MATH 1300, Mathematical Explorations

Koch Snowflake

Activity

- Review - Put some numbers up on the board for students to compare:

$$1,\ 0.9999\ldots,\ \cos 0,\ \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots,\ \frac{4}{17-13},\ \text{area of a square of side-length 1.}$$

- Discuss: Two numbers are the same if their difference is smaller than any positive number you can give.
- Then work through area & perimeter of the Koch snowflake (Section 1.5 from Calculus book) - do it on the board, instead of with a handout.
  - If the initial equilateral triangle has area 1, then what are the areas of the three smaller triangle that we attach to it? How many are there?
  - Same question for the next layer of yet-smaller triangles.
  - Write the area of Koch's snowflake as an infinite series. $1+3\cdot\left(\frac{1}{3}\right)^2+3\cdot 4\cdot\left(\frac{1}{3}\right)^4+3\cdot 4^2\cdot\left(\frac{1}{3}\right)^6+\cdots=1+\frac{1}{3}+\frac{1}{1-\frac{4}{9}}=\frac{8}{5}$.
  - Suppose the perimeter of the initial equilateral is P. What's the perimeter after attaching the first layer? After the second? Etc. What's the perimeter of the whole thing?
  - Surprised? How much paint would it take to draw the perimeter? How much to fill it?
- Could conclude by mentioning the three dimensional version of this paradox: the cake you can eat but cannot frost.

If time:

- Have students compare

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\cdots$$

  with

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\cdots.$$

- Discuss the alternating Harmonic series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\cdots\ =\ \ln 2.$$

- Do you think it converges or not?
- Plot the partial sums. (Why are they bounded above? Why are they bounded below? The odd ones are decreasing. The even ones are increasing. Why do the odd and even partial sums converge to the same thing?)
- And Euler's
$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots \;=\; \pi^2/6$$
(proved using quadratic residues).

Notes

Some knowledge of geometric series is helpful for this module. (See Zeno's Paradox module for possible activities.)

References and resources

Discovering the Art of Mathematics: Calculus

Math Explorer's Club Spring 2017 and Fall 2019: Infinity and Paradoxes

Cake You Can Eat but Cannot Frost

Follow-on activities

Hilbert Hotel

Zeno's Paradox