Limits and One-sided Limits (2.1, 2.2 & 2.4 on Thomas)

Expected Skills.
At the end of these sections, the students will be able to:

- explain in their own words the definition of a limit and one-sided limit,
- compute limits and one-sided limits using limit laws for polynomial and rational functions as well as common methods studied in class,
- give examples that illustrate the different cases where a limit or a one-sided limit fails to exist,
- know important limits such as $\lim_{x\to 0} \dfrac{\sin x}{x}$ and $\lim_{x\to 0} \sin(1/x)$,
- appropriately use the squeeze theorem to compute limits. This includes being able to:
  - give the statement of the theorem,
  - recognize situations in which the theorem applies and can be useful,
  - follow a procedure to use the theorem in order to compute the limit of a given function,
- explain the relationship between the existence of a limit and one-sided limits.

Note: for the following activities we assume that the students have already done the activities on Brilliant.org motivating the study of limits.

Pre-Class Activity (ch2-limits-1-limits-1-pc)**.** The goal of this pre-class activity is to introduce the students to the concept of a limit and have them realize the following points:

- even though the functions presented in the exercise are not defined everywhere, if we plug in points that are close to the point where they are undefined, we observe that these functions do go to a specific value (in that sense, the function follow a "pattern"),
- even though the function never attains a certain value (e.g. $f(x)$ never equals 3), we can have $f(x)$ get as close as we want to 3.
  In other words, we can decide on a pre-determined level of precision with which we want the function to "attain" the value 3. We can make this level of precision as small as we want (make sure to underline this last point in class!),
- in general, the function can return values that are both below and above the limit (i.e. in general a limit is not an upper or lower bound),
- the interval for which we can assure that the function is going to be in this pre-determined level of precision we want depends on both the function and the level of precision.

Aspects that are treated here and that will need to be investigated in class include:

- for the limit to exist, we need to functions to go to a *single* value no matter how we go to the point (as opposed to cases where the left-hand side and right-hand side limits are different, or where the value of the functions depends on how we approach the point a),

- we can also compute limits for points where the functions is defined (and the actual value of the function does NOT matter),

Worksheet (ch2-limits-1-limits-2-ws)**.** *Linked to pre-class activity 1*
In the pre-class activity we have the students think about the concept of a limit by working on examples where the limit exists. We also introduce there the idea of pre-determined level of precision.
In this activity, we start by giving two examples where the limit does not exist. The goal is to have the students think about what the definition of a limit should and should not be. We then introduce the definition and ask the students to summarize the main points.
Part 4 focuses on the difference between the limit and the value of the function.
Finally we introduce examples where the limit does not exist and introduce the definition of a one-sided limit.

Pre-Class Activity (ch2-limits-1-limits-3-pc)**.** The goal of this activity is to have the students do some computations that will be used in class: factoring and simplifying a rational function and multiplying by the conjugate.
After the computations we ask them questions to have them think about why this works even though the functions are "technically" different. Indeed, many students are confused by these computations and don't understand why sometimes one can plug in numbers and at other times not.
(Indirectly, we are also introducing the idea of continuity that will come next).

Worksheet (ch2-limits-1-limits-4-ws)**.** The goal of this worksheet is to focus on computing limits (in contrast of the first worksheet that focuses on the concept of a limit). We thus cover "standard" techniques (factoring and simplifying a rational function, multiplying by the conjugate).
In the second exercise we focus on the various situations that can happen when computing a limit.
The third exercise look at what happens when the limit goes to infinity or negative infinity.
The fourth exercise is both a recap about $\sin(1/x)$ and a motivation for the Squeeze Theorem. (One should then formally introduce it).

Worksheet (ch2-limits-1-limits-5-ws)**.** This worksheet should be both an opportunity to compute interesting limits and to wrap the topic up.
The first exercise is a recall about the definition of a limit.
In the second exercise, we ask the students to compute $\lim_{t\to 0} \frac{\sin t}{t}$. While students should only know the result (but not the process), it is interesting to have them compute it once (but don't spend too much time on it).
The last exercise is really here to have them reflect on the various computations we have done about limits. (It may reorder the exercises).

If it is too short for a whole class, one could use of of Maria Tyrell's *Good Questions* (available here: http://www.math.cornell.edu/~GoodQuestions/materials.html) or start with the next pre-class activity on continuity.

Supplemental Activity (ch2-limits-1-limits-6-sup-limits)**.** After this activity, students will be able to

- identify cases where limits do not exist (contradicting one-sided limits, oscillations, unboundedness),
- explain in their own words and diagrams what one-sided limits are,
- explain the relationship between one-sided limits and two-sided limits.

The activity has the students investigate three functions that showcase the three main cases where limits do not exist. Students are asked to characterize each function and describe the behavior of the function that disallows a limit to exist. Students are then introduced to the notion of one-sided limits and are asked to connect these definitions to two-sided limits. The last section of the activity are questions aimed to have students ponder extensions to reinforce their understanding of limits.

It is suggested that instructors assign one of the three functions to each student and have each individual work through Problem 1. Students are then pair up with other students with the same function to discuss their solutions. After paired sharing, students are then grouped such that every group has each function represented. Students then share their solutions to the group for Problem 1 and can continue to address Problem 2.

Alternatively, students can be grouped and the functions distributed within each group. Students would address Problem 1 individually and share their responses to their group. Students would then continue to individually address Problem 2 for their assigned function and share their responses to their group again.

Problem 3 can be posed to the entire classroom, where the discussion can be used to debrief and conclude the activity.

Supplemental Activity (ch2-limits-1-limits-6-sup-squeeze)**.** After this activity, students will be able to

- explain the squeeze theorem and how to apply it in their own words and diagrams,

The activity restates the squeeze theorem for students and challenges students to identify appropriate circumstances and then apply the theorem. The activity then considers the squeeze theorem in the context of a geometric formulation to compute a limit.

It is suggested that instructors assign a limit to each student for Problem 1 for individual work. Students are then paired so that each pair has both limits to share their responses. Students should continue to work in their pairs to solve Problem 2. It is also suggested that instructors have students fully write up a solution for Problem 2c individually after working through the problem in pairs. Within pairs, students would then explain their partner's work to their partner. The instructor can conclude the activity by asking for a student to share/defend their partner's solution to the classroom.

Here we will look at the concept of a limit from a "general point of view". In class we will see more precisely what the definition of a limit is and what its various subtleties are.

Consider the following functions:

$$f(x) = \frac{x^2 + x - 2}{x - 1}, \qquad g(x) = \frac{\sqrt{x^2 + 4} - 2}{x^2}, \qquad h(x) = \frac{\sin x}{x}.$$

1. For each function, determine its domain and list the point at which it is *not* defined.

2. If a function $f(x)$ is undefined at a point $x = a$, it means that we cannot plug in this point into the function. In other words, we cannot compute $f(a)$. Nevertheless, we *can* compute the value of the function for points that are close (and even very close) to a.

   For each function above, compute the value of the function for points that very close to the point where it is undefined. In each case, what do you notice? Do you see a "pattern"?

A conclusion we can draw from the previous points is that even though these functions are not defined everywhere, if we plug in points that are close to the point where they are undefined (in this case $x = 1$ for $f(x)$ and $x = 0$ for both $g(x)$ and $h(x)$), we see that these functions do go to a specific value.
We will investigate this aspect more precisely now.

3. Use Geogebra or Desmos to graph the above functions. Do the graphs correspond to your computations? Are the graphs correct (warning: graphing softwares have the bad tendency of "filling" holes)?

4. For the function $f(x)$ there is no point a such that $f(a) = 3$ (in other words, $f(x)$ never gives the value 3). But then, one important question is the following: even though $f(x)$ never gives the value 3, can the function get as close as we want to 3? Or put differently, can we pre-determine a level of precision for which the function will "attain" 3?
Let's say we want $f(x)$ to "attain" the value 3 with a level of precision of 0.1. Find several values of x for which $2.9 < f(x) < 3.1$ (i.e. for which $f(x)$ is between 2.9 and 3.1).

5. Can you find an interval around $x = 1$ for which you can assure that $f(x)$ is going to be between 2.9 and 3.1?

6. Let us now do the same with a level of precision of 0.01, in other words, we now want $f(x)$ to be between 2.99 and 3.01. Can you find an interval around $x = 1$ for which you can assure that $f(x)$ is going to be between 2.99 and 3.01? Does the interval you found at the previous point also work here?

7. Let us finally inquire the same question with the function $h(x) = \dfrac{\sin x}{x}$. By computing the value of $h(x)$ for points that are close to $x = 0$, we see that $h(x)$ goes to 1. Let's say we want a precision of 0.1, i.e. we want $h(x)$ to be between 0.9 and 1.1.
   Try to find an interval around $x = 0$ for which $h(x)$ is in that range. Does the interval you found for $f(x)$ for a precision of 0.1 work here? Would a bigger interval also work?

1. What are the 3 or 4 main elements you retain from the pre-class activity?

   - *even though the functions presented in the exercise are not defined everywhere, if we plug in points that are close to the point where they are undefined, we observe that these functions do go to a **specific value** (in that sense, the function follow a "**pattern**"),*
   - *even though the function never attains a certain value (e.g. $f(x)$ never equals 3), we can have $f(x)$ get **as close as we want** to 3.*
     *In other words, we can decide on a **pre-determined precision** with which we want the function to attain the value 3. We can make this level of precision as small as we want (make sure to underline this last point in class!),*
   - *in general, the function can return values that are **both below and above** the limit value (i.e. in general a limit is not an upper or lower bound),*
   - *the interval for which we can assure that the function is going to be in the range we want **depends on both the function and the level of precision.***

Let us keep in mind this idea of having a pre-determined level of precision around *one* value. We will investigate a few more examples and then will get to the definition of a limit.

2. Consider the following functions:

$$f(x) = \frac{x}{|x|}, \qquad g(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n}, \text{ for } n \text{ non-zero integers,} \\ 0 & \text{if } x \neq \frac{1}{n}, \text{ for } n \text{ non-zero integers.} \end{cases}$$

   (a) What are the domains of definition of these two functions? Sketch their graphs.

   If you want to be more precise, you can say that f(x) is undefined at 0. Nevertheless, it would be more interesting to see if students come up with this question (and if not to prompt them about it).

   (b) Concerning $f(x)$, what value(s) does the function go to if we take points close to $x = 0$?

   It depends what side one approaches 0 from!
   The function $f(x)$ goes to -1 from the left-hand side and 1 from the right-hand side.

   (c) Can we take a pre-determined level of precision and then find an interval around 0 such that we can assure the function will be in that pre-determined range?
   No, BUT we can do that for each side individually.

   Key point(s) of this example

   For some functions, the behavior of the function depends one the side one approaches from.
   We will thus need more than one notion to capture these different behaviors (here one-sided limits).

   (d) Let us now look at $g(x)$. What value(s) does the function go to if we take points close to $x = 0$?

   It depends on "how" you approach 0!
   If one approaches 0 following points that are all different from $1/n$ we always find $g(x) = 0$ whereas if one approaches 0 following $1/n$ we always get $g(x) = 1$.

(e) Can we take a pre-determined level of precision and then find an interval around 0 such that we can assure the function will be in that pre-determined range?

Explain why it is not possible (this will be a case where the "limit does not exist".

Key point(s) of this example

For some functions one ***cannot*** *find an interval such that the function remains in a pre-determined range* .

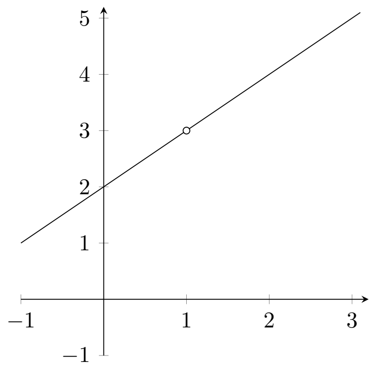
Limits
We can think of the *limit of* $f(x)$ *as* x *approaches* a in the following way: choose *any* pre-determined level of precision. Then the limit $\lim_{x \to a} f(x)$ equals L if we can find an interval around a, such that for any x different from a in this interval, the function $f(x)$ approaches L with the desired pre-determined level of precision.
To make it short, we use the notation $\lim_{x \to a} f(x)$ for the limit of $f(x)$ as x approaches a.

3. What are the key elements of this definition?

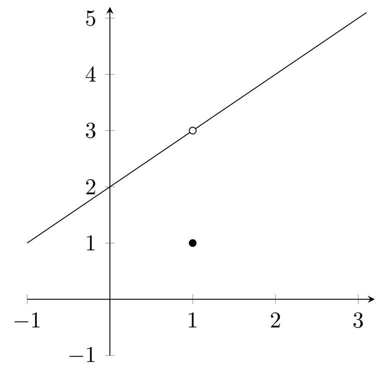
   - *the limit is a* ***single number****,*
   - *there are cases where the limit* ***does not exist*** *(we will investigate that soon), have we already encountered such cases?*
   - *since $x \neq a$, the function* ***may or may not be defined*** *at $x = a$ (so far we have mainly looked at examples where the function was not defined at a). If it is defined, the value $f(a)$ of the function at $x = a$* ***does NOT matter*** *for the limit*
   - *we can make the level of prevision* ***as small as we want****. Once we have done that, what we need to do if find an interval around a for which all values of $f(x)$ are inside this level of precision (except at a itself)*

   Using the graphs of the previous functions could be helpful here!

4. Let us now look at what this means graphically. For each of the following example, determine $\lim_{x \to 1} f(x)$ as well as $f(1)$.



a)



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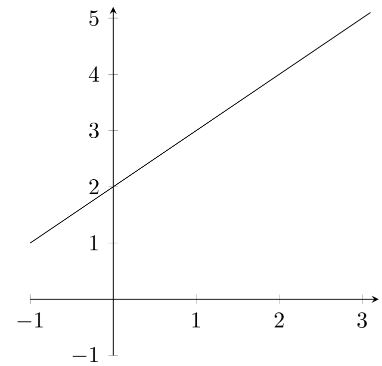
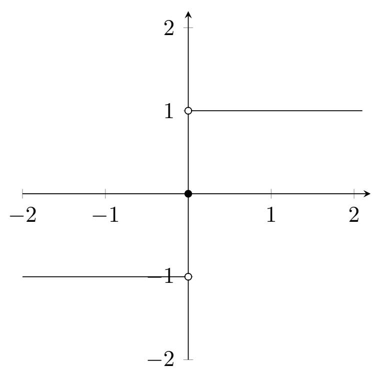


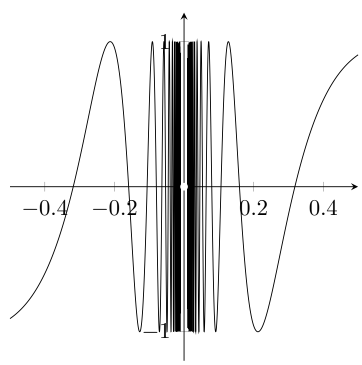
c)

Conclusion: whether or not the function is defined has no influence on the limit.
Moreover, $f(a)$ and $\lim\limits_{x\to a} f(x)$ may be different.

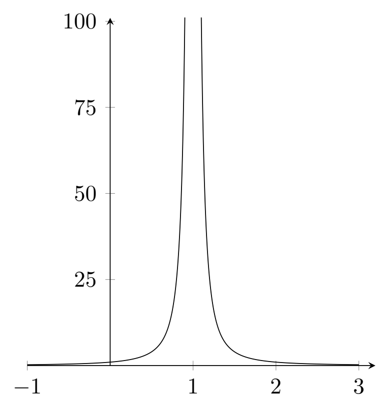
5. Let us now consider the following functions. For i) and ii), determine if the limit $\lim\limits_{x\to 0} f(x)$ exists and if so, what it is. Determine also $f(0)$. For iii), same questions but for $\lim\limits_{x\to 1} f(x)$ and $f(1)$.



i)



ii)



iii)

For ii), indicate that it oscillates more and more as it approaches 0 (the function being $\sin(1/x)$*).*

i) $f(0) = 0$ and $\lim_{x\to 0}$ f(x) does not exist because the one-sided limits are different,
ii) $f(0)$ is undefined and $\lim_{x\to 0}$ f(x) does not exist because of the oscillations,
iii) $f(1)$ is undefined and $\lim_{x\to 1}$ f(x) does not exist because it is unbounded (will be studied again later).

Overall, these are illustrations of the following ways in which a limit can fail to exist.

The first example above motivates the following definition of **one-sided limits**.
For *any* pre-determined level of precision we choose,

- the *limit of $f(x)$ as x approaches a from the left*, written $\lim\limits_{x\to a^-} f(x)$, is the number L that the function $f(x)$ approaches when x is in an open interval (b, a) with $b < a$, in other words with x strictly *smaller* than a.
- the *limit of $f(x)$ as x approaches a from the right*, written $\lim\limits_{x\to a^+} f(x)$, is the number L that the function $f(x)$ approaches when x is in an open interval (a, b) with $a < b$, in other words with x strictly *greater* than a.

1. What are the 3 or 4 main elements you retain from the pre-class activity?

Let us keep in mind this idea of having a pre-determined level of precision around *one* value. We will investigate a few more examples and then will get to the definition of a limit.

2. Consider the following functions:

$$f(x) = \frac{x}{|x|}, \qquad g(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n}, \text{ for } n \text{ non-zero integers,} \\ 0 & \text{if } x \neq \frac{1}{n}, \text{ for } n \text{ non-zero integers.} \end{cases}$$

   (a) What are the domains of definition of these two functions? Sketch their graphs.

(b) Concerning $f(x)$, what value(s) does the function go to if we take points close to $x = 0$?

(c) Can we take a pre-determined level of precision and then find an interval around 0 such that we can assure the function will be in that pre-determined range?

Key point(s) of this example

(d) Let us now look at $g(x)$. What value(s) does the function go to if we take points close to $x = 0$?

(e) Can we take a pre-determined level of precision and then find an interval around 0 such that we can assure the function will be in that pre-determined range?

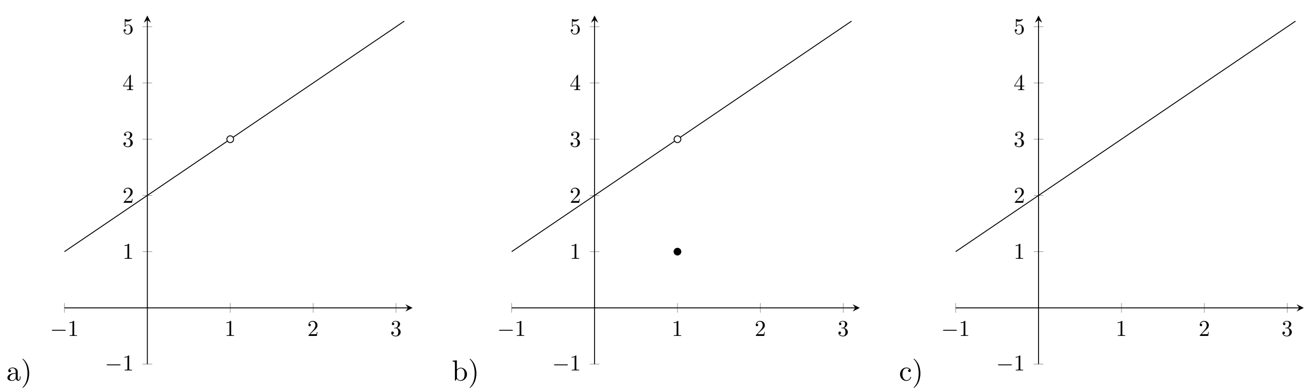
Key point(s) of this example

Limits

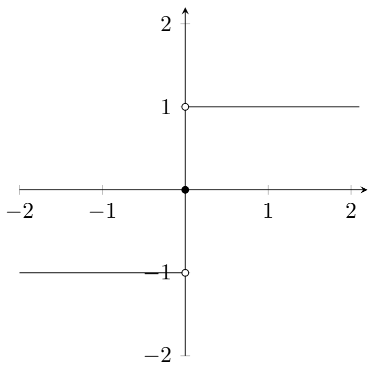
We can think of the *limit of* $f(x)$ *as* x *approaches* a in the following way: choose *any* pre-determined level of precision. Then the limit $\lim_{x\to a} f(x)$ equals L if we can find an interval around a, such that for any x different from a in this interval, the function $f(x)$ approaches L with the desired pre-determined level of precision. We use the notation $\lim_{x\to a} f(x)$ for the limit of $f(x)$ as x approaches a.

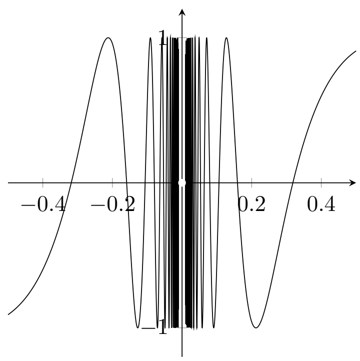
3. What are the key elements of this definition?

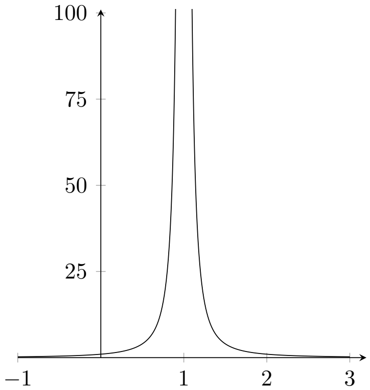
4. Let us now look at what this means graphically. For each of the following example, determine $\lim_{x\to 1} f(x)$ as well as $f(1)$.



5. Let us now consider the following functions. For i) and ii), determine if the limit $\lim_{x\to 0} f(x)$ exists and if so, what it is. Determine also $f(0)$. For iii), same questions but for $\lim_{x\to 1} f(x)$ and $f(1)$.

i) 

ii) 

iii) 

The first example above motivates the following definition of **one-sided limits**.
For *any* pre-determined level of precision we choose,

- the *limit of $f(x)$ as x approaches a from the left*, written $\lim_{x\to a^-} f(x)$, is the number L that the function $f(x)$ approaches when x is in an open interval (b, a) with $b < a$, in other words with x strictly *smaller* than a.
- the *limit of $f(x)$ as x approaches a from the right*, written $\lim_{x\to a^+} f(x)$, is the number L that the function $f(x)$ approaches when x is in an open interval (a, b) with $a < b$, in other words with x strictly *greater* than a.

In class we will focus on how to “algebraically” compute limits. In this activity, we will look at some functions and computations that we will use in class.

1. Consider the function $f(x) = \dfrac{x^2 - 2x - 3}{x - 3}$. Our goal is to compute $\lim_{x \to 3} f(x)$.
   (a) What is the domain of $f(x)$? What is $f(3)$?

   (b) Can we simplify this fraction (for example by factoring the numerator)? What do you get?

   (c) Let us call this new function $g(x)$. What is $g(3)$? Do we have $f(x) = g(x)$? Why or why not?

   (d) Draw the graphs of $f(x)$ and $g(x)$. What are the similarities and what are the differences? (Warning: if you use a graphing software to help you, be aware of their tendency to “fill holes”).

(e) Keeping in mind what we have just observed about $f(x)$ and $g(x)$, what is the relationship between $\lim_{x\to 3} f(x)$ and $\lim_{x\to 3} g(x)$? Why is that so?

2. Let us now look at the function $f(x) = \dfrac{\sqrt{x^2+25}-5}{x^2}$. Our goal will be to compute $\lim_{x\to 0} f(x)$.

   (a) What is the domain of this function? What is $f(0)$?

   (b) Our goal is now to “simplify” this expression. What do you get if you multiply by the conjugate (if you don’t know what “multiplying by the conjugate” means, look it up in the textbook or online, for example here: https://youtu.be/WVj284EvgBI).

   (c) Let us call this new expression $g(x)$. What is $g(0)$? Do we have $f(x) = g(x)$? Why or why not?

(d) Draw the graphs of $f(x)$ and $g(x)$. How similar or dissimilar are they? What is the implication of this about $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$?

(e) Something we will discuss in class: why do think in 1. we looked at the limit as x goes to 3 and not as x goes to 0 or 1?

1. Using the computations you have done in the pre-class activity as well as the Limit Laws (Theorem 1, p. 66), compute the following limits. For and justify each step of your computations.

   (a) $\lim_{x\to 1} \frac{x^2+x-2}{x-2}$,

   (b) $\lim_{x\to 1} \frac{x^2+x-2}{x-1}$,

   (c) $\lim_{h\to 0} \frac{\sqrt{7h+9}-3}{h}$,

   (d) $\lim_{t\to -1} \frac{t^2+3t+2}{t^2-1}$,

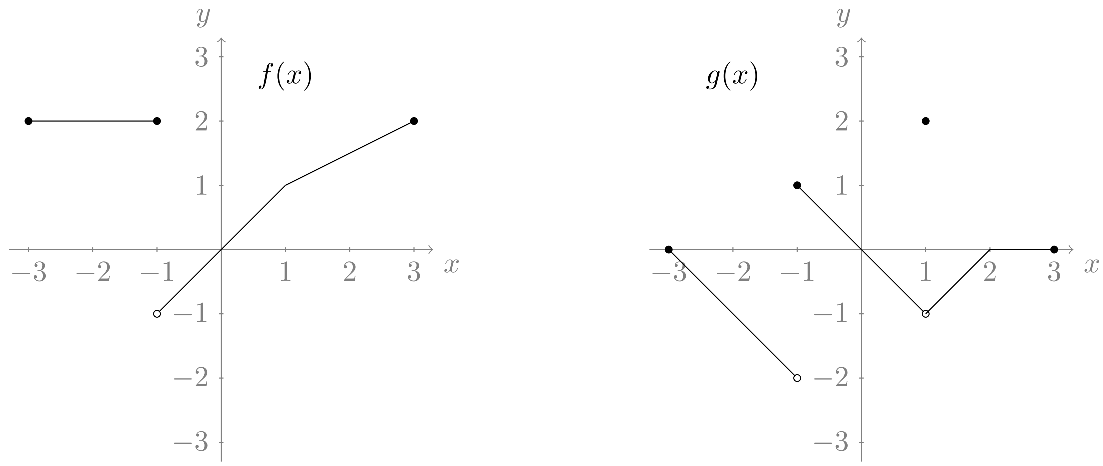
   (e) $\lim_{x\to -2} \frac{x+2}{\sqrt{x^2+5}-3}$,

   Here point out the step where we "simplify the fractions" and explain why the whole process works (thus making the link with the pre-class activity and graphs the students drew). Also point out that we could NOT have plugged the numbers in at the very start.

OVERALL, make sure to address the following questions/topics at some point:

- *Why can we sometime just plug in the value in the limit and at other times not?*
- *What is the difference between the left-hand and right-hand limit?*
- *Make sure the students can explain in words what is happening when computing limits.*

2. Here are the graphs of the functions f and g. Compute the limits indicated below.



(a) $\lim_{x \to 1} g(x)$ *lim $g(x) \neq g(a)$*

(b) $g(1)$ *lim $g(x) \neq g(a)$*

(c) $\lim_{x \to -1} f(x)$ *DNE*

(d) $\lim_{x \to -2} \frac{f(x)}{g(x)}$ *the limit exists*

(e) $\lim_{x \to 1} \frac{f(x)}{g(x)}$ *lim $f(x)/g(x) \neq f(a)/g(a)$*

(f) $\lim_{x \to 0} \frac{f(x)}{g(x)}$ *lim $f(x)/g(x)$ exists even though $g(x)$ (and $f(x)$) go to 0*

(g) $\lim_{x \to 2} \frac{f(x)}{g(x)}$ *the left-hand limit goes to infinity and the right-hand limit is undefined*

(h) $\lim_{x \to -1} \frac{f(x)}{g(x)}$ *the limt exists even though the individual limits do not*

Make sure to have defined what going to infinity means for a limit.

3. Compute the following limits. If a limit does not exists because the right-hand and left-hand limits differ, evaluate them separately.

$$\text{a) } \lim_{x\to 1} \frac{x^2+3x+2}{(x-1)^2} \qquad \text{b) } \lim_{x\to 1} \frac{x^2+3x+2}{x-1}$$

Underline that for a) the limit goes to infinity whereas for b) it depends from which side one comes.

4. Compute the following limits using the definition and/or the limit laws. Drawing the graphs of the functions may be helpful here.

   (a) $\lim_{x \to 0} \sin(1/x)$, *DNE, the graph of this function is on Worksheet 1.*

   (b) $\lim_{x \to 0} x \sin(1/x)$. *Don't let students work too long on this one.*
   Underline why the limit laws does NOT apply here.
   Use the graph of the function as a motivation for the Squeeze Theorem and then formally introduce the theorem.

1. Using the computations you have done in the pre-class activity as well as the Limit Laws (Theorem 1, p. 66), compute the following limits. For and justify each step of your computations.

   (a) $\lim_{x\to 1} \dfrac{x^2+x-2}{x-2}$,

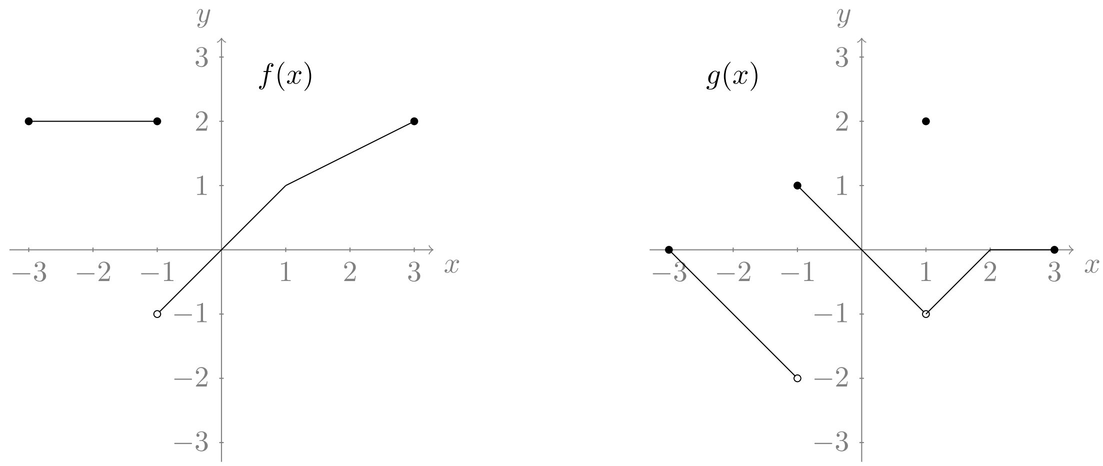
   (b) $\lim_{x\to 1} \dfrac{x^2+x-2}{x-1}$,

   (c) $\lim_{h\to 0} \dfrac{\sqrt{7h+9}-3}{h}$,

   (d) $\lim_{t\to -1} \dfrac{t^2+3t+2}{t^2-1}$,

   (e) $\lim_{x\to -2} \dfrac{x+2}{\sqrt{x^2+5}-3}$,

2. Here are the graphs of the functions f and g. Compute the limits indicated below.



(a) $\lim_{x\to 1} g(x)$

(b) $g(1)$

(c) $\lim_{x\to -1} f(x)$

(d) $\lim_{x\to -2} \frac{f(x)}{g(x)}$

(e) $\lim_{x\to 1} \frac{f(x)}{g(x)}$

(f) $\lim_{x\to 0} \frac{f(x)}{g(x)}$

(g) $\lim_{x\to 2} \frac{f(x)}{g(x)}$

(h) $\lim_{x\to -1} \frac{f(x)}{g(x)}$

3. Compute the following limits. If a limit does not exists because the right-hand and left-hand limits differ, evaluate them separately.

a) $\lim_{x\to 1} \frac{x^2+3x+2}{(x-1)^2}$ b) $\lim_{x\to 1} \frac{x^2+3x+2}{x-1}$

4. Compute the following limits using the definition and/or the limit laws. Drawing the graphs of the functions may be helpful here.

   (a) $\lim_{x \to 0} \sin(1/x)$,

   (b) $\lim_{x \to 0} x \sin(1/x)$.

1. Consider the function $f(x) = \begin{cases} x & \text{if } x = \frac{1}{n}, \text{ for } n \text{ non-zero integers,} \\ 0 & \text{if } x \neq \frac{1}{n}, \text{ for } n \text{ non-zero integers.} \end{cases}$ Draw the graph of the function.

   The goal here is to have the students think again about the definition of a limit and also about the difference between a limit and the value of the function.

   (a) Compute $\lim\limits_{x \to \frac{1}{3}} f(x)$.

   Example with $\lim_{x\to a} f(x) \neq f(a)$.

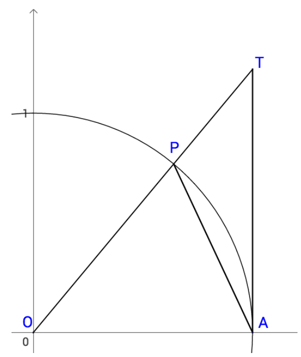
   (b) Compute $\lim\limits_{x \to \frac{2}{5}} f(x)$. Compare with your previous answer.

   Example with $\lim_{x\to a} f(x) = f(a)$.

   (c) Compute $\lim\limits_{x \to 0} f(x)$. Justify your answer.

   $\lim_{x\to 0} f(x) = 0$.

2. The goal of this section is to compute the limit $\lim_{t \to 0} \dfrac{\sin t}{t}$. In order to do so, we will use a geometrical argument (and the figure below).



(a) The figure above represents the unit circle and a given angle t.
Determine the areas of: i) the triangle OPA, ii) the area sector OPA, and iii) the triangle OTA.

The goal of this exercise is to have the students compute the limit $\lim_{t\to 0} \frac{\sin t}{t}$*.*
Looking at the areas of the triangles will also prepare the them for related rates/optimization problems. At the end we want the students to know the result but not to remember how we find the result.

(b) "Rank" them in increasing orders (i.e. area 1 $<$ area 2 $<$ area 3). Then multiply the inequalities by $\frac{2}{\sin t}$.

Underline that here we can multiply by $2/\sin t$ *without changing the inequalities because* $\sin t$ *is positive.*

(c) Finally take the reciprocals. What do you get? What can you conclude about $\lim_{t \to 0} \dfrac{\sin t}{t}$? What limit laws or theorem have you used?

Be sure to underline the fact that we need to use the Sandwich Theorem here (p. 70 in the textbook).

A question that will undoubtedly come up from the students is: "Do we have to know this?".
They don't need to know how to redo the above computations BUT they need to know the result and be able to use it (such as in the following exercise).

These computations are done on pp. 86-7 in the textbook.

(d) Using the previous result. Compute the limits:

$$\text{i. } \lim_{x\to 0} \frac{\sin(x^2)}{x^2} \qquad \text{ii. } \lim_{x\to 0} \frac{\sin(2x)}{x} \qquad \text{iii. } \lim_{x\to 0} \frac{\sin(x^2)}{x}$$

Part ii. can be done either by the change of variable $2x = t$ *or using the trig identity* $\sin(2x) = 2\sin x \cos x$ *(the former is somewhat favored as it doesn't imply knowing any "extra material" such as this identity).*

3. Find the value of the constant k for the following limits to exist.
The goal of this exercise to recap the main points and techniques we have used computing limits.

(a) $\displaystyle\lim_{x\to 5} \frac{x^2 - k^2}{x - 5}$,

(b) $\displaystyle\lim_{x\to -2} \frac{x^2 + 4x + k}{x + 2}$,

(c) $\displaystyle\lim_{x\to 1} \frac{\sqrt{x^2 + k} - 4}{x - 1}$.

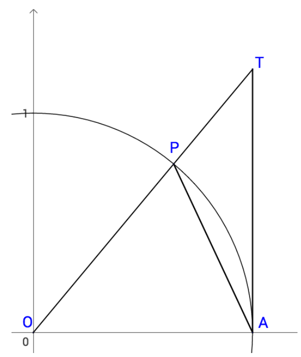
1. Consider the function $f(x) = \begin{cases} x & \text{if } x = \frac{1}{n}, \text{ for } n \text{ non-zero integers,} \\ 0 & \text{if } x \neq \frac{1}{n}, \text{ for } n \text{ non-zero integers.} \end{cases}$ Draw the graph of the function.

   (a) Compute $\lim_{x \to \frac{1}{3}} f(x)$.

   (b) Compute $\lim_{x \to \frac{2}{5}} f(x)$. Compare with your previous answer.

   (c) Compute $\lim_{x \to 0} f(x)$. Justify your answer.

2. The goal of this section is to compute the limit $\lim_{t\to 0} \dfrac{\sin t}{t}$. In order to do so, we will use a geometrical argument (and the figure below).



(a) The figure above represents the unit circle and a given angle t.
Determine the areas of: i) the triangle OPA, ii) the area sector OPA, and iii) the triangle OTA.

(b) "Rank" them in increasing orders (i.e. area 1 < area 2 < area 3). Then multiply the inequalities by $\frac{2}{\sin t}$.

(c) Finally take the reciprocals. What do you get? What can you conclude about $\lim_{t\to 0} \dfrac{\sin t}{t}$? What limit laws or theorem have you used?

(d) Using the previous result. Compute the limits:

i. $\lim_{x\to 0} \dfrac{\sin(x^2)}{x^2}$ ii. $\lim_{x\to 0} \dfrac{\sin(2x)}{x}$ iii. $\lim_{x\to 0} \dfrac{\sin(x^2)}{x}$

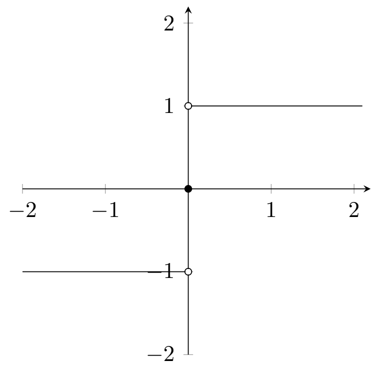
3. Find the value of the constant k for the following limits to exist.

(a) $\lim_{x\to 5} \dfrac{x^2 - k^2}{x - 5}$,

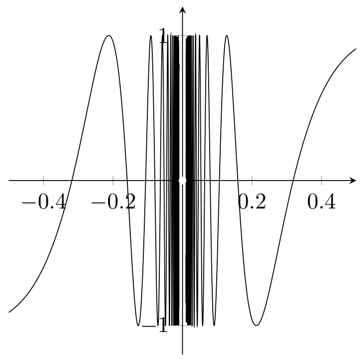
(b) $\lim_{x\to -2} \dfrac{x^2 + 4x + k}{x + 2}$,

(c) $\lim_{x\to 1} \dfrac{\sqrt{x^2 + k} - 4}{x - 1}$.

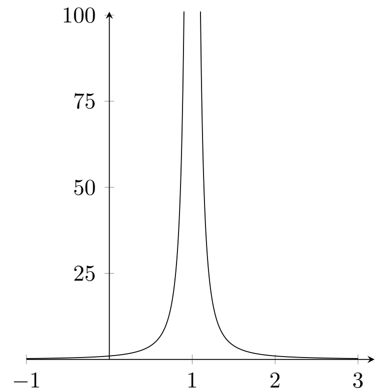
We will be exploring three functions that provide us with test cases for nuanced situations where limits do and do not exist.



$f(x)$



$g(x)$



$h(x)$

1a) For each function, describe its graph, domain, and range. Also identify any points that may be of interest with respect to limits.

1b) Of the points you've identified, determine if the limit exists and if so, what it is. Also, what is the value of the function at this point?

The first example above motivates the following definition of **one-sided limits**.
For *any* pre-determined level of precision we choose,

- the *limit of $f(x)$ as x approaches a from the left*, written $\lim_{x \to a^-} f(x)$, is the number L that the function $f(x)$ approaches when x is in an open interval (b, a) with $b < a$, in other words with x strictly *smaller* than a.
- the *limit of $f(x)$ as x approaches a from the right*, written $\lim_{x \to a^+} f(x)$, is the number L that the function $f(x)$ approaches when x is in an open interval (a, b) with $a < b$, in other words with x strictly *greater* than a.

2a) For each function, determine where the left-limits exist and where the right-limits exist. Also determine the values of these limits where they do exist.

2b) How would you summarize when a one-sided limit does not exist?

2c) What is the relationship between one-sided limits and the two-sided limit? Can you write a theorem that mathematically formalizes this relationship?

Extensions of Limits

3a) How would we deal with unbounded limits? Propose a solution and investigate some consequences of your solution.

3b) How would you extend the definition of a limit to a function of two variables?

3c) What would be the analogues of one-sided limits?

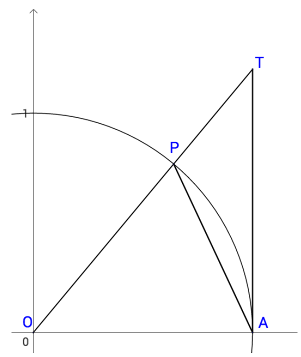
The Squeeze Theorem is another way to compute limits, often employed when limit laws cannot be used.

1a) Explain why you cannot use limit laws to determine the limits $\lim\limits_{t\to 0} t^2 \sin\left(\frac{1}{t}\right)$ and $\lim\limits_{t\to 0} \left| t\cos\left(\frac{1}{t}\right)\right|$?

1b) Explain how you would use the squeeze theorem to compute the limit of $\lim\limits_{t\to 0} t^2 \sin\left(\frac{1}{t}\right)$.

1c) Explain how you would use the squeeze theorem to compute the limit of $\lim\limits_{t\to 0} \left| t\cos\left(\frac{1}{t}\right)\right|$.

In this section, we will compute the limit $\lim_{t\to 0} \frac{\sin t}{t}$. In order to do so, we will use a geometrical argument (and the figure below).



2a) The figure above represents the unit circle and a given angle t. Determine and rank the areas of: i) the triangle OPA, ii) the area sector OPA, and iii) the triangle OTA.

2b) How can you use this relation to compute the limit $\lim_{t\to 0} \frac{\sin t}{t}$? What limit laws or theorem have you used?

2c) Write a solution detailing how to compute the limit $\lim_{t\to 0} \frac{\sin t}{t}$.

Continuity (2.5 in Thomas)

Expected Skills.
At the end of this section, students will be able to:

- explain with their own words the definition of the continuity of a function,
- list the different types of discontinuity shown in the textbook,
- list examples of continuous and discontinuous functions,
- prove the continuity of a function at a given point using the definition and/or the theorems,
- use the Intermediate Value Theorem (IVT) to show the existence of solutions to given equations. This includes being able to:
  - state the theorem,
  - recognize when we can apply the theorem,
  - follow a procedure to show the existence of a root using the theorem.

Pre-Class Activity (ch2-limits-2-continuity-1-pc)**.** in this activity we ask the students look at graphs of functions and determine which ones are continuous (using an intuitive definition of continuity). The goal is to have the students think about their "intuitive" definition of continuity on the one hand and the relationship between the limit and value of the function at that point on the other hand. Looking at that, we will then follow up in class by asking what the formal definition of continuity "should" be.
We also suggest the idea of left-hand and right-hand continuity.

Worksheet (ch2-limits-2-continuity-2-ws)**.** The goal of this worksheet is multifold.
First, using the pre-class activity, we introduce the definition of continuity, ask the students to think about what this definition means and how it is related to "the intuitive definition" of continuity.
We then have the students prove the continuity or discontinuity of functions using the definition.
Then we look at exercises where the students have to properly glue functions together.
Finally, using the previous part we introduce the Intermediate Value Theorem and some applications.

Supplemental Activity (ch2-limits-2-continuity-3-sup-dctsparameter)**.** After this activity, students will be able to

- explain with their own words and diagrams the definition of a function being continuous at a point
- identify points of discontinuity from a function's graph
- give examples of continuous and discontinuous functions
- prove that a function is continuous at a point

The activity has the students investigate instances of continuity and discontinuity for two families of functions parameterized by constants a and b. For each family of functions, students are asked to draw a graph for a given parameter, prove that the function for the given parameter is discontinuous, and determine a parameter value for which the function is continuous and/or invertible.

It is suggested that instructors assign each student a function (f or g) and have them pair with a student with the same function to work through the four parts of the problem. Students should then form new pairs with students of different functions to share their results to their partner. The instructor can conclude the activity by having 2 students go to the board to present their findings and hold a classroom discussion.

Supplemental Activity (ch2-limits-2-continuity-3-sup-dctstype)**.** After this activity, students will be able to

- explain with their own words and diagrams the definition of a function being continuous at a point
- identify points of discontinuity from a function's graph
- give examples of continuous and discontinuous functions
- prove that a function is continuous at a point

The activity has the students investigate 6 functions for continuity and classify any discontinuities that they find. Students are then asked to formalize their findings using limits. Finally students are asked to determine ways to adjust the given functions to recover continuity.

It is suggested that instructors periodically debrief the classroom after each group has completed a problem. Problem 4 can be used as a way to conclude the activity by having the classroom brainstorm ways to change the given functions so that their altered forms would be continuous.

Supplemental Activity (ch2-limits-2-continuity-3-sup-ivt)**.** After this activity, students will be able to

- explain the IVT in their own words and figures
- determine where the IVT can be applied
- apply the IVT to determine roots of functions

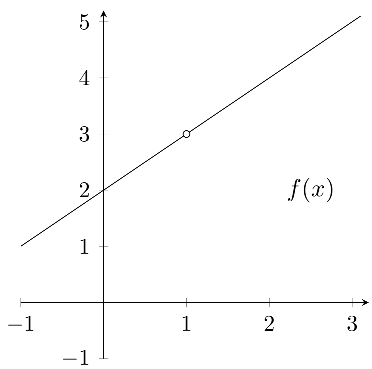
The activity has the students address the Intermediate Value Theorem by considering the logic of the statement, expressing the statement in their own words, and applying it to find an intersection point of two graphs.

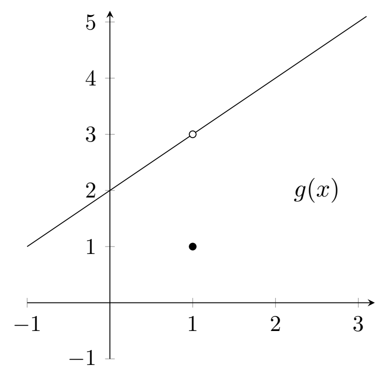
It is suggested that instructors write the IVT in formal terms on the board to introduce the activity. Students should be working through Problem 1 in groups. For Problem 2, instructors should make it clear that students should take the time to write their individual response. Students then will have to share/explain their neighbor's response to the group.

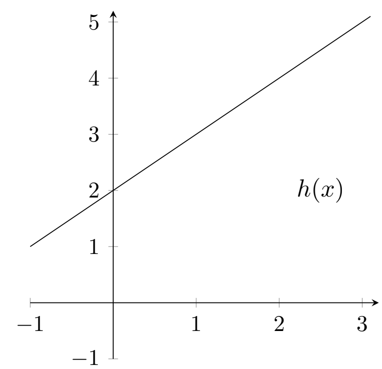
Our goal here is to inquire the notion of continuity.

1. What are two "real life" phenomena that are continuous and two that are not continuous?

2. Look at the following graphs (that come from the first worksheet on limits), which one(s) look continuous to you?

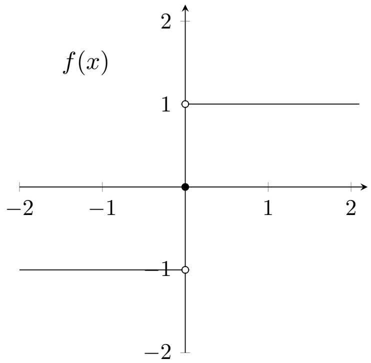
a) 

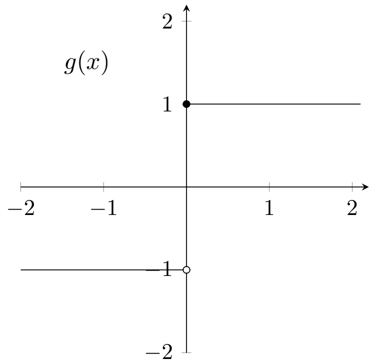
b) 

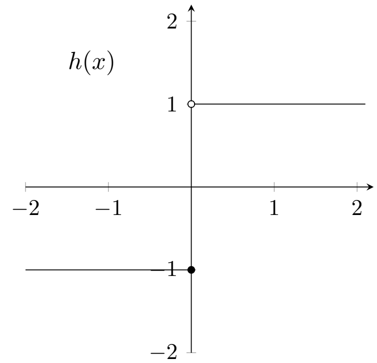
c) 

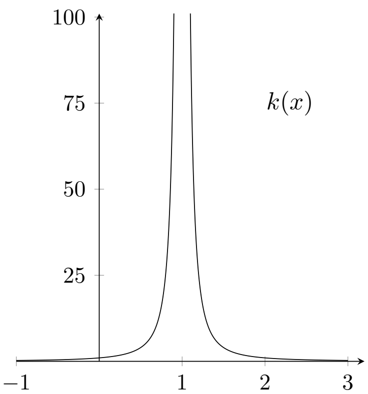
3. In each case, compute the limit as x goes to 1 as well as the values of the functions at one.
   Do you notice any pattern?

4. Again, looking at the following graphs, which one(s) look continuous to you? Do some of them look or "half-continuous"?

i) 

ii) 

iii) 

iv) 

5. For $f(x), g(x)$ and $h(x)$ compute the limit as x goes to 0 as well as the value of these functions at 0. Does the side you come from make a difference?
   For $k(x)$, compute the limit as x goes to 1 as well as the value of the function at 1.

1. The goal of this first part is to determine the formal definition of continuity.

   (a) Based on the pre-class activity, what relationship have you observed between the notions of continuity at a point a, limit of the function at a and value of the function at a?

   (b) Based on that, what "should" the formal definition of continuity be?
   Be sure to give the definition of continuity for a point and also for an interval.

   (c) What are the three implications of this definition?

   For a function to be continuous at a:

   - *$f(a)$ must exist, i.e. the function is defined at a,*
   - *the limit $\lim_{x \to a} f(x)$ must exist, and thus we must have $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$,*
   - *the value of the function must equal the value of the limit, i.e. $f(a) = \lim_{x \to a} f(x)$.*
     This means that when computing the limit of a continuous function, one can just plug in the number.

2. (a) Which of the following functions are continuous? Drawing their graphs can be helpful. (for $g(x)$, it may be helpful to factor the numerator).

$$f(x) = x+1, \qquad g(x) = \begin{cases} \frac{x^2-1}{x-1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}, \qquad h(x) = \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}, \qquad k(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases},$$

$$l(x) = \frac{1}{x}, \qquad m(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases},$$

The goal here is to show the different types of discontinuity that can happen.

(b) Looking at the discontinuities of the previous functions, what "types" of discontinuities do you see? How can you relate these discontinuities with the definition?

The types are: removable, jump, infinite discontinuities and oscillating (p. 93 in the textbook).

3. (a) Consider the function $f(x) = \begin{cases} 1 + x & \text{if } x < 2 \\ x^2 + 3 & \text{if } x \geqslant 2 \end{cases}$. Prove that there is a point where it is discontinuous.

Here one should probably mention Theorem 8, p. 94 in the textbook that lists classes of continuous functions. This will answer the question "how do we know that each part is continuous" and also indirectly "what point(s) should we look at"?).

(b) By what constant should you replace the "1" in the $1 + x$ part of the definition of $f(x)$ to make the function continuous?

Point out here that there are many different "ways" to glue two functions in a continuous manner. E.g. one could also replace x by $6x$ in $1 + x$.

(c) Where is $g(x) = \begin{cases} x^2 + 3 & \text{if } x \leqslant 0 \\ 1 + \sin(3x^2) & \text{if } x > 0 \end{cases}$ discontinuous?

One can ask the students how we know that $1 + \sin(3x^2)$ is continuous?
Mention that trig functions are continuous and also that composition of continuous functions are continuous (Theorem 9, p. 95 in the textbook).

4. (a) Look at the functions $f(x)$ and $g(x)$ on the previous page. For what value(s) of x do we have $f(x) = 4$? And for what value(s) of x do we have $g(x) = 2.5$?

   Introduce here the idea of intermediate value property.

   (b) Would this also happen with the function you wrote down in 3b? Why or why not?

   Here introduce the Intermediate Value Theorem (Thm 11, p.97).

5. Here is an application of the Intermediate Value Theorem (IVT).

   (a) Does the equation $x^3 + 2x^2 - x = 1$ have a solution between $x = 0$ and $x = 1$.

   Indicate that you need to put everything on one side of the equation. Then it is a straightforward application of the IVT.

   (b) Show that the equation $-x^2 + 6x - 7 = 0$ has a solution between $x = 0$ and $x = 5$.

   This equation actually has TWO solutions between 0 and 5. Thus underline that when the IVT says "for some c in $[a, b]$", it doesn't mean that there is only one *such point.*
   An extra question could be, prove there are two solutions between 0 and 5.

When talking about the intermediate value property, one can also mention "real world" examples, such as the height of a person (even though this specific example is debated: is height a continuous or discreet function?).

1. The goal of this first part is to determine the formal definition of continuity.

   (a) Based on the pre-class activity, what relationship have you observed between the notions of continuity at a point a, limit of the function at a and value of the function at a?

   (b) Based on that, what "should" the formal definition of continuity be?

   (c) What are the three implications of this definition?

2. (a) Which of the following functions are continuous? Drawing their graphs can be helpful. (for $g(x)$, it may be helpful to factor the numerator).

$$f(x) = x + 1, \qquad g(x) = \begin{cases} \frac{x^2-1}{x-1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}, \qquad h(x) = \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}, \qquad k(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases},$$

$$l(x) = \frac{1}{x}, \qquad m(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases},$$

(b) Looking at the discontinuities of the previous functions, what "types" of discontinuities do you see? How can you relate these discontinuities with the definition?

3. (a) Consider the function $f(x) = \begin{cases} 1+x & \text{if } x < 2 \\ x^2+3 & \text{if } x \geqslant 2 \end{cases}$. Prove that there is a point where it is discontinuous.

(b) By what constant should you replace the "1" in the $1+x$ part of the definition of $f(x)$ to make the function continuous?

(c) Where is $g(x) = \begin{cases} x^2+3 & \text{if } x \leqslant 0 \\ 1+\sin(3x^2) & \text{if } x > 0 \end{cases}$ discontinuous?

4. (a) Look at the functions $f(x)$ and $g(x)$ on the previous page. For what value(s) of x do we have $f(x) = 4$? And for what value(s) of x do we have $g(x) = 2.5$?

   (b) Would this also happen with the function you wrote down in 3b? Why or why not?

5. Here is an application of the Intermediate Value Theorem (IVT).

   (a) Does the equation $x^3 + 2x^2 - x = 1$ have a solution between $x = 0$ and $x = 1$.

   (b) Show that the equation $-x^2 + 6x - 7 = 0$ has a solution between $x = 0$ and $x = 5$.

$f(x) = \begin{cases} a + x, & x < 2 \\ x^2 - 3 & x \geq 2 \end{cases}$ describes a family of functions that are specified by the constant a.

1a) Draw the graph of $f(x)$ when $a = 0$.

1b) Prove that $f(x)$ is discontinuous when $a = 0$.

1c) For what value of a would the resulting $f(x)$ continuous? Explain your reasoning.

1d) For which values of a is $f(x)$ a function? For which values of a is $f(x)$ invertible?

$g(x) = \begin{cases} bx, & x < \pi/2 \\ 1 - 3\sin 2x & x \geq \pi/2 \end{cases}$ describes a family of functions that are specified by the constant b.

2a) Draw the graph of $g(x)$ when $b = 0$.

2b) Prove that $g(x)$ is discontinuous when $b = 0$.

2c) For what value of b would the resulting $g(x)$ continuous? Explain your reasoning.

2d) For which values of b is $g(x)$ a function? For which values of b is $g(x)$ invertible?

1) Which of the following functions are continuous?

$$f(x) = x + 1$$

$$g(x) = \begin{cases} \frac{x^2-1}{x-1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$$

$$h(x) = \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

$$k(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

$$\ell(x) = \frac{1}{x}$$

$$m(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

2) Of the discontinuities of the previous functions, what types of discontinuities do you see?

3) How can you relate these discontinuities with the definition?

4) Which of the functions could you adjust to make continuous? What kinds of adjustments would you make?

1) Given $f(1) = 4$ and $f(2) = -3$ which of the following is true by the Intermediate Value Theorem?

   a) there exists a constant c such that $-4 < c < 3$ and $f(0) = c$.

   b) $f(x)$ has a root between $[1, 2]$.

   c) $f(2) \leq f(x) \leq f(1)$ for any $1 < x < 2$.

   d) for any value $-3 \leq y \leq 4$, there is some x-value $1 \leq x \leq 2$ so that $f(x) = y$.

   e) for any value $1 \leq x \leq 2$, there is some y-value $-3 \leq y \leq 4$ so that $f(x) = y$.

2) Explain in your own words and diagrams what the Intermediate Value Theorem entails.

3) Do the graphs of $y = x^3$ and $y = 1 + x - 2x^2$ intersect at a positive value of x?

Limits Involving Infinity and Asymptotes (2.6 in Thomas)

Expected Skills.
At the end of this section, students will be able to:

- compute limits at infinity using limit laws and methods studied in class,
- explain in words what an asymptote is,
- compute the equation of a horizontal or vertical asymptote.

Pre-Class Activity (ch2-limits-3-asymptotes-1-pc)**.** The goal of the pre-class activity is to have the students reflect about what the definition of $\lim_{x\to\infty} f(x)$.
We then give them two examples (one where the limit exits and one where it doesn't) to "test" their definition.
Worksheet (ch2-limits-3-asymptotes-2-ws)**.** This activity follows the following plan:

1. have the students come up with the definition of limit at infinity (based on the pre-class activity),
2. compute typical limits of rational functions,
3. compute horizontal asymptotes of functions,
4. compute limits where one gets "$\infty - \infty''$
5. compute vertical and horizontal asymptotes of functions. Note that oblique asymptotes have been "left out".

Supplemental Activity (ch2-limits-3-asymptotes-3-sup-asymptotes)**.** After this activity, students will be able to

- explain in their own words and diagrams what $\lim_{x\to\infty} f(x)$ means
- compute limits of rational functions
- compute horizontal asymptotes and identity vertical asymptotes of functions

The activity begins by asking the students to explain the distinction and connection between $\lim_{x\to\infty} f(x)$ and unbounded limits. The activity then asks the students to connect these limit notions to the graphical features of asymptotes. Finally the activity challenges the students to formulate a procedure to determine horizontal and vertical asymptotes and test this methodology against several functions.

It is suggested that instructors debrief the entire classroom after each group of students has addressed each of the first two problems. Students should be encouraged to collaboratively design a methodology for their group to determine asymptotes. Students should be given enough time to test their methodology against at least the first two given functions. Instructors can conclude the activity with a final classroom debrief where students from different groups present their methodologies on different functions.

Here we will inquire about the notion of limit when x goes to infinity or negative infinity.

1. Look at the definition of a limit $\lim_{x\to a} f(x)$ that we have defined in class.
   Having this definition in mind, how can we define the limit $\lim_{x\to\infty} f(x)$? In other words, how can modify the definition to make it work for x going to infinity while keeping the notion pre-determined level of precision?

2. Look at the graph of $f(x) = \dfrac{6x^2 - 5x + 2}{2x^2}$. Use this definition and the graph to determine $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Do your answers make sense with respect to what you see on the graph?

3. Again, looking at the graph of $g(x) = \sin(\pi x)$ and at your definition, determine $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$?

1. (a) Based on your pre-class activity reflections, what should the definition of $\lim_{x\to\infty} f(x)$ "look like"?

   Limit at infinity
   We can think of the limit of $f(x)$ as x approaches infinity *in the following way: choose* any *pre-determined level of precision. Then the limit $\lim_{x\to\infty} f(x)$ equals L if we can find a number $N > 0$ such that for any $x > N$ the function $f(x)$ approaches L with the desired pre-determined level of precision.*
   To make it short, we use the notation $\lim_{x\to a} f(x)$ for the limit of $f(x)$ as x approaches a.

   (b) And what about the definition of $\lim_{x\to-\infty} f(x)$?

   Just replace $x > N$ by $x < -N$.

   (c) What are the key elements of this definition?
   This somewhat *redundant with what was done on the first worksheet on limits but as it is important and a source of misunderstanding for the students, it seems a good idea to repeat it here.*

   - *the limit is a* single number*,*
   - *there are cases where the limit does not exist, have we already encountered such cases?*
   - *we can make the level of prevision as small as we want. Once we have done that, what we need to do if find the number N for which all values of $f(x)$ are inside this level of precision for $x > N$ (resp. $x < -N$),*
   - *Directly related to this definition (but part of it): be careful that ∞ and $-\infty$ are not numbers! We cannot just plug them into the function!*

2. Compute the limits as x goes to infinity and negative infinity for the following functions:

   (a) $f(x) = \dfrac{2x^4 + 2x^2 - 3}{3x^4 + x^3 - 2x^2}$

   Maybe show an example on the blackboard before starting with this example.

   (b) $g(x) = \dfrac{2x^5 + 2x^2 - 3}{3x^4 + x^3 - 2x^2}$

   (c) $h(x) = \dfrac{2x^4 + 2x^2 - 3}{3x^6 + x^3 - 2x^2}$

   What is the general rule one sees from these three cases?

   (d) $j(x) = \dfrac{\sqrt{2x^6 + 2x^2 - 3}}{3x^3 - 2x^2}$

3. Compute the horizontal asymptotes of the following functions:

(a) $f(x) = \dfrac{\sqrt[3]{x} - 4x + 7}{3x + x^{2/3} - 1}$

(b) $g(x) = \dfrac{1}{x} \sin x$ (compare your answer with $\lim\limits_{x \to \infty} \sin(\pi x)$ that you computed in the pre-class activity).

Underline that to compute $\lim\limits_{x\to\infty} \dfrac{1}{x}\sin x$, one MUST use the squeeze theorem.
Also underline that this example shows that a function CAN CROSS its horizontal asymptote (maybe show the graph).
Finally it illustrates that we can have $f(a)$ closer to $\lim\limits_{x\to\infty} f(x)$ than $f(b)$ even though $a < b$, e.g. $f(100) = 0$ whereas $f(100.5) = 1$.

(c) $f(x) = \dfrac{-x^2 + 5x - 1}{2x + 3}$

Here the limits do not exists as this function has in fact an oblique asymptote (which we don't explicitly cover).
One can also point out that the limit goes to $-\infty$ when $x \to \infty$ and to ∞ when $x \to -\infty$.

4. Compute the following limits:

   (a) $\lim_{x\to\infty}(x^2 - x)$

   Underline that we don't have $\infty - \infty = 0$!

   (b) $\lim_{x\to\infty}(\sqrt{x^2+2} - x)$

   (c) $\lim_{x\to\infty}(\sqrt{x^2+ax} - \sqrt{x^2+bx})$, where a and b are constants

   Based on what you have just computed, can you find a limit for which "$\infty - \infty$" gives 3? What about 10? What about any other number?

   The goal here is to show to the students that $\infty - \infty$ can actually be equal to any number.

5. Determine the vertical and horizontal asymptotes of the following functions:

   (a) $f(x) = \dfrac{2x^2+1}{3x-5}$

   (b) $f(x) = \dfrac{2x^2+5}{x^2-5x}$

   This example also shows clearly that a horizontal asymptote can be crossed by the function (to illustrate this show the graph).

1. (a) Based on your pre-class activity reflections, what should the definition of $\lim_{x\to\infty} f(x)$ "look like"?

   (b) And what about the definition of $\lim_{x\to-\infty} f(x)$?

   (c) What are the key elements of this definition?

   -
   -
   -
   -

2. Compute the limits as x goes to infinity and negative infinity for the following functions:

   (a) $f(x) = \dfrac{2x^4 + 2x^2 - 3}{3x^4 + x^3 - 2x^2}$

   (b) $g(x) = \dfrac{2x^5 + 2x^2 - 3}{3x^4 + x^3 - 2x^2}$

   (c) $h(x) = \dfrac{2x^4 + 2x^2 - 3}{3x^6 + x^3 - 2x^2}$

   (d) $j(x) = \dfrac{\sqrt{2x^6 + 2x^2 - 3}}{3x^3 - 2x^2}$

3. Compute the horizontal asymptotes of the following functions:

   (a) $f(x) = \dfrac{\sqrt[3]{x} - 4x + 7}{3x + x^{2/3} - 1}$

   (b) $g(x) = \dfrac{1}{x} \sin x$ (compare your answer with $\lim\limits_{x \to \infty} \sin(\pi x)$ that you computed in the pre-class activity).

   (c) $f(x) = \dfrac{-x^2 + 5x - 1}{2x + 3}$

4. Compute the following limits:

   (a) $\lim_{x\to\infty}(x^2 - x)$

   (b) $\lim_{x\to\infty}(\sqrt{x^2+2} - x)$

   (c) $\lim_{x\to\infty}(\sqrt{x^2+ax} - \sqrt{x^2+bx})$, where a and b are constants

   Based on what you have just computed, can you find a limit for which "$\infty - \infty$" gives 3? What about 10? What about any other number?

5. Determine the vertical and horizontal asymptotes of the following functions:

   (a) $f(x) = \dfrac{2x^2+1}{3x-5}$

   (b) $f(x) = \dfrac{2x^2+5}{x^2-5x}$

1) Explain in your own words and diagrams what $\lim_{x\to\infty} f(x)$ means. How does this differ from when we say that a limit is unbounded?

2) What is relationship between limits and asymptotes (horizontal and/or vertical)?

3) Design a methodology to compute the horizontal asymptotes and vertical asymptotes of functions. Test your methodology with the following functions:

   a)
   $$f(x) = \frac{2x^2+5}{x^2-5x}$$

   b)
   $$g(x) = \frac{\sqrt[3]{x}-4x+7}{3x+x^{2/3}-1}$$

   c)
   $$h(x) = \sqrt{x^2+x} - \sqrt{x^2-2x}$$

   d)
   $$\ell(x) = \frac{1}{x\sin x}$$

The Continuity Review is a single-session extended activity targeting the following student learning objectives:

- students can differentiate between the concept of a limit and of continuity,
- students can identify and use the definition of continuity to verify continuity of functions, and
- students can provide examples of discontinuous functions and explain in words why these functions have discontinuities.

The Review consists of two activities primed by an initial discussion of function continuity. The instructor should begin by polling the class and writing the definition on the board before transitioning to the first activity.

The first activity has the students devise examples of discontinuities. The instructor should divide the students into groups of 3. The instructor can also assign one student from each group the role of spokesperson during the classroom debrief segment. Within each group, the students are assigned a type of discontinuity and are tasked with generating the algebraic equation of a function, draw its graph, and provide a short explanation of why the generated function is an example of the assigned discontinuity. Students are then to share their examples to their group and make and adjustments if necessary. The groups are then to report back to the classroom their findings. The instructor should have a groups share their example of a specified type to the classroom on the board.

The second activity has the students group the eight functions on the tear-away slip on the fourth page of their handout. The students are free to group the functions however they would like on the slip, which are anonymously collected by the instructor and placed into a receptacle. The instructor randomly pulls a slip from the receptacle and replicates the grouping on the board for the classroom. The instructor then encourages the students to sleuth the reasoning behind the grouping of the function. The instructor is encouraged to fill the remainder of the session by continuing this process of drawing a new slip and discussing the grouping with the class.

- **Priming the Session** (5 min):
  - The instructor asks the class to define continuity at a point.
  - The instructor writes the definition on the board.
- **Examples of Discontinuous Functions** (30-40 min):
  - Students are divided into groups of 3.
  - Students are assigned different types of discontinuity and are to individually generate examples. (5-10 minutes)
  - Students share their examples to their groups. (5-10 min)
  - Groups report an example to the classroom. (10-15 min)
- **Function Grouping** (10+ min):
  - Students submit groupings of the 8 functions on their handout.
  - The instructor draws groupings and leads a discussion on how the groupings were made.

Type of Discontinuity:

Generate a function $f(x)$ that exhibits the type of discontinuity above. Draw its graph below and write a short explanation why the function exhibits the required discontinuity.

$f(x) =$

$a(x) = 2\cos(x^2) + 1$	$b(x) = \dfrac{x^2 + 3x - 5}{x - 1}$
$c(x) = x^3 - x + 1$	$d(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$
$e(x) = \dfrac{x^2 - 1}{x + 1}$	$f(x) = \sin\left(\dfrac{1}{x}\right)$
$g(x) = \dfrac{e^x}{x - 1}$	$h(x) = \dfrac{\lvert x \rvert}{x}$

Write your grouping here!