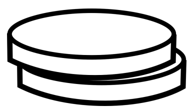
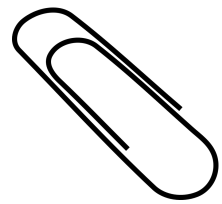
Pennies vs Paperclips

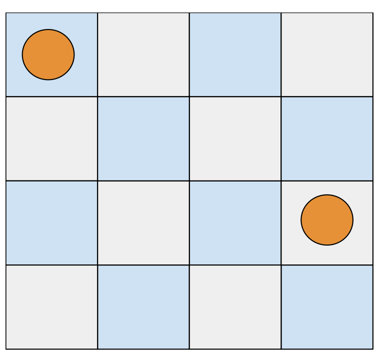
Today we will take part in a daring game, a clash of copper and steel. Today we play the game: pennies versus paperclips.





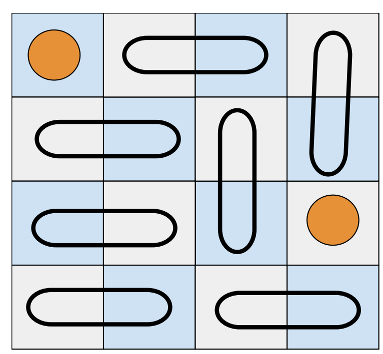
Battle begins on a $2k$ by $2m$ (where k and m are natural numbers) sized grid, colored as a chessboard. There are two players:

Player A: Player A goes first by placing two pennies on the given board one per square.



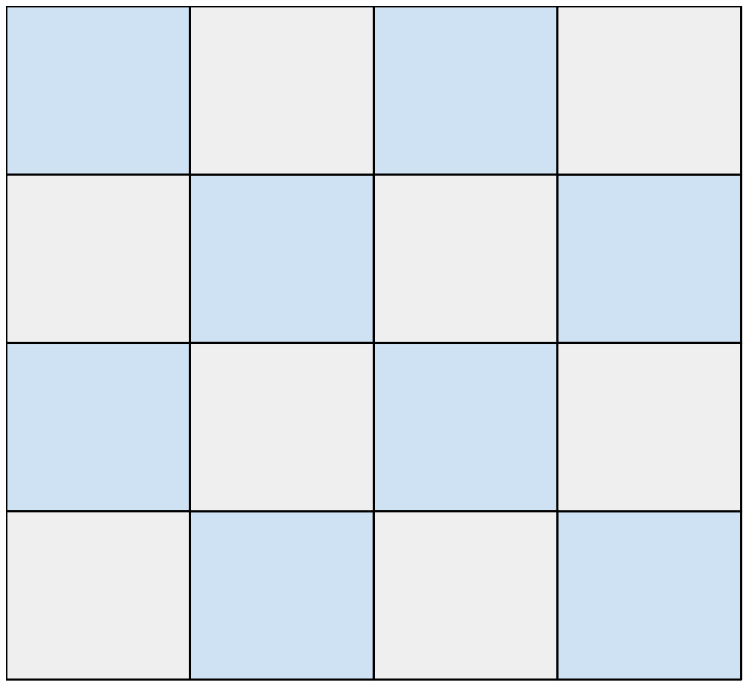
Lets assign coordinates to our board by calling the lower left box $(1, 1)$. In this given example of a game, player A placed pennies on squares $(1, 4)$ and $(4, 2)$.

Player B: Player B moves next, trying to place paperclips on the board to cover the remaining area with the caveat that each paperclip must occupy two adjacent white and grey squares.



We say that player B wins this round because they were able to cover the remaining area of the board with paperclips. If B cannot cover the board, then player A wins.

With your group, play a few games of pennies versus paperclips on the provided board. Complete the tasks below as you play.



1. Each time you play, record the results of your game, including the placement of the pennies.

Game	Penny 1	Penny 2	Winner
Ex.	(1,4)	(4,2)	Paper
1			
2			
3			
4			
5			

Game	Penny 1	Penny 2	Winner
6			
7			
8			
9			
10			
11			

2. Do you notice any patterns that give winning strategies for the players? If so, test them by playing a few more games.

3. State a conjecture which determines precisely when pennies will win based on their placement on the board.

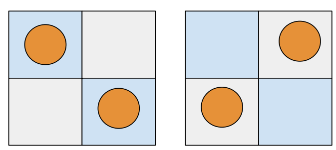
A Solution to Pennies vs Paperclips: Induction

Conjecture. *On a $2k$ by $2m$ board, paperclips win if and only if the two pennies cover different colored squares.*

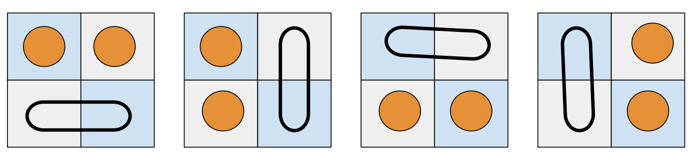
Induction is a way of proving claims that depend on dimension by proving it is true for a small case and using this small case to show it is true for an arbitrary dimension by assuming the property is true for all previous smaller cases. Due to time constraints, we will prove the following simpler claim today.

Claim. *On a $2k$ by 2 board, paperclips win if and only if the two pennies cover different colored squares.*

Proof. **Base Case: Let k=1**. We will show that on a 2 by 2 board, paperclips win if and only if the two pennies cover different colored squares. There are 6 possible cases, depending on where player A places the two pennies. Penny will win in 2 of these configurations, precisely when the 2 pennies cover squares of the same color:



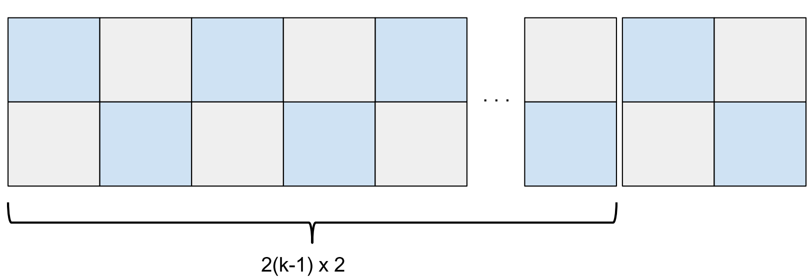
Paperclips will win with the 4 other configurations, precisely when the 2 pennies cover squares with differing color:



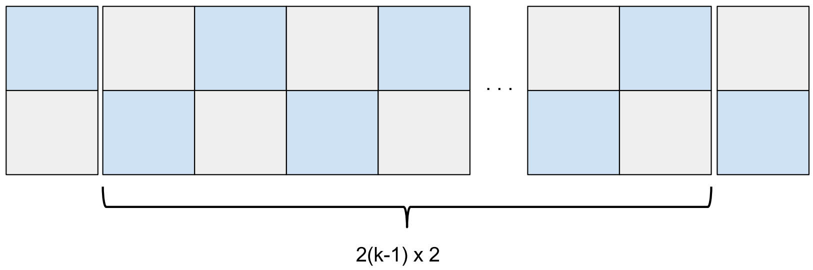
Induction step: We will assume the claim is true for dimensions 2 by $2(k-1)$. We will show that is also true for boards of dimension $2k$ by 2. There are $2k(4k-1)$ possible cases, depending on where player A places the two pennies.

First assume the pennies occupy a common $2(k-1)$ by 2 subgrid. This provides three cases:

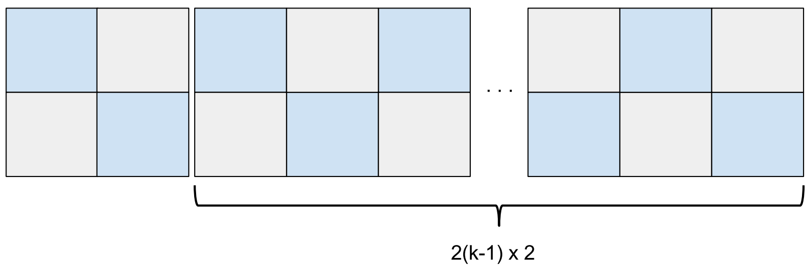
Case A.



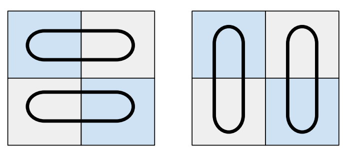
Case B.



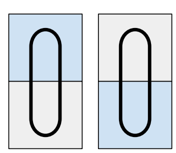
Case C.



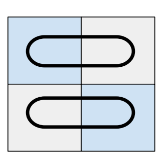
For cases A and C, notice that the only possible way for paperclips to cover the leftover 2 by 2 part of the board is by arranging their paper clips stacked vertically or horizontally:



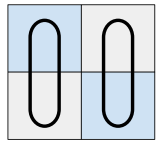
For case B, if paperclips can win on the subgrid, then paperclips can complete the board by covering the ends with paperclips as so:



If paperclips can't win on the subgrid, then they must have to use the ends to complete the board. But then either far end would appear as



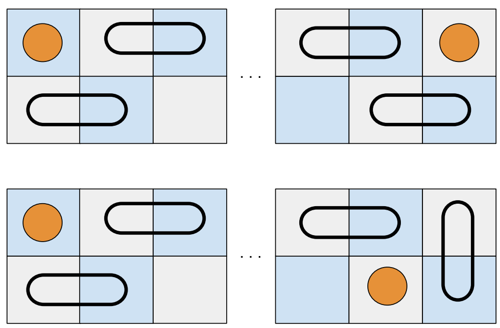
which could equivalently be covered as



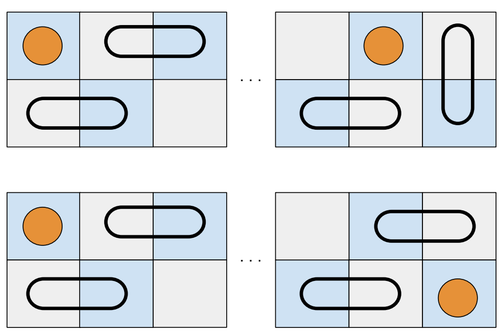
Thus to win paperclips would have to be able to win on the $2(k-1)$ by 2 board.

We will now consider the remaining cases. Without loss of generality we can assume that one penny appears in the upper left hand box. There are 4 other possible ways to arrange the second penny.

Paperclips wins if the penny is placed on a square of the opposite color:



In the final two arrangements, where the second penny is on a square of the same color, paperclips cannot cover the board because the uncovered portion of the top and bottom rows are odd length.

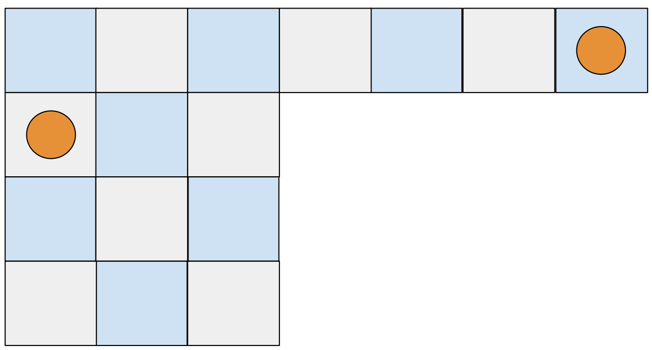


Thus on a $2k$ by 2 board, paperclips win if and only if the two pennies cover different colored squares. $\square$

Pennies vs Paperclips Redux

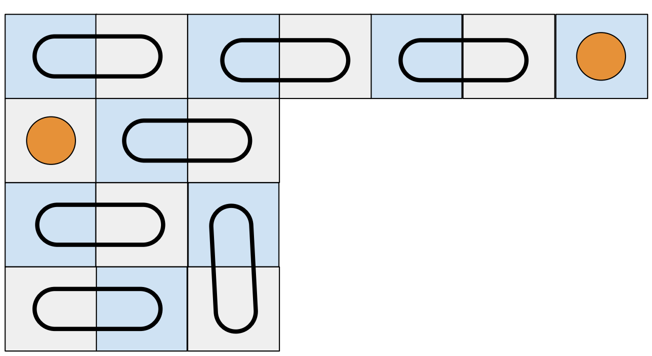
In our previous game of pennies versus paperclips, penny seemed to have a unfair advantage. Today we play again! To make things interesting, we will play on any connected configuration of $2k$ squares colored as a chessboard. Again there are two players:

Player A: Player A goes first by placing two pennies on the given board one per square.



Lets assign coordinates to our board by calling the lower left box $(1,1)$. In this given example of a game, player A placed pennies on squares $(1,3)$ and $(7,4)$.

Player B: Player B moves next, trying to place paperclips on the board to cover the remaining area with the caveat that each paperclip must occupy two adjacent white and grey squares.



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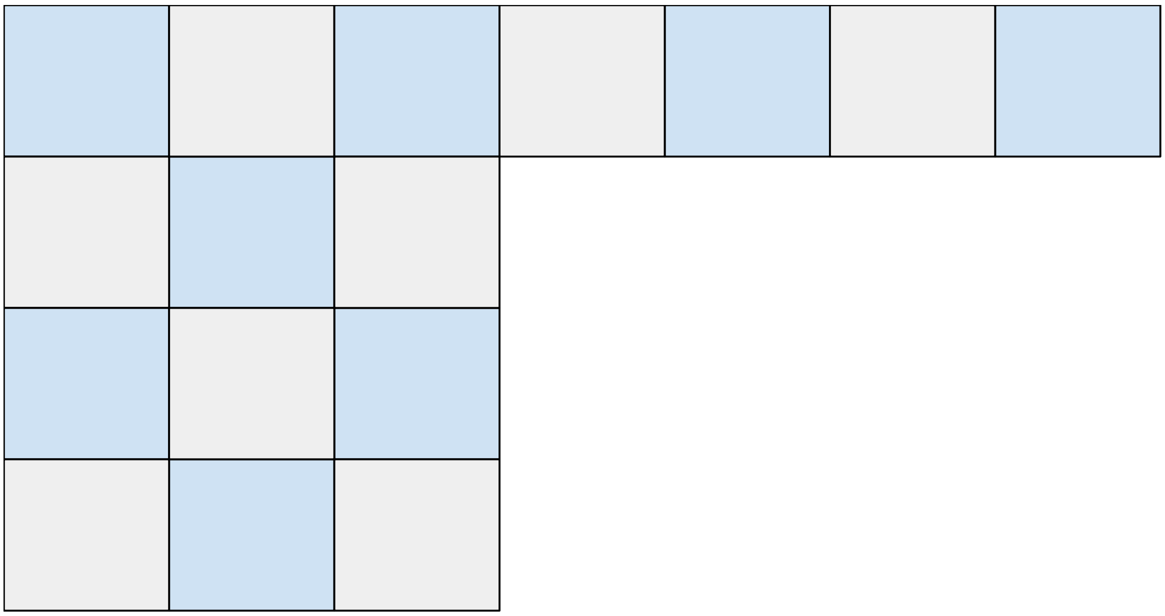

1. Each time you play, record the results of your game, including the placement of the pennies.

Game	Penny 1	Penny 2	Winner
Ex.	(1,3)	(7,4)	Paper
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2. Do you notice any patterns that give winning strategies for the players? If so, test them by playing a few more games.

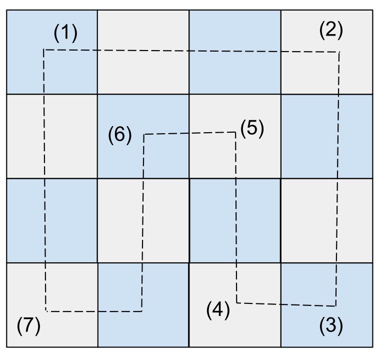
3. State a conjecture which determines precisely when pennies will win based on their placement on the board.

A Solution to Pennies vs Paperclips: Hamiltonian Circuits

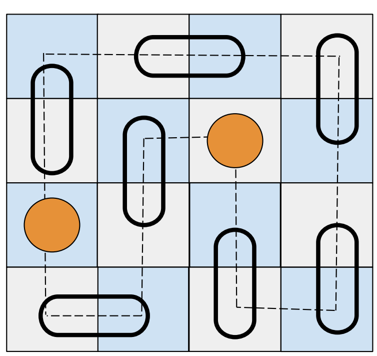
Conjecture. *On a $2k$ by $2m$ board, paperclips win if and only if the two pennies cover different colored squares.*

Proof. In the pennies and paperclip game, each paperclip must cover a square of each color. If the pennies cover 2 squares of the same color, then there will remain two squares of the same color that a paperclip cannot possibly cover.

So assume the pennies are covering two squares of opposite color. We can associate to every board a corresponding graph. Each square is a vertex. Two vertices share an edge if their corresponding squares share a side. There exists a Hamiltonian circuit in this graph. For example, we can start in the upper left hand box (1). We travel right until we can't anymore (2), then turn down until we hit the bottom of the board (3). We then turn left and head up the next column (4) until we get to the last unvisited vertex in this column (5). We turn left and head back down the next column (6), repeating this pattern until we've reached the lower left hand box (7). We head straight up completing the circuit.



If the pennies cover vertices of opposing color, then it has divided the circuit into a path or paths of even distance. Thus these paths can be covered by paperclips, completely covering the remaining vertices of the board.



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