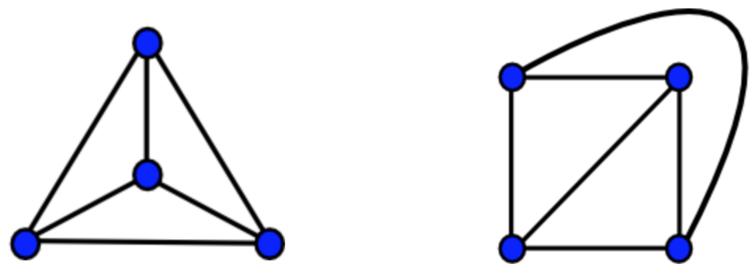
Euler's Formula for Graphs

A *graph* is a collection of vertices (or nodes) connected by edges (or arcs or lines). Here are two examples:



Here are three 'real world' examples:

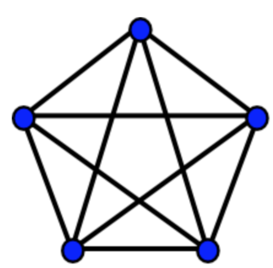
- The vertices could be countries with two joined by an edge when they share a common land border.
- The vertices could represent movie actors. Two are joined by an edge when they have starred in the same movie.
- Family trees: vertices are people and two are joined by an edge when they are married or when one is a parent of the other.

Task 1. Think of some more 'real world' examples.

A graph is *embedded in the plane* when it is drawn in the plane in such a way that no edges meet except at vertices. The two graphs shown above are examples.

Task 2. In fact, the two examples above are the *same* graph embedded in the plane in two different ways. How so?

We're used to seeing family trees embedded in the plane. Also the countries and borders example will give a graph embedded in the plane (why?). But a graph of movie actors graph often won't be embeddable in the plane. Indeed, if you take five actors all of whom have starred in a movie together, then here's the graph you get:



Edges running through the interior of the pentagon cross, so this is not an embedding of this graph in the plane. In fact, there is no way to embed it in the plane.

Task 3. Try to convince yourself that there's no way of drawing this graph in the plane without crossings.

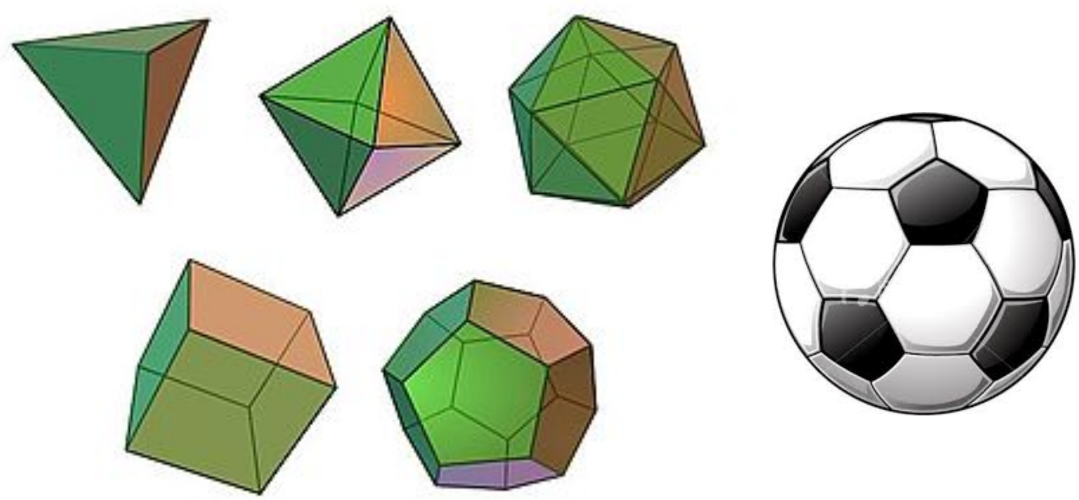
Suppose G is a graph. For the rest of this investigation let's assume the number V of vertices and E of edges it has are both finite—we say G is a *finite* graph.

We call G *connected* when you can travel from any one vertex to any other by traversing some sequence of edges. For example the graph 'common land border' graph of all countries of the world is not connected: there's no way to get from Australia to France across only land borders.

If G is embedded in the plane, it will subdivide the plane into regions called *faces*. (There is an outside region which counts as one face.) Let F be the number of faces.

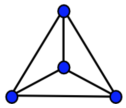
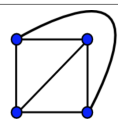
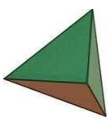
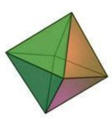
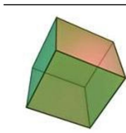
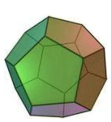
Task 4. Calculate V, E, F and $V - E + F$ for the two graphs pictured at the start of this handout. Put your answers in the table on the next page. (I've done the first one for you—check you agree.)

Task 5. Draw three of your own examples of connected finite graphs embedded in the plane and calculate V, E, F and $V - E + F$ for each one. Enter the results in the table on the next page.



Task 6. What are V, E, F and $V - E + F$ for the five Platonic solids (first row of the figure: tetrahedron, octahedron, icosahedron; second row: cube, dodecahedron)? What about for a soccer ball? How are these examples like graphs in a plane?

Euler's Formula confirms what you will hopefully notice from your completed table: for a connected finite graph embedded in the plane, it is always the case that $V - E + F = 2$.

	V	E	F	$V - E + F$
	4	6	4	2
				
Your first example				
Your second example				
Your third example				
				
				
				
				
				
				

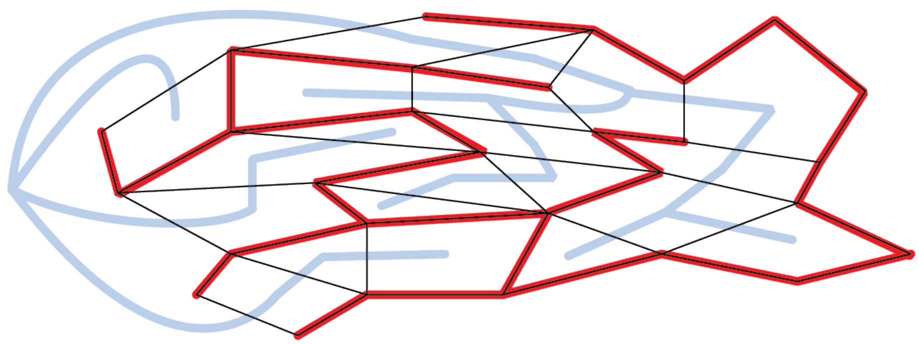
Proving Euler's Formula

Proof is an idol before which the mathematician tortures himself.
Arthur Eddington (British astrophysicist; 1882–1944)

An elegantly executed proof is a poem in all but the form in which it is written.
Morris Kline (American mathematician and educator; 1908–1992)

A good proof is one that makes us wiser.
Yuri I. Manin (Russian mathematician; 1937–)

We've talked about the important role proof plays in mathematics. Euler's Formula has many proofs. We are going to work through one of them. Let's see if it measures up to the three quotes above! It's based on what are called *interdigitating trees.* This picture hints at where we are heading:



Before we come to our proof, I want to introduce *spanning trees.* A collection T of edges in a graph G forms a *spanning tree* when is satisfies two properties:

(a) you can get from any vertex in G to any other by following edges in T, and
(b) there are no loops in T.

For example, the red edges in the black graph pictured above form a spanning tree.

Task 7. Draw three examples of planar connected graphs and highlight spanning trees in each of them. Do all connected graphs have spanning trees?

Suppose we have a connected graph G embedded in the plane. Construct a new graph G^* known as the *dual graph* as follows. First put a dot inside each face of G (including the outer region). Whenever two faces have a common edge e, connect their dots by an arc crossing e. These dots and arcs are the vertices and edges of G^*.

Task 8. Draw three more connected graphs embedded in the plane. Using a different colour, draw their dual graphs over them. What would you get if you start with one of the dual graphs you've drawn and construct the dual of that?

Task 9. What are the duals of each of the five platonic solids? How is your answer reflected in the table you completed on page 3?

Task 10. Use a different colour for this task. Put a dot inside each face (including the outside region) of your three examples from Task 7. Connect two of these dots with an edge when the corresponding faces have an edge in common which is *not* in T. Comment on the results.

Now we're ready for our proof. We number the steps for ease of reference later. After the proof, you'll find some final exercises to help you understand how it works.

Proof of Euler's Formula. (1) Suppose H is a connected finite graph with no loops. Then H is what is known as a *tree*. (The spanning tree T we discussed above and T^* which we will meet shortly are examples.) Let n be the number of edges H has. By again and again removing a single vertex and a single edge from H it is possible to 'prune' H into a smaller and smaller tree, until eventually it has just a lone vertex and has no edges. So H must have had $n + 1$ vertices.

(2) Suppose G is a finite graph embedded in the plane. Suppose T is a spanning tree in G. Let T^* be the collection of edges in G^* that are dual to edges of G that are not in T. (This is what you drew in Task 10.)

We claim that T^* forms a spanning tree in G^*. We'll check the properties (a) and (b) in turn. First (a): you can get from any vertex in T^* to any other by following edges in T^*. This is because if some portion of T^* was cut off from the rest of T^*, then there would be a loop in T surrounding it—but that can't be: T has no loops. And now (b): there can't be a loop in T^*, as otherwise there would be a one vertex of G inside that loop and one outside, and they couldn't be connected by a path in T.

(3) By Step 1, the tree T has $V - 1$ edges and the tree T^* has $F - 1$. So in total they have $V + F - 2$ edges, but this is equal to E as every edge in G is either in T or is dual to an edge in T^* (and never both). It follows that $V - E + F = 2$, as required.

■

Task 11. Draw an example of a tree H and go through the 'pruning' process described in Step 1.

Task 12. In Step 2 the fact that a loop in the plane has an inside and an outside is used. It may surprise you to learn that this result, which is known as the Jordan Curve Theorem, is not straight-forward. Look it up and read about it. (It's also vital to Task 3.)

Please complete the tasks in this handout in your journals.

Acknowledgements. I have adapted the proof above from David Eppstein's Geometry Junkyard, where it is attributed to a 1958 book by D. M. Y. Sommerville who, in turn cites an 1847 book by G. K. C. Von Staudt. The first three images of graphs are from boost.org. The interdigitating trees picture is from a paper I wrote with William Thurston. The quotes about proofs are from the *Discovering the Art of Mathematics* book *Truth, Reasoning, Certainty, and Proof* by Julian F. Fleron and Philip K. Hotchkiss.

TRR, November 2017