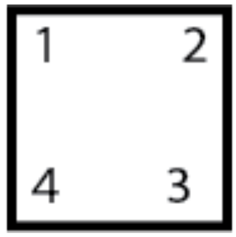
Flipping a square mattress: a non-commutative group

We are going to look at the group of symmetries of a square mattress. In place of a real mattress, which would be cumbersome to experiment with, get or make a square piece of paper. Label the four corners 1, 2, 3, and 4:



1. A square mattress has eight symmetries. Four are rotations of the mattress without flipping it over: just turn it 0° (i.e. do nothing), 90°, 180°, or 270°. The other are four ways of flipping it over—describe these.

2. A convenient way of keeping track of symmetries of the mattress is by recording where the corners go. For example a 90°-clockwise rotation moves corner 1 to corner 2, corner 2 to corner 3, corner 3 to corner 4, and corner 4 to corner 1. Let's write this more concisely as a list 2 3 4 1, recording the destinations of corners 1, 2, 3, and 4, in order. Describe the remaining seven symmetries in the same way.

3. List 1, 2, 3 and 4 in some order that does not arise in the Step 2.

4. Find a pair of mattress symmetries a and b that do not commute—that is, performing a then b is different from b then a.

Permutations and change ringing

5. A permutation of 1 2 3 is a way of writing them in a different order—for example, 3 1 2. There are six possibilities. Why? What are they?

6. Explain why there are exactly 24 permutations of the list of four numbers 1 2 3 4.

7. How many permutations of the list of n numbers $1\ 2\ 3 \cdots\ n$ are there?

Imagine a bell tower with n bells, labelled $1, 2, \ldots, n$ from highest note to lowest. A *change* (or *peal*) is the bellringing term for what we called a permutation above—it's an order in which to ring the bells. For example, 3 2 1 4 is the change of four bells where we ring bell 3, then bell 2, then bell 1 and finally bell 4. To *ring the changes* means to ring a sequence of changes, whilst obeying three rules:

(i). From one change to the next, any bell can move by at most one position in its order of ringing.

(ii). The sequence starts and ends with the change $1\ 2\ \cdots\ n$.

(iii). Except for $1\ 2\ \cdots\ n$ as the first and last changes, no change is repeated.

If, additionally, every possible change is included, it's called an *extent*. For example, you can check that this is an extent with three bells:

$$\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \\ 3 & 1 & 2 \\ 1 & 3 & 2 \\ 1 & 2 & 3. \end{array}$$

8. It takes about 2 seconds to ring one change. Estimate how long it would take to perform an extent with 3 bells. What about with 4 bells? 5 bells? 6 bells? 7 bells? 8 bells? 9 bells? 10 bells? 11 bells? And finally 12 bells?

9. With one ringer per bell, probably the largest extent that is humanly possible is an extent on eight bells. This has apparently been achieved only once, at the Loughborough Bell Foundry in 1963. The ringing began at 6.52am on July 27, and finished at 12.50am on July 28, after 40,320 changes and 17 hours 58 minutes of continuous ringing.

   (a) Check an extent on eight bells indeed has 40,320 changes.

   (b) How many seconds did they take per change?

   (c) How many complaints were made to the police?

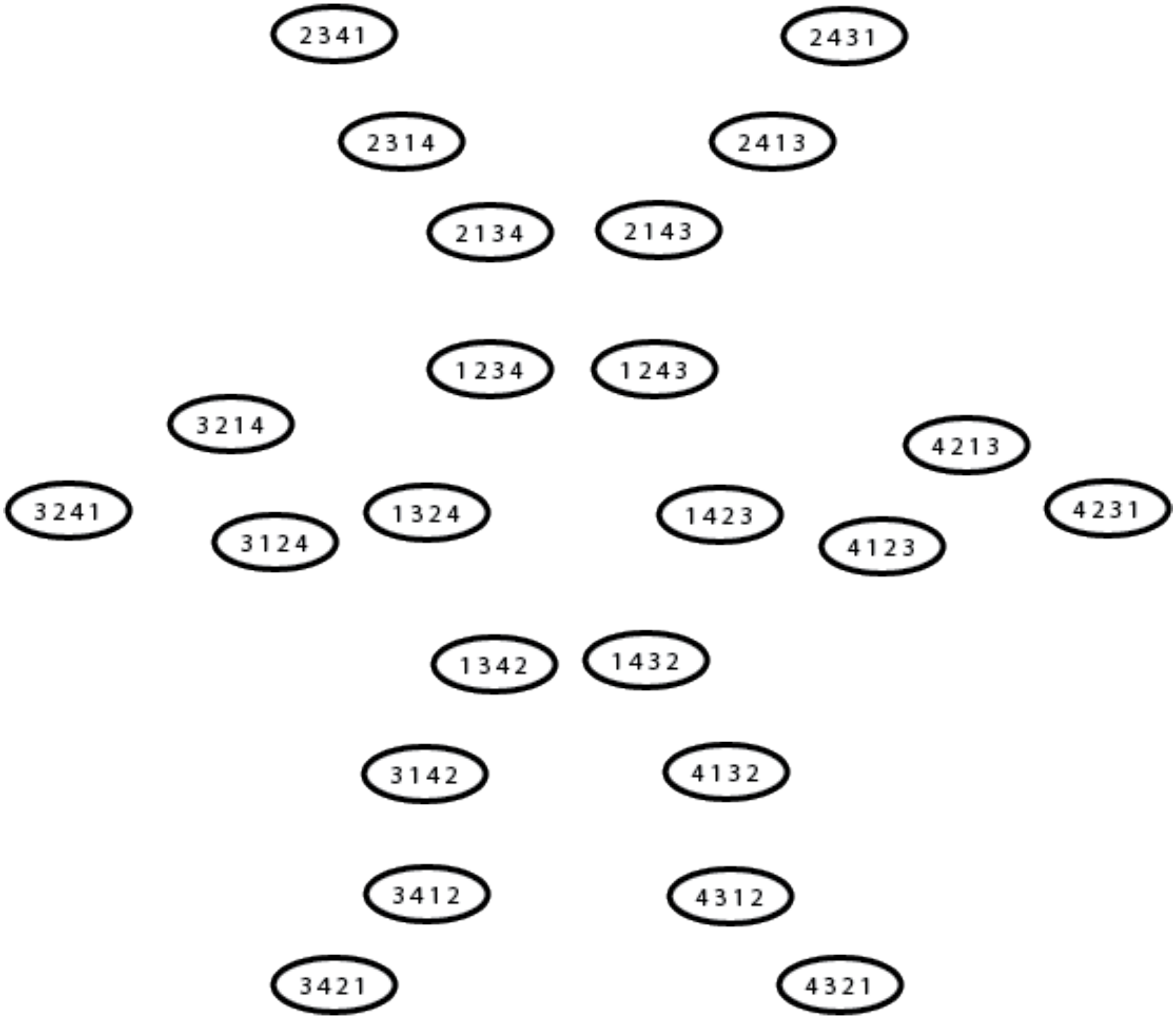
10. The figure on the next page shows all the changes on four bells. Draw a line connecting one change to another when you can move between them obeying rule (i).

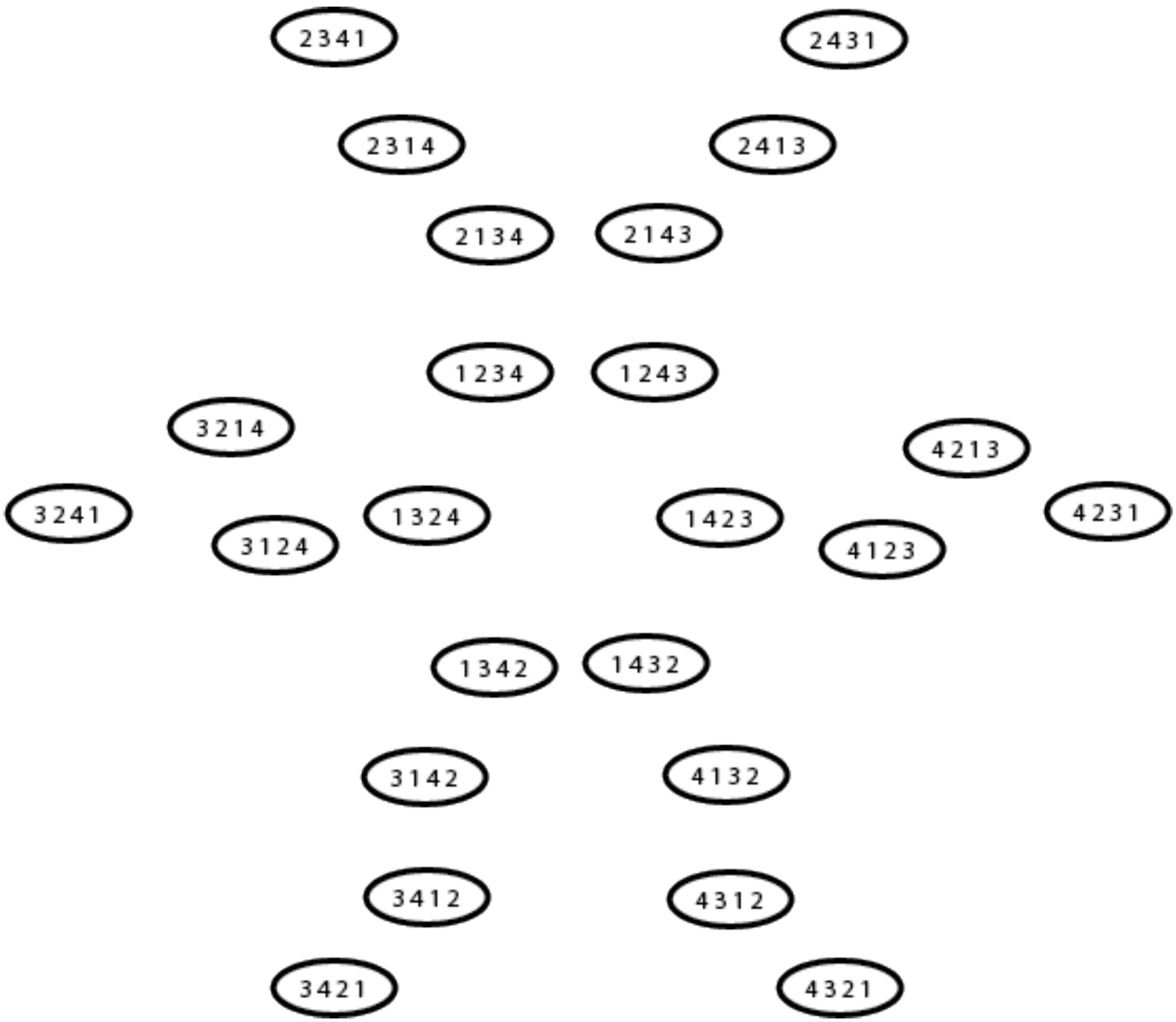
11. Use the resulting figure to find an extent with four bells. (Does this remind you of something we saw a couple of weeks ago?)

12. The eight permutations we found in Step 2 can be thought of as changes. Shade them in the figure.

The permutations of $1, 2, \ldots, n$ form a group. It's called a *symmetric group*. (What do you think the binary operation is?) What we have found in Step 12 is that the group of symmetries of a square mattress can be found within the group of permutations of $1, 2, 3, 4$.

Acknowledgement. This worksheet is based on a +plus magazine blog posts by Burkard Polster and Marty Ross.


2341
2431
2314
2413
2134
2143
1234
1243
3214
4213
3241
1324
1423
4231
3124
4123
1342
1432
3142
4132
3412
4312
3421
4321


2341
2431
2314
2413
2134
2143
1234
1243
3214
4213
3241
1324
1423
4231
3124
4123
1342
1432
3142
4132
3412
4312
3421
4321