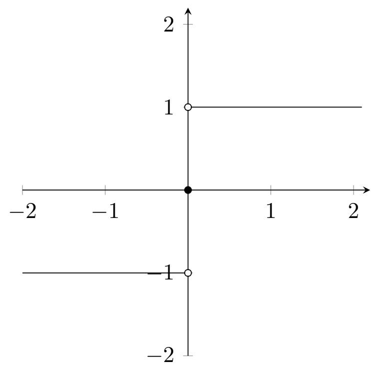
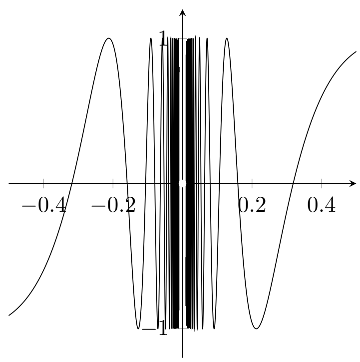
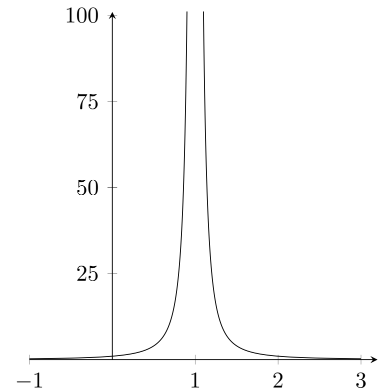
We will be exploring three functions that provide us with test cases for nuanced situations where limits do and do not exist.



$f(x)$



$g(x)$



$h(x)$

1a) For each function, describe its graph, domain, and range. Also identify any points that may be of interest with respect to limits.

1b) Of the points you've identified, determine if the limit exists and if so, what it is. Also, what is the value of the function at this point?

The first example above motivates the following definition of **one-sided limits**.
For *any* pre-determined level of precision we choose,

- the *limit of $f(x)$ as x approaches a from the left*, written $\lim_{x \to a^-} f(x)$, is the number L that the function $f(x)$ approaches when x is in an open interval (b, a) with $b < a$, in other words with x strictly *smaller* than a.
- the *limit of $f(x)$ as x approaches a from the right*, written $\lim_{x \to a^+} f(x)$, is the number L that the function $f(x)$ approaches when x is in an open interval (a, b) with $a < b$, in other words with x strictly *greater* than a.

2a) For each function, determine where the left-limits exist and where the right-limits exist. Also determine the values of these limits where they do exist.

2b) How would you summarize when a one-sided limit does not exist?

2c) What is the relationship between one-sided limits and the two-sided limit? Can you write a theorem that mathematically formalizes this relationship?

Extensions of Limits

3a) How would we deal with unbounded limits? Propose a solution and investigate some consequences of your solution.

3b) How would you extend the definition of a limit to a function of two variables?

3c) What would be the analogues of one-sided limits?