Fabian Stedman: The First Group Theorist?

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1. INTRODUCTION. Fabian Stedman, a son of the Reverend Francis Stedman, vicar of Yarkhill, Herefordshire, England, was baptized there on December 7, 1640. At age fifteen he was apprenticed to the Master Printer Daniel Pakeman in London. In London he joined the Scholars of Cheapside, a bell-ringing society, serving as its Treasurer in 1662. The following year Stedman became a Freeman of the Stationers Company. In 1664 he joined the Society of Colledg Youths, which had been founded in 1637; renamed the Ancient Society of College Youths in the nineteenth century, this bell-ringing society is still active today. There is some evidence that Stedman moved from London to Cambridge in 1664 (see **[3]** and **[5]**, which are the sources for much of this background information), and he might have been working as a printer there and also serving as parish clerk of St. Bene't's. The early 11th-century Saxon tower of St. Benedict's Church is the oldest surviving building in Cambridgeshire.

In 1677 Stedman became Steward of College Youths; five years later he was Master of the Society. Returning to (or staying in) London, he changed profession, becoming a clerk in the office of Audit of Excise. He died in 1713, and was buried at St. Andrew Undershaft on November 16.

Fabian Stedman's claim to fame as at least one of the "fathers of bell ringing" seems beyond doubt; his contributions to the first two books on change ringing, *Tintinnalogia* (1668) **[2]** and *Campanalogia* (1677) **[7]**, will be summarized shortly. What is less well known, and what has occasioned this article, is the group theory latent in his writings and in his compositions—a full century before Lagrange wrote "Reflexions" (1770).

2. CHANGE RINGING. In England church bells are rung not in melody, but in permutations (changes). To a limited extent, this practice has spread to Australia, to Canada, and to the United States. The increase in control facilitated by the mounting of each bell on a circular wheel allowed the inception of change ringing in about 1610. Early forms of change ringing involved one row (one ordering of the bells, denoted by $1, 2, \ldots, n$; here $n = 6$), such as *rounds* (123456), *queens* (135246), or *tittums* (142536) to be rung repeatedly until the conductor (one of the ringers) called for a change; these are known as *call changes*. Due to mechanical considerations arising from the manner of mounting the bells, if the rows are changed constantly then no bell can readily change its order of striking by more than one position. Thus each *change* (a transition from one row to the next) involves one or more disjoint pairs of adjacent bells swapping over. At first *plain changes*, involving one pair only at each step, were in vogue. The four rows in Figure 1a illustrate three successive plain changes on six bells, commencing with rounds. Soon, *cross changes*, allowing more than one swapping pair, replaced plain changes, continuing to the present day. As we will see, Stedman was instrumental in effecting this

123456	
213456	123456
214356	214365
214365	
(a)	(b)

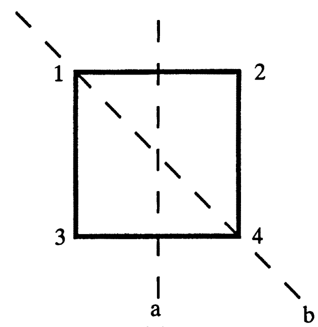
Figure 1. Three Plain Changes and One Cross Change.

transition. In Figure 1b we see how to get from rounds to the last row in Figure 1a by using one cross change, instead of three plain changes.

In about 1621, cross and plain changes were alternated to produce the plain lead on four bells, as shown in Figure 2a. From a modern viewpoint, if we let $a = (12)(34)$ denote the cross change that swaps both the first two and the last two bells and $b = (23)$ the plain change that swaps the middle pair, and if we note that reflections a and b generate the dihedral group D_4 (as the group of symmetries of a square labelled as in Figure 2b), then we see that the eight rows of Figure 2a coincide with the elements of D_4. This lead (it could also be called the *hunting group* **[12]**, as the *treble*—bell 1—is *plain hunting* in this group of rows) is described by the identity word $(ab)^4 = e$ in the symmetric group S_4. In change ringing, every *touch* (on n bells, say) begins and ends with rounds, and thus is described, in modern terms, by an identity word in S_n, where each letter of the word is an involution in S_n consisting of disjoint pairs of adjacent interchanges.

1234
2143
2413
4231
4321
3412
3142
1324
1234

(a)



(b)

Figure 2. The Plain Lead on Four Bells.

3. THE TWO MAIN SOURCES. *Tintinnalogia* **[2]** was published in 1668, written "By a Lover of that ART", and "printed by W. G. for Fabian Stedman, at his shop in St. Dunstans Churchyard in Fleetstreet". It is thought **[3]** that "W. G." stands for the publisher W. Godbid, that Stedman helped to arrange the printing and perhaps helped to supply material for the book, but that the actual author was Richard Duckworth. Duckworth was a fellow of Brasenose College, Oxford and later rector of St. Martin's, Carfax, Oxford. Fabian Stedman's father also was a member of Brasenose College and Fabian's older brother Francis was contemporaneous with Duckworth at Oxford. This might explain the connection between Fabian Stedman of London/Cambridge and Richard Duckworth of Oxford.

At any rate, in *Tintinnalogia* its author describes first plain changes and then, tentatively, cross changes, extending Figure 2a to its full extent, 24 rows now known as *Plain Bob Minimus* (discussed below), and goes on to discuss peals on five and six bells. It is notable that leads of various peals are written out in full, with no omitted rows. In 1671 a Second Edition of *Tintinnalogia* was printed ("for F. S."); only one copy of the Second Edition survives, at the Bodleian Library, Oxford. This was not an updating of the first edition, but a reprinting.

The updating occurred in 1677, with the publication of *Campanalogia* **[7]**, printed "by W. Godbid, for F. S.", and it was substantial. The Epistle Dedicatory for this work, to the "Society of COLLEDG YOUTHS", is signed "A constant Well-wisher to the Prosperity (though an unworthy member) of your Society, F. S." As the College Youths' name book for this period shows only one name with initials F. S.—Fabian Stedman—(see **[4]**), it seems safe to attribute authorship of *Campanalogia* to him. At the outset (page 2) Stedman says

> Although the practick part of *Ringing* is chiefly the subject of this Discourse, yet first I will speak something of the Art of *Changes*, its Invention being Mathematical, and produceth incredible effects, as hereafter will appear.

In the Epistle Dedicatory, Stedman had referred to the plain lead on five bells (ten rows, generated from rounds by $(ab)^5 = e$, where $a = (12)(34)$ and $b = (23)(45)$, analogous to Figure 2(a)) as follows:

> ... it was thought impossible that double changes on five bells could be made to extend further than *ten*...

The blockage was evidently caused by the awareness that the two changes generated a closed system, what we now call the dihedral group D_5, and was relieved by the discovery that one closed system can be enlarged to another by the addition of new elements (in this case adding first $c = (34)$ to produce the *plain course* $[(ab)^4ac]^4 = e$ and then the bob $d = (45)$ to ring all of Plain Bob Doubles: $[((ab)^4ac)^3(ab)^4ad]^3 = e$, as on page 104 of *Campanalogia*—except that Stedman calls the bob an *extream*, and refers to the composition as *Old Doubles*).

After giving lengthy instruction on factorials and discussing the "Practice of Ringing" and plain changes, Stedman describes a number of cross peals coinciding with those in *Tintinnalogia*, with the innovation that only the first two leads are written out in full (a *lead* is a block of rows, such as the first eight in Figure 2a, from one *treble lead*—bell 1 in the first position—to the next); subsequent leads were represented by only their first and last row (both treble leads), as the rows between can be reconstructed from the pattern of the first leads given. The crucial point here is that a subset of rows is being represented by two of its elements. As the last row of a lead can be reconstructed from the first row, the last row is also superfluous to list, but Stedman continued to do so in order to make more readily apparent the change used to get to the first row of the next lead.

The second half of *Campanalogia* contains a large number of new methods, on five, six, seven, and eight bells, including fifty-three of Stedman's own compositions under the heading "London Peals". Other venues represented are Nottingham, Oxford, and Cambridge. Included among the fifty-three London peals is "Stedman's Principle", now known as "Stedman Doubles". Extendable to any odd number of bells (and even numbers as well, by having the tenor (bell n) ring last (*in cover*) in every row), this composition is one of the most popular to this day.

4. IMPLICIT ELEMENTS OF GROUP THEORY. To make the point that group theory is latent in change ringing to a substantial degree, we next analyze two

1234	1342	1423
2143	3124	4132
2413	3214	4312
4231	2341	3421
4321	2431	3241
3412	4213	2314
3142	4123	2134
1324	1432	1243
		1234

Figure 3. Plain Bob Minimus.

compositions—Plain Bob Minimus and Stedman Doubles—in some detail. We then discuss the extent to which Fabian Stedman was aware of these connections.

In Figure 3 we list all twenty four rows of Plain Bob Minimus in three columns; each column gives one lead of the full extent. At the end of the third column, we show the required return to rounds. As for all extents (which ring the full $n!$, on n bells), it is crucial that no other row is repeated.

As before, let $a = (12)(34)$ and $b = (23)$ describe possible changes from one row to the next. Add $c = (34)$ and let e denote the identity element of S_4. As before, a and b generate the dihedral subgroup D_4 of S_4, and the rows of the first lead correspond to $D_4 = \{e, a, ab, aba, (ab)^2, (ab)^2a, (ab)^3, (ab)^3a\}$. (From a modern point of view—see **[8–11]**, for example—$ab = (12)(34)(23) = (1243)$, composing right to left, which we interpret as ringing in position 1 bell 2, in position 2 bell 4, in position 4 bell 3, and in position 3 bell 1; that is, the row 2 4 1 3. This extends in a natural manner to a full correspondence between rows and permutations.) If we follow row $(ab)^3a$ by change b we would regain rounds prematurely, since $(ab)^4 = e$. Thus we employ change $c = (34)$ for the first time, obtaining the second column $\{w, wa, wab, waba, w(ab)^2, w(ab)^2a, w(ab)^3, w(ab)^3a\}$, where $w = (ab)^3ac$. But this is just the left coset wD_4! Using c a second time, we get the third column as the final left coset w^2D_4, and a third and final use of c returns us to rounds.

A composition such as Plain Bob Minimus is required to satisfy six conditions, which we now list, together with an algebraic verification for each one.

(i) The extent must begin and end with rounds. (This follows from $[(ab)^3ac]^3 = e$.)
(ii) No other row is repeated. (The coset decomposition guarantees this.)
(iii) From one row to the next, no bell moves more than one position. (This is forced by our choice of $a = (12)(34)$, $b = (23)$, and $c = (34)$.)
(iv) No bell rests in the same place for more than two successive rows. (The alternation of $a = (12)(34)$ moves every bell appropriately.)
(v) The working bells (here, all but the treble) should all do the same work. (This is guaranteed by $w = (234)$, so that what bell 2 does in the first lead, bell 3 does in the second and bell 4 does in the third, etc.)
(vi) Each lead should be palindromic in its changes. (Examine $(ab)^3a$.)

These *axioms* for a *method*, as it is called, are not formally combined by Stedman in *Campanalogia*. His "Obser. 4" (pp. 38–39) corresponds to (iii); his other "observations" apply to performance, rather than composition. Axioms (ii) and (v) appear in *Campanalogia* on pages 3 and 84 respectively. Axiom (i) is implicit throughout; (iv) and (iv) are more commonly employed in modern times. Modern ringers have more formally set forth these and other requirements (see, for example, [**1**, p. 8] and [**16**]), in what we now recognize as an axiomatic approach.

In summary, the decomposition of S_4 into left cosets of D_4 shown by Figure 3 precisely describes Plain Bob Minimus. (In *Campanalogia* (p. 96) Stedman lists the 24 rows in one column, but he uses letters to show where each block of eight rows (i.e., each coset) changes into the next.) But there are at least four other coset decompositions of interest here. To describe these, it is helpful to think of Figure 3 as an 8×3 matrix, whose entries are the rows of Plain Bob Minimus. In what comes immediately below, the term *row** will refer to a row of this matrix, which consists of three rows of the composition.

(1) The rows* of the matrix are the right cosets of the subgroup $\{e, w, w^2\}$ of S_4.

(2) Rows* 1 and 8 give the subgroup $(S_4)_1 \cong S_3$ of S_4; the set of all treble leads is just the stabilizer in S_4 of (bell) 1. The other right cosets consist of rows* 2 and 7, rows* 3 and 6, and rows* 4 and 5. Note that the row* numbers of each coset are symmetrical about the half lead, and that each half lead constitutes a right transversal of $(S_4)_1$ in S_4 (each of the right cosets is represented exactly once, in each half lead). Note also that each element of the ith coset fixes bell 1 in the ith position, $i = 1, 2, 3, 4$ (in accordance with the plain hunt). These follow from the fact that $(ab)^3a$ is a palindrome (condition (vi)) and are useful in "proving" extents, as we do for Plain Bob Minimus below.

(3) Rows* 1, 2, 5, and 6 give the subgroup A_4 of S_4, consisting of all the even (*in- course*) rows. The other four rows*, which form the other (right or left) coset of A_4 in S_4, consist of all the odd (*out-of-course*) rows.

I believe that Stedman made use of all these decompositions, although of course he lacked the modern terminology for them. Here is one decomposition that was probably not used by Stedman.

(4) In Figure 2 of [**8**], a Cayley color graph $C_\Delta(S_4)$, with $\Delta = \{a, b, c\}$, is shown imbedded in the projective plane with 4-fold symmetry, as generated by $ab =$ (1243). The six right cosets of the corresponding subgroup allow an even simpler depiction of Plain Bob Minimus, as a Schreier coset graph. This idea has been exploited to great advantage in [**9**], [**10**], and [**11**].

Now we turn our attention to Stedman Doubles. The plain course (consisting of 60 rows) is given in Figure 4. For convenience the presentation differs slightly from that given by Stedman in *Campanalogia* (pp. 129–132); both differ from that used by ringers today. But the connection with group theory is unaffected. Letting $a = (12)(45)$, $b = (23)(45)$, and $c = (12)(34)$, we can describe the plain course by $w^5 = e$, where $w = (ab)^2ac(ba)^2bc = (13452)$. The sequence (word) ababa of changes gives a *slow six*; the sequence babab, which is used in alternation, gives a *quick six*. Each yields all the permutations on the front three bells, and thus a subgroup isomorphic to S_3, if we start with rounds. We introduce $c = (12)(34)$, called by Stedman a *parting change*, to link successive sixes—by exchanging one of the back two bells with one of the front three. Stedman notes: "Bt this method the peal will go sixty changes, and to carry it farther *extremes* must be made." We check that, with all changes (a, b, and c) even, the largest subgroup of S_5 we can generate is A_5. Figure 4 displays A_5 decomposed into ten left cosets of the subgroup isomorphic to S_3 given by the first six rows. Or, if we focus on the rows* of the 12×5 matrix, we find twelve right cosets of the subgroup generated by

12345	31452	43521	54213	25134
21354	13425	34512	45231	52143
23145	14352	35421	42513	51234
32154	41325	53412	24531	15243
31245	43152	54321	25413	12534
13254	34125	45312	52431	21543
31524	43215	54132	25341	12453
35142	42351	51423	23514	14235
53124	24315	15432	32541	41253
51342	23451	14523	35214	42135
15324	32415	41532	53241	24153
13542	34251	45123	52314	21435
				12345

Figure 4. Plain Course of Stedman Doubles.

$w = (13452)$, represented by the five rows of Stedman Doubles in the first row* of the matrix. All 60 rows are in-course. (We note in passing that Stedman Doubles, as a *principle*, has no hunt bell (bell 1 was plain hunting in Plain Bob Minimus, where bells 2, 3, and 4 were working alike); now all five bells are working alike, as forced by w being a 5-cycle. With this understanding, all "axioms" (i)–(iv) hold for this principle, just as they did for the method Plain Bob Minimus. However, the subgroup $(S_5)_1$ of S_5 plays no role here, and we have no coset decomposition to match (2) for Plain Bob Minimus.) To get the remaining 60 (out-of-course, i.e., odd) rows of Stedman Doubles, we replace the tenth use of the parting change c, which brought us back to rounds after 60 rows, by an appropriate change, say $d = (34)$—called by Stedman an *extream*, known now as a *single*. (The single (12) would also work here.) This throws us into the other coset of A_5 in S_5, and we get all of Stedman Doubles as $\{[(ab)^2ac(ba)^2bc]^4(ab)^2ac(ba)^2bd\}^2 = e$. Stedman's arrangement in *Campanalogia* (p. 131) clearly reflects this division of S_5 into cosets of A_5. However, he does not emphasize the division of A_5 (and the other coset) into cosets of S_3, as later change ringers have done.

5. STEDMAN A GROUP THEORIST? Certainly a knowledge of group theory helps us analyze (and compose!) pieces of change ringing music such as Plain Bob Minimus and Stedman Doubles for their structure and properties. Group theory, a mathematical discipline developed in the late eighteenth and nineteenth centuries, was of course not available to Fabian Stedman in 1677 and before, when he composed the music he recorded in *Campanalogia*. But was he in reality functioning as a very early group theorist in composing and verifying his compositions?

Of all the six requirements for change ringing given above, (ii) is by far the most difficult to verify: no row is repeated; each appears exactly once (except for rounds, which appears first and last, but nowhere else). The verification of (ii) is called *proof* by ringers, and if two rows that should differ in fact agree, then *falsity* has been established. Not all change ringing compositions correspond to left coset decompositions by a subgroup consisting of the rows of the first lead. But it is interesting to note that the two most popular methods (Plain Bob, on any even number of bells; Grandsire, on any odd number) and the most popular principle (Stedman, on any odd number) all do (except that the bob leads for Grandsire are not quite cosets). As mathematicians, we know that two left cosets are either disjoint or identical, and that no one coset has any internal falsity. Thus if we just note that the three even (in-course) treble leads for Plain Bob Minimus (rows 1234, 1342, and 1423) are distinct, we have proved the composition.

But how might a ringer without explicit knowledge of group theory prove a composition like Plain Bob Minimus? Suppose, for example, the row 2341 appears twice. Since each lead is true, this must be in different leads. Since following 2341, which is out-of-course, by $a = (12)(34)$, $b = (23)$, and then a again gets us to an in-course treble lead (since row 2341 must be in either row* 4 or row* 5, and the palindromic condition (vi) guarantees that moving up by aba (row* 4) or down by aba (row* 5) will reach a treble lead head or a treble lead end respectively), the two leads containing 2341 must be headed by the same in-course treble lead. But a quick inspection shows that 1234, 1342, and 1423 are distinct. Thus 2341 *cannot* appear twice. A similar analysis applies to any other row. In summary, we need compare only one representative from each lead, even if we don't know that that lead is going to be called a coset more than a century later.

Did Stedman reason in this way? Here is what he said, on pages 94 and 95 of *Campanalogia*; for *whole hunt* read "treble;" for *course* read "lead;" for *peal* read "extent" (the full $n!$), for *pricking* read "writing."

> ... every note in a *cross-peal* must of necessity lie as many times in one place, as the rest of the notes are capable of making changes;

(In the 20th century, we would write $|(S_n)_i| = |S_{n-1}|$.)

> and also that two or more of the notes must jointly lie in the same places as many times, as the remaining number are also capable of making changes:

($|(S_n)_{i,j}| = |S_{n-2}|$, etc.)

> this being a certain touchstone to prove all *cross-peals* after they are prickt, and must be held as a principle upon which to ground such methods of pricking, that the course of all the notes may demonstrably tend to produce those effects. And from hence it is, that the whole *hunt* immediately derives the manner of its uniform motion through the courses of each peal. And the changes in every course are as so many guides to conduct the rest of the notes in such sort, that they may be prepared to lie at the last change of the course in apt places for each succeeding course to receive them, and to perform the like. Now as the changes in all the courses of a peal are made alike, ... so in the composing of *cross-peals*, by pricking of one course may soon be discovered, whether a compleat peal will from these arise.

In connection with Stedman Doubles, the composer clearly seemed to know that following 60 true in-course rows (the first 59 "changes are all double", as he said on page 129 of [7]) by an appropriate "extream" would produce 60 true out-of-course rows, that $wx = wy$ means that also $x = y$. And, he seemed aware that the parity (in or out of course) of a row is dependent only on the row itself, not on its position in the composition. The modern theorem is that the parity (even or odd) of the number of transpositions into which a permutation can be decomposed is constant.

If we extend Stedman's principle on five bells, Stedman Doubles, to seven bells, we get Stedman Triples. Letting $a = (12)(45)(67)$, $b = (23)(45)(67)$, and $c = (12)(34)(56)$, we obtain the plain course $w^7 = e$, where $w = (ab)^2ac(ba)^2bc = (1374562)$; this plain course consists of 84 of the 5040 rows. To expand this touch to a full extent, bob $d = (12)(34)(67)$ and single $f = (12)(34)$ have been used effectively, replacing c in either or both of its occurrences in certain subwords w, in order to get beyond the plain course, even as far as the full extent. Until recently, the most famous unsolved problem in bell ringing was the following: Is it possible to ring the full extent of Stedman Triples using only a, b, c, and d? In late 1994, Colin Wyld achieved such a composition **[14]**, using, out of 840 positions where a bob might be called, 705 bobs **[13]**. Then, in early 1995, Andrew Johnson and Philip Saddleton also composed an extent of Stedman Triples using common bobs only (no singles), and one week later their composition was successfully rung by a Cambridge University Guild band, being called (579 bobs) at the first attempt by Philip Agg, at St. John's Waterloo Road **[15]**. See also Saddleton **[6]**. Thus a centuries-old (mathematical!) problem derived from the work of Fabian Stedman has finally been settled. The solution corresponds to a hamiltonian circuit in the Cayley graph for the symmetric group S_7, as generated by the involutions a, b, c, and d above, incorporating slow and quick sixes in alternation, linked by generators c and d.

I have not tried to make the case that Fabian Stedman was using group theory explicitly, but rather that group-theoretical ideas were implicit in his writings and compositions. These ideas, as we have seen, include closed systems, axiomatic systems, coset decomposition (including the ideas of coset representative and disjointness), even and odd permutations, factorials, and stabilizers in permutation groups. We should remember that those usually thought of as the first group theorists (Lagrange, Ruffini, Cauchy, Abel, and Galois) also were operating implicitly in the context of permutation groups, many decades before the definition of an abstract group as a set with a binary operation satisfying certain axioms.

REFERENCES

1. William T. Cook, Introduction to the art of change ringing, *Change Ringing*: *The History of an English Art*, J. Sanderson, ed., The Central Council of Church Bell Ringers, Cambridge, 1987, 1–17.
2. Richard Duckworth, *Tintinnalogia*, W. G., London, 1668.
3. John C. Eisel, Tintinnalogia, *Change Ringing*: *The History of an English Art*, J. Sanderson, ed., The Central Council of Church Bell Ringers, Cambridge, 1987, 50–58.
4. J. C. Eisel, A question of authorship, *The Ringing World*, Guildford, January 29, 1993, 93.
5. Ron Johnston, *Bell-ringing*, Viking, Harmondsworth, 1986.
6. Philip Saddleton, Key unexpectedly simple, *The Ringing World*, Guildford, March 3, 1995, 216.
7. Fabian Stedman, *Campanalogia*, W. Godbid, London, 1677.
8. A. T. White, Ringing the changes, *Math. Proc. Camb. Philos. Soc.* **94** (1983), 203–215.
9. A. T. White, Ringing the cosets, *Amer. Math. Monthly* **94** (1987), 721–746.
10. A. T. White, Ringing the cosets II, *Math. Proc. Camb. Philos. Soc.* **105** (1989), 53–65.
11. A. T. White, Treble dodging minor methods: ringing the cosets, on six bells, *Discrete Math.* **122** (1993), 307–323.
12. Arthur White and Robin Wilson, The hunting group, *Mathematical Gazette* **79** (1995), 5–16.
13. Colin Wyld, A 250-year old problem solved, *The Ringing World*, Guildford, February 24, 1995, 197–198.
14. "Around and About", *The Ringing World*, Guildford, January 27, 1995, 104.
15. "Editorial", *The Ringing World*, Guildford, February 3, 1995, 107.
16. *The Council's Decisions*, The Central Council of Church Bell Ringers, Cambridge, 1989.

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