1. The goal of this first part is to determine the formal definition of continuity.

   (a) Based on the pre-class activity, what relationship have you observed between the notions of continuity at a point a, limit of the function at a and value of the function at a?

   (b) Based on that, what "should" the formal definition of continuity be?
   Be sure to give the definition of continuity for a point and also for an interval.

   (c) What are the three implications of this definition?

   For a function to be continuous at a:

   - *$f(a)$ must exist, i.e. the function is defined at a,*
   - *the limit $\lim_{x\to a} f(x)$ must exist, and thus we must have $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$,*
   - *the value of the function must equal the value of the limit, i.e. $f(a) = \lim_{x\to a} f(x)$.*
     This means that when computing the limit of a continuous function, one can just plug in the number.

2. (a) Which of the following functions are continuous? Drawing their graphs can be helpful. (for $g(x)$, it may be helpful to factor the numerator).

$$f(x) = x+1, \qquad g(x) = \begin{cases} \frac{x^2-1}{x-1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}, \qquad h(x) = \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}, \qquad k(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases},$$

$$l(x) = \frac{1}{x}, \qquad m(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases},$$

The goal here is to show the different types of discontinuity that can happen.

(b) Looking at the discontinuities of the previous functions, what "types" of discontinuities do you see? How can you relate these discontinuities with the definition?

The types are: removable, jump, infinite discontinuities and oscillating (p. 93 in the textbook).

3. (a) Consider the function $f(x) = \begin{cases} 1+x & \text{if } x < 2 \\ x^2+3 & \text{if } x \geqslant 2 \end{cases}$. Prove that there is a point where it is discontinuous.

Here one should probably mention Theorem 8, p. 94 in the textbook that lists classes of continuous functions. This will answer the question "how do we know that each part is continuous" and also indirectly "what point(s) should we look at"?).

(b) By what constant should you replace the "1" in the $1+x$ part of the definition of $f(x)$ to make the function continuous?

Point out here that there are many different "ways" to glue two functions in a continuous manner. E.g. one could also replace x by $6x$ in $1+x$.

(c) Where is $g(x) = \begin{cases} x^2+3 & \text{if } x \leqslant 0 \\ 1+\sin(3x^2) & \text{if } x > 0 \end{cases}$ discontinuous?

One can ask the students how we know that $1+\sin(3x^2)$ is continuous?
Mention that trig functions are continuous and also that composition of continuous functions are continuous (Theorem 9, p. 95 in the textbook).

4. (a) Look at the functions $f(x)$ and $g(x)$ on the previous page. For what value(s) of x do we have $f(x) = 4$? And for what value(s) of x do we have $g(x) = 2.5$?

   Introduce here the idea of intermediate value property.

   (b) Would this also happen with the function you wrote down in 3b? Why or why not?

   Here introduce the Intermediate Value Theorem (Thm 11, p.97).

5. Here is an application of the Intermediate Value Theorem (IVT).

   (a) Does the equation $x^3 + 2x^2 - x = 1$ have a solution between $x = 0$ and $x = 1$.

   Indicate that you need to put everything on one side of the equation. Then it is a straightforward application of the IVT.

   (b) Show that the equation $-x^2 + 6x - 7 = 0$ has a solution between $x = 0$ and $x = 5$.

   This equation actually has TWO solutions between 0 and 5. Thus underline that when the IVT says "for some c in $[a, b]$", it doesn't mean that there is only one *such point.*
   An extra question could be, prove there are two solutions between 0 and 5.

When talking about the intermediate value property, one can also mention "real world" examples, such as the height of a person (even though this specific example is debated: is height a continuous or discreet function?).