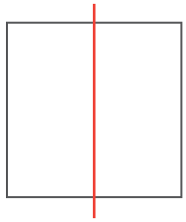
Counting Symmetries

Can you find all the symmetries of the familiar square?

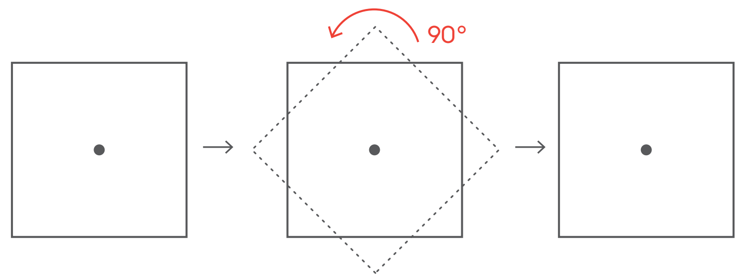
WHAT IS SYMMETRY?

Symmetries are transformations of an object that preserve its size and shape and whose result is indistinguishable from the original.

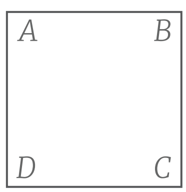


For example, a line cuts a square into two equal parts, each one the mirror image of the other. This is called **line symmetry**.

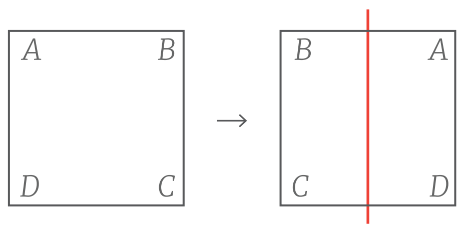
The square also has **rotational symmetry**. After rotating a square counterclockwise about its center point (the intersection of its diagonals) 90 degrees, it looks the same as before.



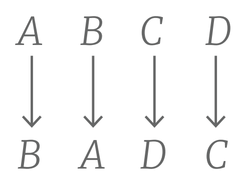
HOW MANY SYMMETRIES DOES A SQUARE HAVE?



Hint: Label the corners A, B, C, and D to specify each symmetry of the square by some arrangement of the four letters.



As an example, reflect the square across a vertical line through its center and watch where the labels go.



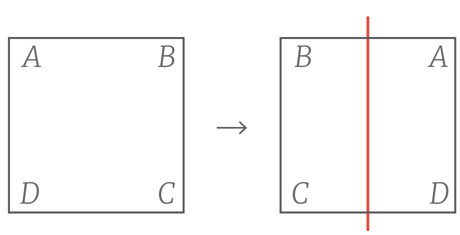
We can denote the resulting line symmetry as $BADC$.

Q **Which four-letter arrangement is an example of rotational symmetry? How many other symmetries can you find?**

HOW TO SOLVE

A logical first step is to ask how many letter arrangements are possible. Once you choose one of the **four** letters to start with, you have only **three** choices for the second. After choosing the second letter, you'll have only **two** choices for the third, and finally there will be only **one** option for the fourth and final letter. An elementary counting argument tells us there are **24 possible arrangements** of $ABCD$.

$$4 \times 3 \times 2 \times 1 = 24$$



Another simple argument tells us a square has far fewer than 24 symmetries. Suppose we know that a symmetry of the square maps A to B. Because the square's size (and distances between points) cannot change, the only option for C is to be swapped for D.

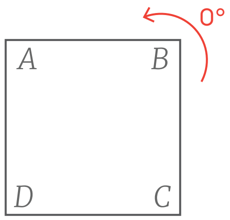
So, there are really only two things to decide: where A goes (**four** choices) and where B goes (**two** choices). This tells us there are **eight possible arrangements** of $ABCD$ that satisfy our symmetry requirements.

Can you draw them all?

$$4 \times 2 = 8$$

ANSWER KEY

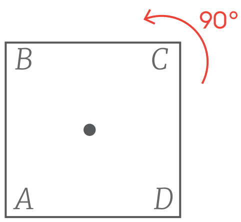
We aren't guaranteed that all eight possibilities are actual symmetries of the square.
But it's a small list, so we can check them and verify that, indeed, they all correspond to legitimate symmetries.



$A \to A$, $B \to B$, $C \to C$, $D \to D$

SYMMETRY 1

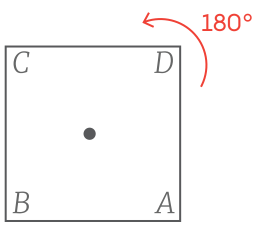
Original / Rotate 0 degrees counterclockwise



$A \to B$, $B \to C$, $C \to D$, $D \to A$

SYMMETRY 2

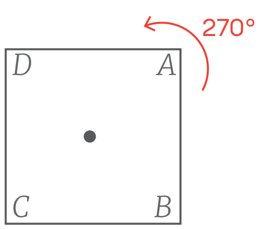
Rotate 90 degrees counterclockwise



$A \to C$, $B \to D$, $C \to A$, $D \to B$

SYMMETRY 3

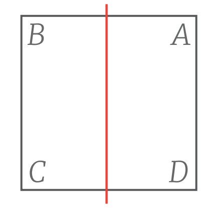
Rotate 180 degrees counterclockwise



$A \to D$, $B \to A$, $C \to B$, $D \to C$

SYMMETRY 4

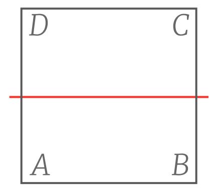
Rotate 270 degrees counterclockwise



$A \to B$, $B \to A$, $C \to D$, $D \to C$

SYMMETRY 5

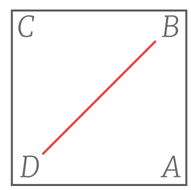
Vertical mirror



$A \to D$, $B \to C$, $C \to B$, $D \to A$

SYMMETRY 6

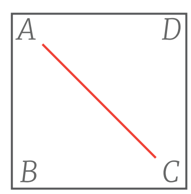
Horizontal mirror



$A \to C$, $B \to B$, $C \to A$, $D \to D$

SYMMETRY 7

Diagonal mirror



$A \to A$, $B \to D$, $C \to C$, $D \to B$

SYMMETRY 8

Diagonal mirror

Q

We've glimpsed the algebraic structure underlying the simple symmetries of a square.
Extend what you've learned to the following questions:

Find all the symmetries of (a) an equilateral triangle; (b) a regular pentagon; (c) a regular *n*-gon.

For *n* = 1, 2, 3, 4, ..., find or create an object that has exactly *n* symmetries.

Find all the symmetries of the cube.