Z}_6$, respectively. We compute the corresponding groups of invariants and co-invariants:

11. **Proposition.** *(a)* $M^{\mathbb{Z}_4} = M^{\mathbb{Z}_6} = M^{\mathbb{Z}_2} = 0$*, and (b)* $M_{\mathbb{Z}_4} = \mathbb{Z}_2$, $M_{\mathbb{Z}_2} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $M_{\mathbb{Z}_6} = 0$.

*Proof.* (a) We note that $M^{\mathbb{Z}_2} = ker(\gamma - 1) = ker \left(\begin{smallmatrix} -2 & 0 \\ 0 & -2 \end{smallmatrix}\right) = 0$, which implies all the assertions of part (a).

(b) If $g$ is a generator of $\mathbb{Z}_n$, $n = 2, 4, 6$, then the group of co-invariants $M_{\mathbb{Z}_n} = coker(g-1)$. So, we compute

$$\gamma - 1 = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \alpha - 1 = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}, \beta = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},$$

which immediately yields (b). $\square$

12. **Proposition.** *(a)* $H^0(SL(2,\mathbb{Z}); M) = 0$. *(b)* $H^1(SL(2,\mathbb{Z}); M) = 0$.

*Proof.* (a) $H^0(SL(2,\mathbb{Z}); M) = M^{SL(2,\mathbb{Z})} \subseteq M^{\mathbb{Z}_2} = 0$.

(b) Use the Mayer-Vietoris sequence for the amalgamated free product $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6 = SL(2,\mathbb{Z})$ :

(12) $0 = M^{\mathbb{Z}_2} = H^0(\mathbb{Z}_2; M) \to H^1(SL(2,\mathbb{Z}); M) \to H^1(\mathbb{Z}_4; M) \oplus H^1(\mathbb{Z}_6; M) \xrightarrow{a} H^1(\mathbb{Z}_2; M)$.

It is well known that $H^1(\mathbb{Z}_n; M) = M_{\mathbb{Z}_n}$, so by Proposition 11, (12) becomes

$$0 \to H^1(SL(2,\mathbb{Z}; M) \to \mathbb{Z}_2 \xrightarrow{a} \mathbb{Z}_2 \oplus \mathbb{Z}_2, \tag{13}$$

so it remains to compute the map $a$. We have the domain of $a$ equal to $coker(\alpha - 1)$, with the map $a : coker(\alpha - 1) \to coker(\alpha^2 - 1)$ induced by $\alpha + 1 = \left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right)$. Now $coker(\alpha - 1)$ is generated mod $(im(\alpha - 1))$ by $(1, 0)$ in $M$, which maps to $(1, -1)$ under $\alpha + 1$. Clearly, $(1, -1)$ represents a non-zero class in $coker(\alpha^2 - 1) = coker \left(\begin{smallmatrix} -2 & 0 \\ 0 & -2 \end{smallmatrix}\right)$. So, $a$ is injective, which yields (b).

□

13. **Proposition.** $H^2(SL(2,\mathbb{Z}) = \mathbb{Z}_2$.

*Proof.* We again use the Mayer-Vietoris sequence for the amalgamated free product $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6 = SL(2,\mathbb{Z})$, now shifted up one dimension:

$$0 \to \mathbb{Z}_2 \xrightarrow{a} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to H^2(SL(2,\mathbb{Z}); M) \to H^2(\mathbb{Z}_4; M) \oplus H^2(\mathbb{Z}_6; M).$$

As is well known,

$$H^2(\mathbb{Z}_n; M) = coker(M_{\mathbb{Z}_n} \xrightarrow{N} M^{\mathbb{Z}_n}),$$

where $N$ denotes the norm map. Since $M^{\mathbb{Z}_n} = 0$, for $n = 2, 4, 6$, by Proposition 11, the Mayer-Vietoris sequence becomes

$$0 \to \mathbb{Z}_2 \xrightarrow{a} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to H^2(SL(2,\mathbb{Z}); M) \to 0,$$

which implies the proposition. □

14. **Proposition.** $H^2(GL(2,\mathbb{Z}); M) = \mathbb{Z}_2$.

*Proof.* We use the Leray-Hirsch-Serre spectral sequence for the exact sequence of groups

$$SL(2,\mathbb{Z}) \rightarrowtail GL(2,\mathbb{Z}) \twoheadrightarrow \mathbb{Z}_2,$$

which has $E_2^{p,q}$-term

$$H^p(\mathbb{Z}_2; H^q(SL(2,\mathbb{Z}); M))$$

and converges to $H^*(GL(2,\mathbb{Z}); M)$. We need the $E_2^{p,q}$-terms for $(p,q) = (0,2), (1,1), (2,0), (2,1)$, and $(3,0)$.

(a) $(p,q) = (0,2)$: $E_2^{0,2} = H^2(SL(2,\mathbb{Z}), M)^{\mathbb{Z}_2} = (\mathbb{Z}_2)^{\mathbb{Z}_2} = \mathbb{Z}_2$, since $Aut(\mathbb{Z}_2) = \{1\}$.

(b) By Proposition 11, the $E_2$-terms for $(p,q) = (1,1), (2,0), (2,1)$ and $(3,0)$ are 0.

So, $E_\infty^{0,2} = E_2^{0,2} = \mathbb{Z}_2$, and $E_\infty^{1,1} = E_\infty^{2,0} = 0$, implying the desired result. □

*Remark.* The rationalization $\mathbb{Z}^2 \otimes \mathbb{Q} = \mathbb{Q}^2$ also receives canonical left $SL(2,\mathbb{Z})$ and left $GL(2,\mathbb{Z})$ module structures, and it is not difficult to show that the corresponding cohomology groups $H^2(SL(2,\mathbb{Z}); \mathbb{Q}^2)$ and $H^2(GL(2,\mathbb{Z}); \mathbb{Q}^2)$ are both *trivial* (cf. [2]). Thus, any extension of either of these groups by $\mathbb{Q}^2$ is split. In particular, this gives a cohomology proof for the splitting of the *fibrewise rationalization* of the group $Aut(\mathcal{H})$over $GL(2,\mathbb{Z})$. Of course, this fact also follows directly from the splitting constructed here for $Aut(\mathcal{H})$.

## References


[1] Kenneth S. Brown, *Cohomology of Groups*, Springer-Verlag, New York, 1982, MR0672956(83k:20002), Zbl 0584.200036.

[2] Peter J. Kahn *Symplectic torus bundles and group extensions* New York J. Math. 11 (2005), 35-55.

[3] John W. Milnor *Introduction to Algebraic K-theory*, Annals of Mathematical Studies 72, Princeton University Press, Princeton, N.J., MR0349811(50:2304).

[4] Jean-Pierre Serre *Trees* Springer-Verlag, New York, 1980, MR0607504(82c:20083), Zbl 0548.20018