1. What are the 3 or 4 main elements you retain from the pre-class activity?

   - *even though the functions presented in the exercise are not defined everywhere, if we plug in points that are close to the point where they are undefined, we observe that these functions do go to a* ***specific value*** *(in that sense, the function follow a "****pattern****"),*
   - *even though the function never attains a certain value (e.g. $f(x)$ never equals 3), we can have $f(x)$ get* ***as close as we want*** *to 3.*
     In other words, we can decide on a ***pre-determined precision*** *with which we want the function to attain the value 3. We can make this level of precision as small as we want (make sure to underline this last point in class!),*
   - *in general, the function can return values that are* ***both below and above*** *the limit value (i.e. in general a limit is not an upper or lower bound),*
   - *the interval for which we can assure that the function is going to be in the range we want* ***depends on both the function and the level of precision****.*

Let us keep in mind this idea of having a pre-determined level of precision around *one* value. We will investigate a few more examples and then will get to the definition of a limit.

2. Consider the following functions:

$$f(x) = \frac{x}{|x|}, \qquad g(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n}, \text{ for } n \text{ non-zero integers,} \\ 0 & \text{if } x \neq \frac{1}{n}, \text{ for } n \text{ non-zero integers.} \end{cases}$$

   (a) What are the domains of definition of these two functions? Sketch their graphs.

   If you want to be more precise, you can say that f(x) is undefined at 0. Nevertheless, it would be more interesting to see if students come up with this question (and if not to prompt them about it).

   (b) Concerning $f(x)$, what value(s) does the function go to if we take points close to $x = 0$?

   It depends what side one approaches 0 from!
   The function $f(x)$ goes to -1 from the left-hand side and 1 from the right-hand side.

   (c) Can we take a pre-determined level of precision and then find an interval around 0 such that we can assure the function will be in that pre-determined range?
   No, BUT we can do that for each side individually.

   Key point(s) of this example

   For some functions, the behavior of the function depends one the side one approaches from.
   We will thus need more than one notion to capture these different behaviors (here one-sided limits).

   (d) Let us now look at $g(x)$. What value(s) does the function go to if we take points close to $x = 0$?

   It depends on "how" you approach 0!
   If one approaches 0 following points that are all different from $1/n$ we always find $g(x) = 0$ whereas if one approaches 0 following $1/n$ we always get $g(x) = 1$.

(e) Can we take a pre-determined level of precision and then find an interval around 0 such that we can assure the function will be in that pre-determined range?

Explain why it is not possible (this will be a case where the "limit does not exist".

Key point(s) of this example

*For some functions one **cannot** find an interval such that the function remains in a pre-determined range* .

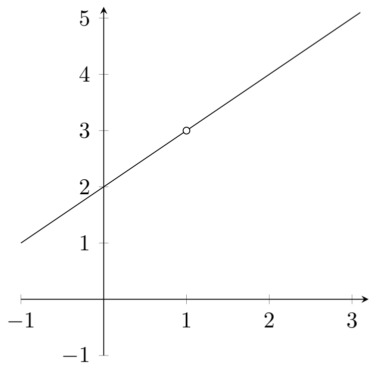
Limits
We can think of the *limit of* $f(x)$ *as* x *approaches* a in the following way: choose *any* pre-determined level of precision. Then the limit $\lim_{x \to a} f(x)$ equals L if we can find an interval around a, such that for any x different from a in this interval, the function $f(x)$ approaches L with the desired pre-determined level of precision.
To make it short, we use the notation $\lim_{x \to a} f(x)$ for the limit of $f(x)$ as x approaches a.

3. What are the key elements of this definition?

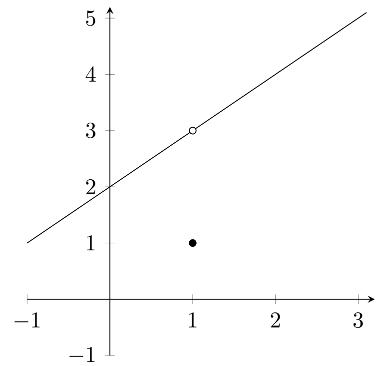
- *the limit is a **single number**,*
- *there are cases where the limit **does not exist** (we will investigate that soon), have we already encountered such cases?*
- *since $x \neq a$, the function **may or may not be defined** at $x = a$ (so far we have mainly looked at examples where the function was not defined at a). If it is defined, the value $f(a)$ of the function at $x = a$ **does NOT matter** for the limit*
- *we can make the level of prevision **as small as we want**. Once we have done that, what we need to do if find an interval around a for which all values of $f(x)$ are inside this level of precision (except at a itself)*

Using the graphs of the previous functions could be helpful here!

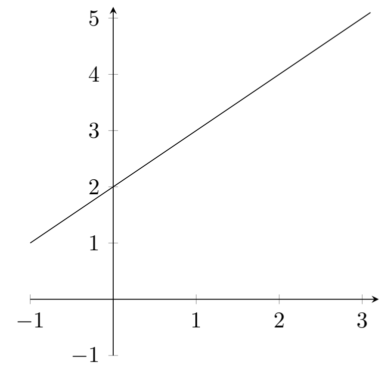
4. Let us now look at what this means graphically. For each of the following example, determine $\lim_{x \to 1} f(x)$ as well as $f(1)$.



a)



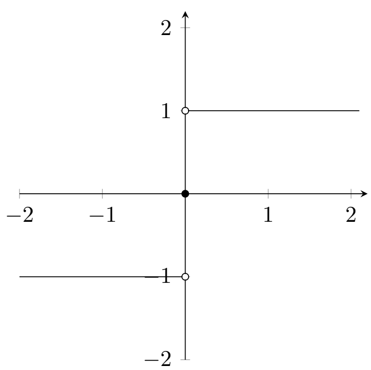
b)



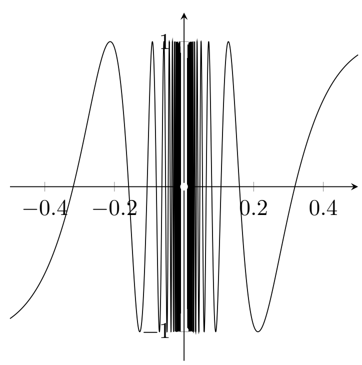
c)

Conclusion: whether or not the function is defined has no influence on the limit.
Moreover, $f(a)$ and $\lim_{x\to a} f(x)$ may be different.

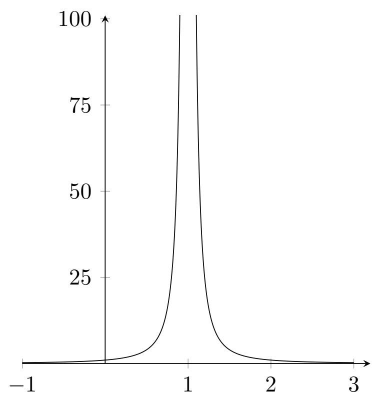
5. Let us now consider the following functions. For i) and ii), determine if the limit $\lim_{x\to 0} f(x)$ exists and if so, what it is. Determine also $f(0)$. For iii), same questions but for $\lim_{x\to 1} f(x)$ and $f(1)$.



i)



ii)



iii)

For ii), indicate that it oscillates more and more as it approaches 0 (the function being $\sin(1/x)$*).*

i) $f(0) = 0$ *and* $\lim_{x\to 0}$ *f(x) does not exist because the one-sided limits are different,*
ii) $f(0)$ *is undefined and* $\lim_{x\to 0}$ *f(x) does not exist because of the oscillations,*
iii) $f(1)$ *is undefined and* $\lim_{x\to 1}$ *f(x) does not exist because it is unbounded (will be studied again later).*

Overall, these are illustrations of the following ways in which a limit can fail to exist.

The first example above motivates the following definition of **one-sided limits**.
For *any* pre-determined level of precision we choose,

- the *limit of $f(x)$ as x approaches a from the left*, written $\lim_{x\to a^-} f(x)$, is the number L that the function $f(x)$ approaches when x is in an open interval (b, a) with $b < a$, in other words with x strictly *smaller* than a.
- the *limit of $f(x)$ as x approaches a from the right*, written $\lim_{x\to a^+} f(x)$, is the number L that the function $f(x)$ approaches when x is in an open interval (a, b) with $a < b$, in other words with x strictly *greater* than a.