MATH 1300, Mathematical Explorations

Zeno's Paradox and Geometric Series

Activity

- *Achilles and the tortoise.* Achilles races the tortoise and gives the tortoise a head start. By the time Achilles reaches the tortoise's starting point x_1, the tortoise will have advanced a small distance to x_2. So now Achilles will have to run to x_2. But by then the tortoise has run to x_3. And so on, ad infinitum. So, however small the tortoise's head-start, Achilles cannot ever overtake it. More devastatingly, the possibility of motion implies an absurdity.
  - Discuss the paradox at tables
  - Work through an example: Achilles is running 10 m/s which is 10 times as fast as the tortoise (who is running 1m/s). Assume the tortoise is given a 9m head start

    When Achilles gets to x_1, how much further will have the tortoise advanced? .9m.
    How long does this take? 0.9s.
    When Achilles gets to x_2, how much further will have the tortoise advanced? 0.09m.
    How long does this take? 0.09s.

    When does Achilles 'get to' the Tortoise? 0.99999999... s
  - Discuss: Why does 0.99999999.... = 1?

    $$0.99999999.... = 3 \cdot 0.333333.... = 3 \cdot 1/3$$

    Let $x = 0.99999999.....$ Then $10x = 9.999999.....$ So $9x = 9.9999... - 0.999.... = 9$. So $x = 9/9 = 1$.

- *Dichotomy paradox.* We assume that a finite distance can be traversed in a finite time - for example an arrow being fired at a target. To traverse the distance from point x to point y, you must first reach the midpoint between the two, point A. Then to traverse the distance from A to y, you must reach the midpoint between A and y, which is point B. But to traverse the distance from B to y, you must reach the midpoint between B and y, point C. This continues forever, so that you can never traverse the distance between x and y. But since this distance has this feature in common with all other distances, it is not possible to traverse the distance between any two points, and so no motion is possible.
  - Discuss the paradox at tables.
  - Work through an example as a class:

    If x and y are 1 meters apart, and the arrow is moving at 1 m/s, how long does it take to get half way, then half the remainder, etc....

    $$\frac{1}{2}s + \frac{1}{4}s + \frac{1}{8}s + \cdots \ = \ 1s.$$

    Copy the proofs above to show this sums to 1.

- Idea: if two real numbers are "infinitesimally close," they are the same—if their difference is smaller than any positive number you can give, they are the same. How would that help us argue that 0.99999999.... = 1 or that $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$?
- Handout on proof-without-words on geometric series. Explain the sums coming from the square and the equilateral triangle. Then have students work through the questions on the other side.

Notes

Students who are having difficulty understanding why .99999...=1 might benefit from being reminded of other (more familiar) examples of multiple representations of two numbers such as $1/4 = .25$

References and resources

Math Explorer's Club Spring 2017 and Fall 2019: Infinity and Paradoxes

Worksheet: Geometric Series Proof Without Words

Visual Proofs of Finite Series

Follow-on activities

Hilbert Hotel

Koch Snowflake

Thompson's Lamp