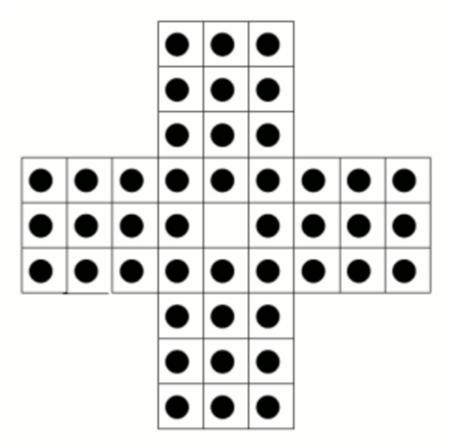
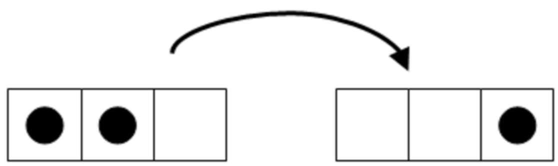
Lonely pursuits with groups

Solitaire, as its name suggests, is a game for one person. The board is in the shape of a plus sign, where each branch of the plus sign, as well as the centre, is composed of 9 holes arranged in a 3 × 3 square. At the beginning of the game, there is a counter, often a marble, in every hole except for the centre one.



That's *(4 × 9) + (1 × 8) = 44* marbles in all. A *move* in the game consists of picking up a marble, and jumping it horizontally or vertically (but *not* diagonally) over a single marble into a vacant hole, removing the marble that was jumped over.



A marble cannot jump over more than one marble at a time, and cannot be jumped into a hole that already has a marble in it. The game is won by finishing with a single marble left on the board, in the central hole — that's the one that starts out as the only empty space.

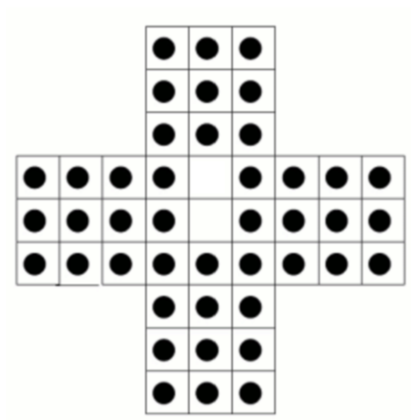
Let's imagine that we just spent a frustrating afternoon, trying but failing to win at a game of Solitaire. We might try to make a new, easier, version of the game for ourselves, by saying that we'll count it as a win if we finish with a single marble *anywhere* on the board, rather than a single marble in the middle hole. The question is: would that really make the game easier?

Solitaire and the Klein 4-group

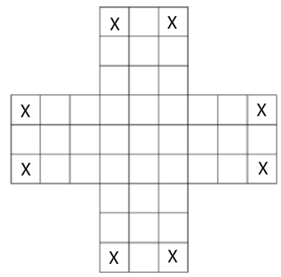
As we've seen before, the Klein-4 group has four elements, which today we will call e, f, g and h. Its multiplication table is:

	e	f	g	h
e	e	f	g	h
f	f	e	h	g
g	g	h	e	f
h	h	g	f	e

1. You have been given a copy of a solitaire board. Label each hole of the board with one of the elements f, g and h of the Klein 4-group, in such a way that
   - the middle square is labelled h,
   - for any three consecutive cells that lie in a horizontal or a vertical line, the cells are labelled f, g and h in some order.
2. Define the *total value* of the board to be the element of our group that we get when we multiply together the labels of all of the holes that have marbles in them. A worry about this definition might be that the order in which we multiply them together might matter. Why is this, in fact, not a problem?
3. What is the total value in the finishing position if you've won the game? Try putting down a small number of marbles and calculating what the total value of the board then is. What is the total value at the start of a normal solitaire game?
4. Explain why the total value of the board never changes as you play the game.
5. Suppose you start a game of solitaire with one marble missing like this:

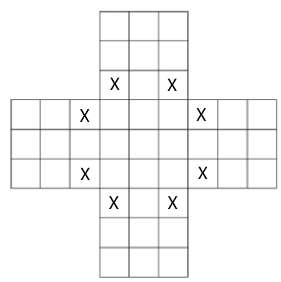


   What is the total value of the board? Is it possible to win the game of solitaire starting like this?
6. Explain why, if you play solitaire and your game ends with a single marble, that marble must be on a square you labelled h in step 1.
7. Pick one of the squares marked X here:

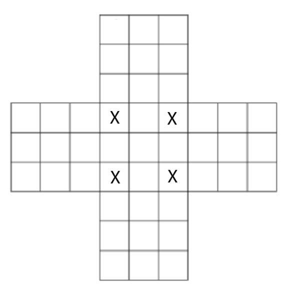


Suppose it's impossible for the game to end with a single marble in your square. Explain why it's then impossible for the game to end with a single marble in any of the squares marked X.

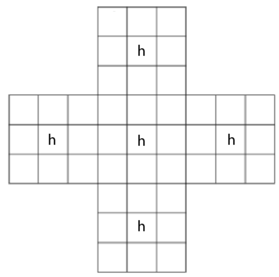
8. Repeat the previous step for these squares



and then for these squares:



9. Explain now that if a solitaire game ends with a single marble, then that marble must be in one of these squares labelled h:



10. Explain why, if you could play a solitaire game ending with a single marble in one of the squares labelled h, then you could adapt it so as to end at the winning middle square.

The conclusion is that the "easier" version of Solitaire turns out to be no easier at all!

Acknowledgement. This worksheet is based on a +plus magazine blog posts by Colva Roney-Dougal. She credits Sinead Lyle for telling her about the application of the Klein 4-group to solitaire.