MATH 1300, Mathematical Explorations

Brussels Sprouts and Euler Characteristic

Two 75 minute classes.

Activity

Day 1:

- Introduce the game of Brussels Sprouts and the following questions.
  - How much strategy seems to be involved in a game of Brussels Sprouts?
  - If you start from a single seed (any number of edges), how long will the game last?
  - What if there are two seeds to begin with? More than two?
- Have students do some calculations of $V - E + F$ on completed games of Brussels Sprouts. (Remind them to count the outside as a region!)
- Have students calculate $V - E + F$ on some polyhedra (e.g. cubes, soccer balls, etc.).
- What is the relationship between the calculations on the plane and on a sphere/solid?
- Give students a model proof, and ask them to work through and explain it. Calculate $E - V + F$ and draw duals for some nice examples such as platonic solids. Make it clear that the proofs are in generality, not just for the examples.
- Talk through the Interdigitating trees proof of $V - E + F = 2$.
- Have students think about the following: Can you tell, from its starting position, how long a game of Brussels Sprouts will last? If you know how long the game will last, you know who will win! How does the number of edges E in the finished Brussels Sprouts game relate to the number of moves? Can you figure out how to predict the number of vertices (sprouts) in a finished Brussels Sprouts game? How many faces R are left at the end of the game?

Day 2:

- Review games as trees HW and then explain how the proof using interdigitating trees needs the Jordan Curve Theorem: Every non-self-intersecting continuous loop in the plane divides the plane into an 'interior' region bounded by the curve and an 'exterior' region containing all of the nearby and far away exterior points, so that any continuous path connecting a point of one region to a point of the other intersects with that loop somewhere.

- Is this obvious? Draw some complicated examples. Remind the students (if they've seen it) of the Koch snowflake curve, which is weird. The correctness of Camille Jordan's 1887 proof was questioned! A 6,500-line 'formal' proof was found in 2005. Strange things happen: the Jordan–Schoenflies theorem states that two regions are homeomorphic to the inside and outside of a standard circle in the plane, however this is false for 2-spheres (the Alexander Horned Sphere)!
- Use the formula for Euler characteristic of a planar graph $2 = V - E + F$ to predict how long the game will last. Can you predict who will win the game by looking at the initial position?
- Let C be the initial number of sprouts.
  Let S be the initial number of shoots (the spikes coming off the sprouts).
  Let T be the number of shoots when the game ends.

  Students should work through the following at their tables:

  - Relate S to T. (Answer: $S = T$.)
  - Relate V to C and M. (Answer: $V = C + M$.)
  - Relate E to M. (Answer: $E = 2M$.)
  - Relate F to T. (Answer: $F = T$.)
  - Use $V - E + F = 2$ to find M in terms of C and S. (Answer: $M = C + S - 2$.)
- Play Brussels Sprouts on a torus, Klein bottle, or projective plane (viewed as pieces of paper with edge identifications).

Assignments

1. Games on Trees Assignment

References and resources

Wikipedia: Brussels Sprouts

Math Explorers Club: Brussels Sprouts and Euler Characteristic

Video on Euler's Formula and Graph Duality

Video on Brussels Sprouts

Euler Characteristic Handout