Limits and One-sided Limits (2.1, 2.2 & 2.4 on Thomas)

Expected Skills.
At the end of these sections, the students will be able to:

- explain in their own words the definition of a limit and one-sided limit,
- compute limits and one-sided limits using limit laws for polynomial and rational functions as well as common methods studied in class,
- give examples that illustrate the different cases where a limit or a one-sided limit fails to exist,
- know important limits such as $\lim_{x\to 0}\frac{\sin x}{x}$ and $\lim_{x\to 0}\sin(1/x)$,
- appropriately use the squeeze theorem to compute limits. This includes being able to:
  - give the statement of the theorem,
  - recognize situations in which the theorem applies and can be useful,
  - follow a procedure to use the theorem in order to compute the limit of a given function,
- explain the relationship between the existence of a limit and one-sided limits.

Note: for the following activities we assume that the students have already done the activities on Brilliant.org motivating the study of limits.

Pre-Class Activity (ch2-limits-1-limits-1-pc)**.** The goal of this pre-class activity is to introduce the students to the concept of a limit and have them realize the following points:

- even though the functions presented in the exercise are not defined everywhere, if we plug in points that are close to the point where they are undefined, we observe that these functions do go to a specific value (in that sense, the function follow a "pattern"),
- even though the function never attains a certain value (e.g. $f(x)$ never equals 3), we can have $f(x)$ get as close as we want to 3.
  In other words, we can decide on a pre-determined level of precision with which we want the function to "attain" the value 3. We can make this level of precision as small as we want (make sure to underline this last point in class!),
- in general, the function can return values that are both below and above the limit (i.e. in general a limit is not an upper or lower bound),
- the interval for which we can assure that the function is going to be in this pre-determined level of precision we want depends on both the function and the level of precision.

Aspects that are treated here and that will need to be investigated in class include:

- for the limit to exist, we need to functions to go to a *single* value no matter how we go to the point (as opposed to cases where the left-hand side and right-hand side limits are different, or where the value of the functions depends on how we approach the point a),

- we can also compute limits for points where the functions is defined (and the actual value of the function does NOT matter),

Worksheet (ch2-limits-1-limits-2-ws)**.** *Linked to pre-class activity 1*
In the pre-class activity we have the students think about the concept of a limit by working on examples where the limit exists. We also introduce there the idea of pre-determined level of precision.
In this activity, we start by giving two examples where the limit does not exist. The goal is to have the students think about what the definition of a limit should and should not be. We then introduce the definition and ask the students to summarize the main points.
Part 4 focuses on the difference between the limit and the value of the function.
Finally we introduce examples where the limit does not exist and introduce the definition of a one-sided limit.

Pre-Class Activity (ch2-limits-1-limits-3-pc)**.** The goal of this activity is to have the students do some computations that will be used in class: factoring and simplifying a rational function and multiplying by the conjugate.
After the computations we ask them questions to have them think about why this works even though the functions are "technically" different. Indeed, many students are confused by these computations and don't understand why sometimes one can plug in numbers and at other times not.
(Indirectly, we are also introducing the idea of continuity that will come next).

Worksheet (ch2-limits-1-limits-4-ws)**.** The goal of this worksheet is to focus on computing limits (in contrast of the first worksheet that focuses on the concept of a limit). We thus cover "standard" techniques (factoring and simplifying a rational function, multiplying by the conjugate).
In the second exercise we focus on the various situations that can happen when computing a limit.
The third exercise look at what happens when the limit goes to infinity or negative infinity.
The fourth exercise is both a recap about $\sin(1/x)$ and a motivation for the Squeeze Theorem. (One should then formally introduce it).

Worksheet (ch2-limits-1-limits-5-ws)**.** This worksheet should be both an opportunity to compute interesting limits and to wrap the topic up.
The first exercise is a recall about the definition of a limit.
In the second exercise, we ask the students to compute $\lim_{t\to 0} \frac{\sin t}{t}$. While students should only know the result (but not the process), it is interesting to have them compute it once (but don't spend too much time on it).
The last exercise is really here to have them reflect on the various computations we have done about limits. (It may reorder the exercises).

If it is too short for a whole class, one could use of of Maria Tyrell's *Good Questions* (available here: `http://www.math.cornell.edu/~GoodQuestions/materials.html`) or start with the next pre-class activity on continuity.

Supplemental Activity (ch2-limits-1-limits-6-sup-limits)**.** After this activity, students will be able to

- identify cases where limits do not exist (contradicting one-sided limits, oscillations, unboundedness),
- explain in their own words and diagrams what one-sided limits are,
- explain the relationship between one-sided limits and two-sided limits.

The activity has the students investigate three functions that showcase the three main cases where limits do not exist. Students are asked to characterize each function and describe the behavior of the function that disallows a limit to exist. Students are then introduced to the notion of one-sided limits and are asked to connect these definitions to two-sided limits. The last section of the activity are questions aimed to have students ponder extensions to reinforce their understanding of limits.

It is suggested that instructors assign one of the three functions to each student and have each individual work through Problem 1. Students are then pair up with other students with the same function to discuss their solutions. After paired sharing, students are then grouped such that every group has each function represented. Students then share their solutions to the group for Problem 1 and can continue to address Problem 2.

Alternatively, students can be grouped and the functions distributed within each group. Students would address Problem 1 individually and share their responses to their group. Students would then continue to individually address Problem 2 for their assigned function and share their responses to their group again.

Problem 3 can be posed to the entire classroom, where the discussion can be used to debrief and conclude the activity.

Supplemental Activity (ch2-limits-1-limits-6-sup-squeeze)**.** After this activity, students will be able to

- explain the squeeze theorem and how to apply it in their own words and diagrams,

The activity restates the squeeze theorem for students and challenges students to identify appropriate circumstances and then apply the theorem. The activity then considers the squeeze theorem in the context of a geometric formulation to compute a limit.

It is suggested that instructors assign a limit to each student for Problem 1 for individual work. Students are then paired so that each pair has both limits to share their responses. Students should continue to work in their pairs to solve Problem 2. It is also suggested that instructors have students fully write up a solution for Problem 2c individually after working through the problem in pairs. Within pairs, students would then explain their partner's work to their partner. The instructor can conclude the activity by asking for a student to share/defend their partner's solution to the classroom.