# Derivative at a Point & The Derivative as a Function (3.1 & 3.2)

**Expected Skills.**
At the end of this section, students will be able to:

- explain in words what the definition of the derivative means,
- use the definition of the derivative to compute the derivative of a function,
- use the definition of the derivative to compute the slope and equation of the tangent line at a given point,
- given the graph of a function, qualitatively draw the graph of its derivative, and conversely, given two graphs recognize the graph of a function and of its derivative,
- using the definition, determine on which intervals a function is differentiable and on which it is not. This implies being able to compute one-sided derivatives and be able to determine when it does not exist
- list the cases where a function is not differentiable and draw the corresponding graphs,
- recognize on a graph where a function fails to be differentiable.

**Pre-Class Activity** (ch3-derivatives-1-derivatives-1-pc)**.** The goal of this activity is to have the students think about the ways to approximate the slope of the tangent line at a point. The goal is to "bring" them to the idea of taking secant lines that "become" tangent.
Note that we ask the students to give *as many ways as possible* to do that. The goal is two fold. First, to avoid that those who already know (and hopefully understand) the derivative just say "take the derivative" or "take the limit". On the other hand, the goal is also to have them think about the limitations of each "type" of approximation and to compare them to one another.

**Pre-Class Activity** (ch3-derivatives-1-derivatives-4-pc)**.** This second pre-class activity can be used in conjunction or instead of the other one (or it could also be used in class).

This activity is more structured in the sense that it asks students to draw and compute the slopes of a series of secant lines. These lines become closer and closer to the tangent line.
The goal of the last two points is to have the students write down a formula that is close to the one used to compute the derivative. (unfortunately, the textbook uses mainly the one with $f(x_0+h)-f(x_0)$ and not $f(z)-f(x_0)$).

**Worksheet** (ch3-derivatives-1-derivatives-2-ws)**.** *This activity is not more related to the pre-class activity a than the pre-class activity b*
The goal of this activity is to first have the student evaluate the derivative of a function using the tangent line of the graph. We then ask them to compute the derivative of the function at a point using the definition of the derivative. Finally, we ask the students to do that for any point.
Here we thus try to "connect" the geometric interpretation of the derivative to its definition. We also want

to have the students see how computing the derivative at one point is essentially the same as computing it at any point.

**Worksheet** (ch3-derivatives-1-derivatives-3-ws)**.** *This activity is not more related to the pre-class activity b than the pre-class activity a*
In this activity we focus on connecting the graph of a function to the graph of its derivative. Using *contrasting cases* of increasing complexity we ask the students to qualitatively draw the graph of the derivative of a function based on the graph of the function.

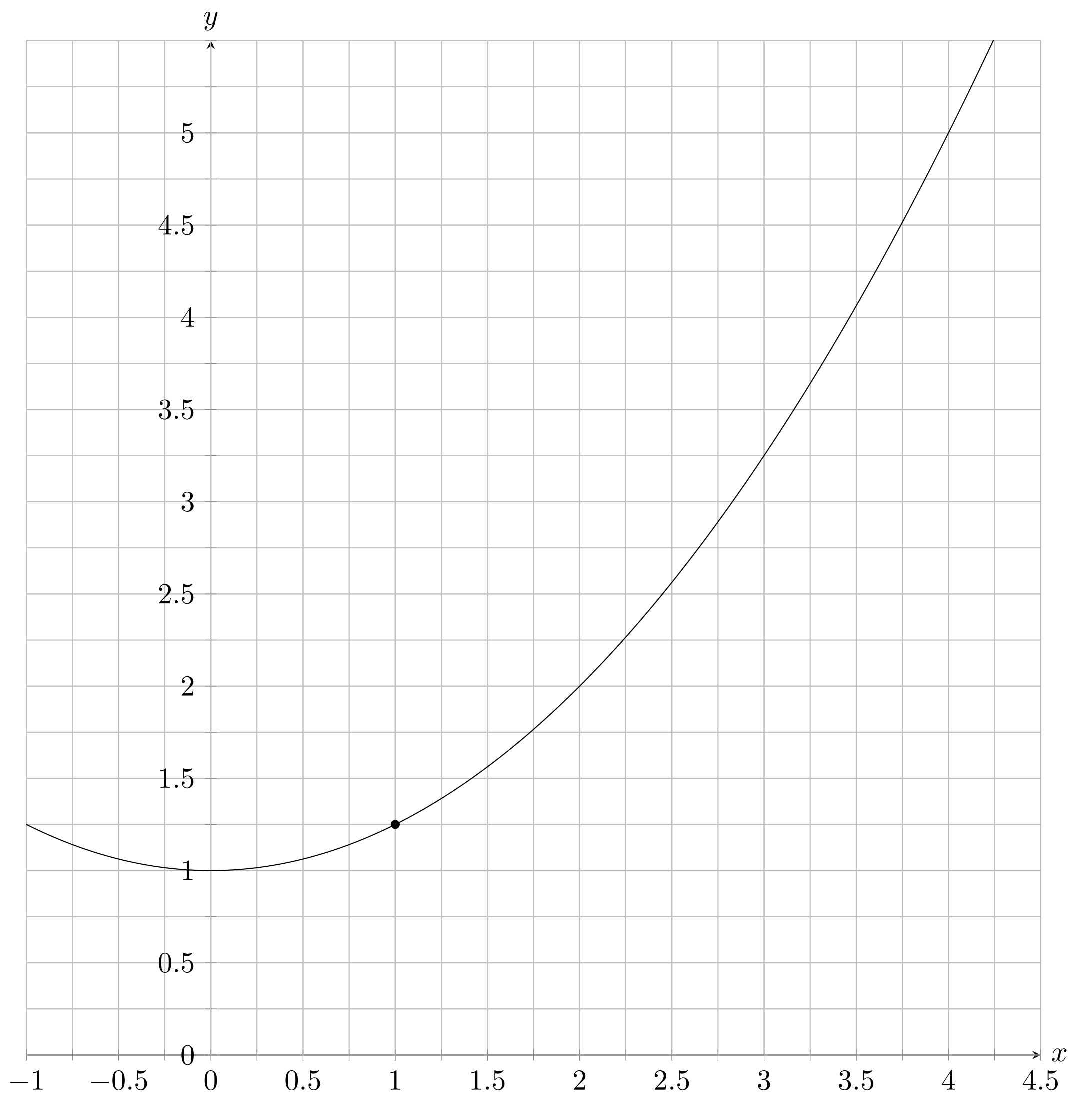
I guess that the first case (constant function) may actually be difficult for the student.

**Worksheet** (ch3-derivatives-1-derivatives-5-ws)**.** The goal of this activity is to help the students see when the derivative fails to exist (cf. Thomas [**?**, p. 129]). To do that we connect to prior knowledge about limits and ask the students in which cases a limit can fail to exist. The cases with the cusp as well as the two cases of discontinuity are probably harder to find as they are "embedded" in other cases.
See next activity for something that focuses more on the "geometry".

**Worksheet** (ch3-derivatives-1-derivatives-6-ws)**.** The goal of this activity is to help the students see when the derivative fails to exist (cf. Thomas [**?**, p. 129]). Here we focus on developing students' intuition about when a derivative fails to exist by looking at graphs of functions.
Only in the second step, do we make the connection with the limit definition.

The goal of this exercise is to see how we can determine the slope of the tangent line to a function at a given point.

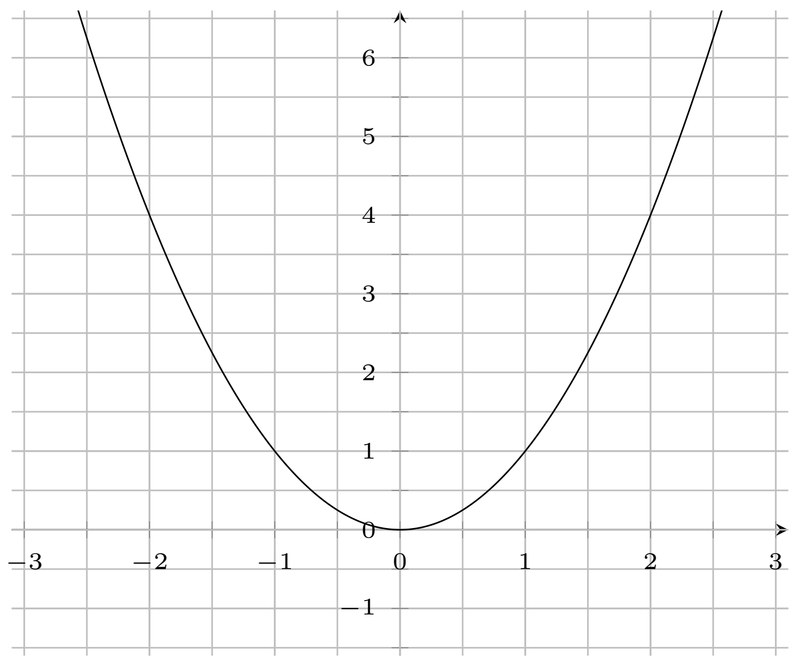
Let us consider the function $f(x)$, whose graph is shown below, and the point $x_0 = 1$. The point $(x_0, f(x_0)) = (1, f(1))$ is indicated on the graph.
Find as many ways you can to approximate the slope of the tangent line at that point.
What are the pros and cons of each method?
What method gives the "best" approximation (if any)? Why?



### Definition of the Derivative

For a function $f(x)$, its derivative is written $f'(x)$ (we call it "*f prime of x*"). Write down the two equivalent formulas for $f'(x)$.

Let us consider the function $f(x)$ below. Drawing the tangent line, we want to estimate the value of the derivative $f'(x)$ at the points $x_0 = -1$, $x_0 = 0$, $x_0 = 1$, $x_0 = 2$.




i) What is your estimate for $f'(-1)$ (this notation stands for the derivative of $f(x)$ at the point $x_0 = -1$)? What about $f'(0)$, $f'(1)$ and $f'(2)$?

ii) As you may have guessed, the function above is $f(x) = x^2$.
Using the definition of the derivative (that you have written above), compute the derivative $f'(1)$.
Does your answer correspond your estimate?

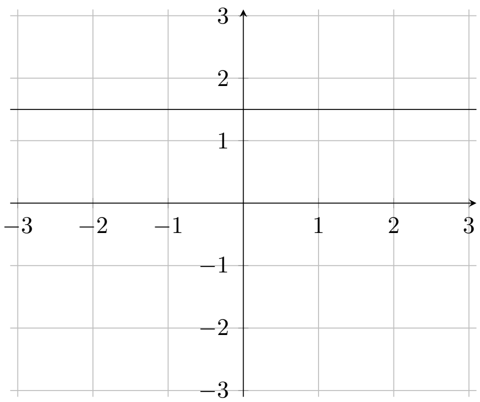
iii) Compute the derivative $f'(2)$ using the definition.

iv) Still using the definition of the derivative, compute $f'(x)$ for any $x$.

We have seen that the derivative of a function at a point is the slope of the tangent line. Keeping this in mind, for each of the following functions, sketch the graph of the derivative of the function. Your sketch doesn't have to be extremely precise, nevertheless the following aspects should be taken into account:

- where is the derivative positive, or negative,
- where is the derivative zero,
- where is the derivative increasing, or decreasing.

i)

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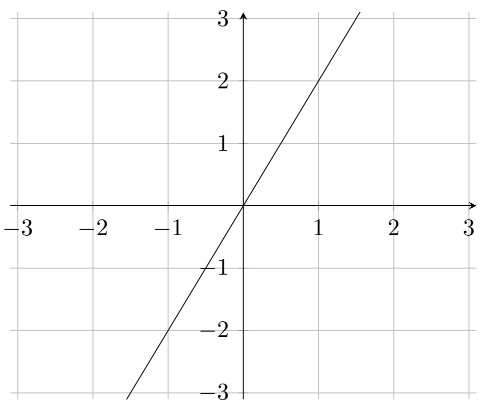


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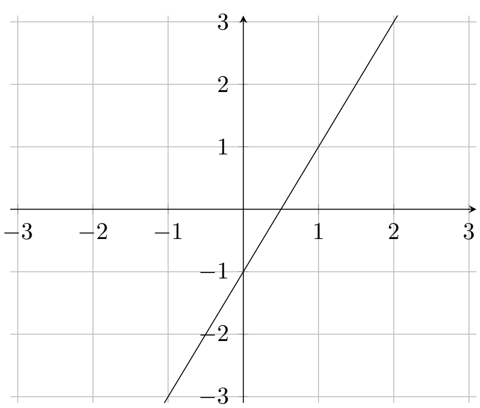

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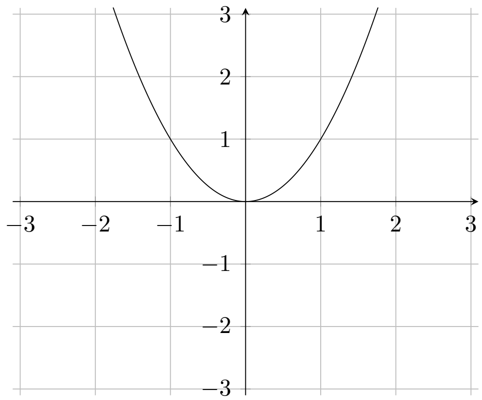


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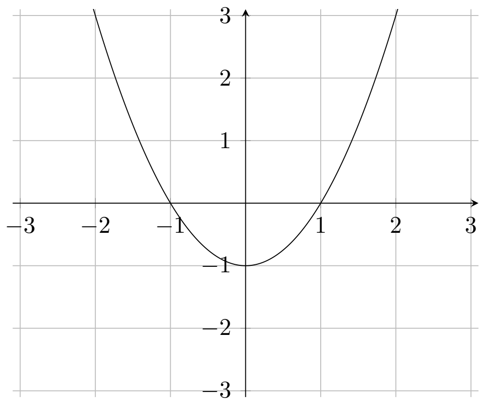
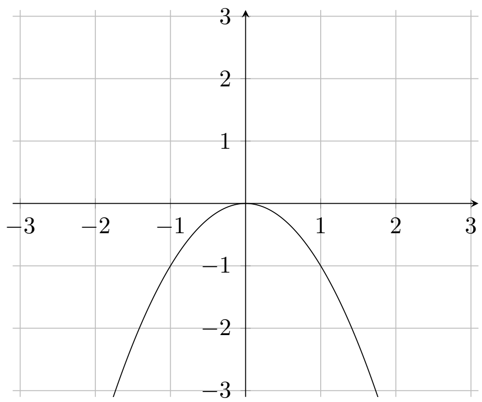
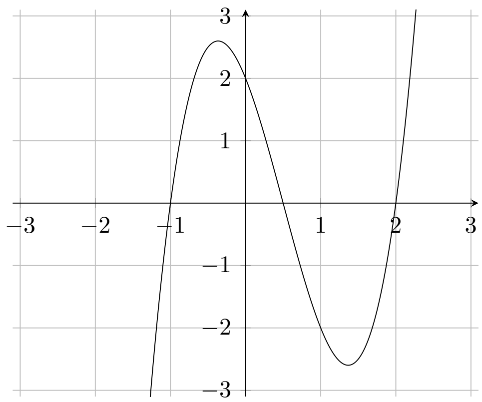



vi)




vii)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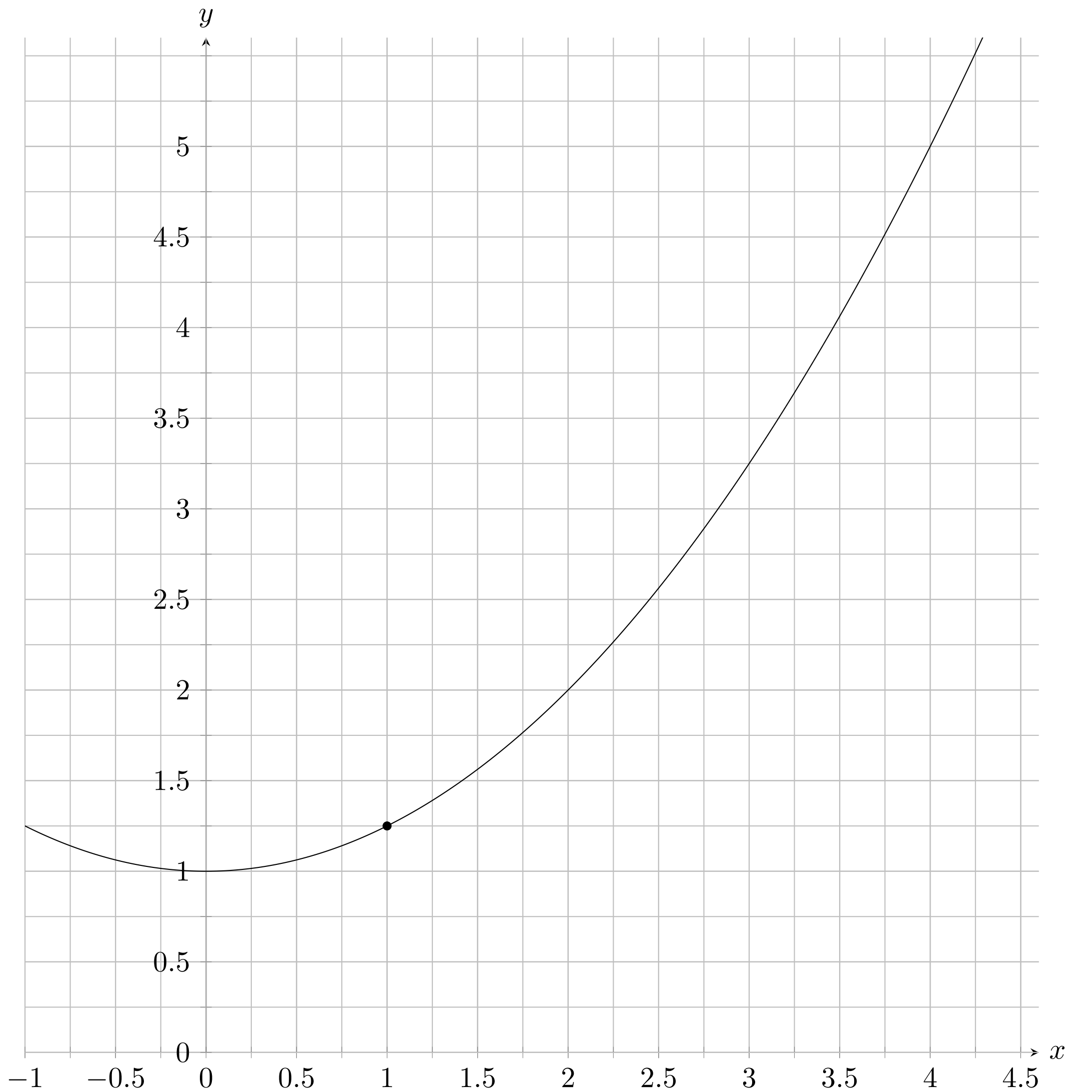
3

The goal of this exercise is to see how we can determine the slope of the tangent line to a function at a given point.

For this exercise, we consider a function $f(x)$ and we are looking for the slope of its tangent line at the point $x_0 = 1$ . To do so, we will look at secant lines that pass through the point $(x_0, f(x_0)) = (1, f(1))$ on the function.

1. Draw each of the lines below and estimate their slopes using the grid pattern of the graph.
   (a) The line that passes through $(1, f(1))$ and $(4, f(4))$,
   (b) The line that passes through $(1, f(1))$ and $(3, f(3))$,
   (c) The line that passes through $(1, f(1))$ and $(2, f(2))$,
   (d) The line that passes through $(1, f(1))$ and $(1.5, f(1.5))$,



2. Draw the tangent line to the function $f(x)$ at $x_0$.
3. Which of the secant lines you have drawn is the closest to the actual tangent line at $x_0$? How could you improve this process of approximating the tangent line further?
4. Let us now look at the secant line that passes through $(1, f(1))$ and $(4, f(4))$.
   (a) What is its slope?
   (b) Rewrite the formula of the slope using only $1, f(1), 4$, and $f(4)$).
   (c) Rewrite the formula of the slope using only $x_0, f(x_0), 4$, and $f(4)$).
5. For the line that passes through $(1, f(1))$ and $(2, f(2))$, write the formula of the slope using only $x_0, f(x_0), 2$, and $f(2)$).

The goal of this exercise is to explore when a function fails to have a derivative at a point.
We have seen that the definition of the derivative at a point $x$ is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}. \tag{1}$$

i) Explain in terms geometrical terms (more precisely, in terms of the graph of the function $f(x)$, as well as secant and tangent lines) what the limits in () "measure".

ii) List the cases for which a limit may fail to exist.

iii) For each of the cases you have listed at the previous point, what does it mean in terms of the slope of the tangent line that the limit fails to exist? For each case, how does the function look like (you draw a sketch for each case)? Why does such a shape prevent the derivative to exist?
**Warning**: some cases actually contain subcases.

*This last part may not be clear for the students. Try to be more precise orally.*

The goal of this exercise is to explore when a function fails to have a derivative at a point.
We have seen that the definition of the derivative at a point $x$ is given by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{z\to x} \frac{f(z)-f(x)}{z-x}. \tag{2}$$

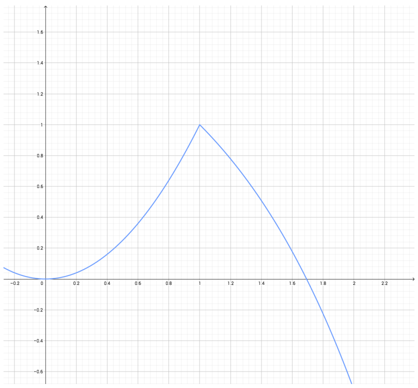
i) Explain in terms geometrical terms (more precisely, in terms of the graph of the function $f(x)$, as well as secant and tangent lines) what the limits in () “measure”.

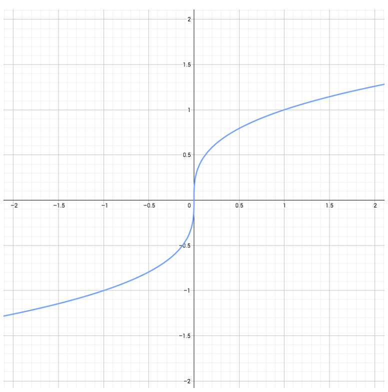
ii) List the cases for which a limit may fail to exist.

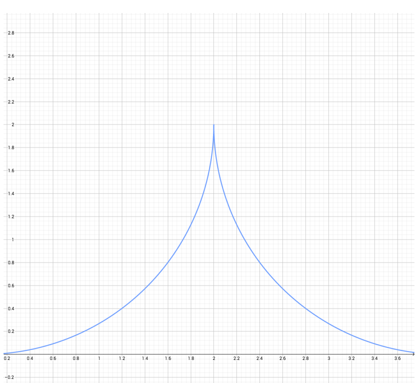
iii) For each of the cases you have listed at the previous point, what does it mean in terms of the slope of the tangent line that the limit fails to exist? For each case, how does the function look like (you draw a sketch for each case)? Why does such a shape prevent the derivative to exist?
**Warning**: some cases actually contain subcases.

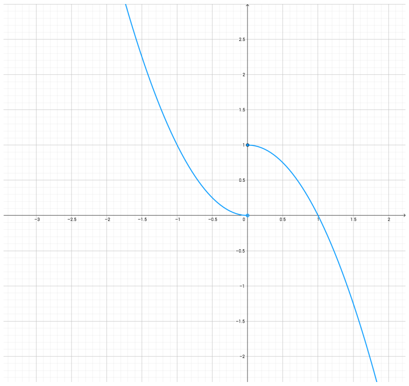
The goal of this exercise is to explore and determine the cases where a function fails to have a derivative at a point.

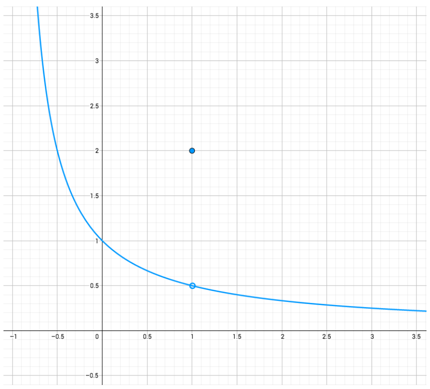
Consider the following graphs. For each of them, indicate the point(s) for which the function fails to be differentiable and give a short explanation of why it fails to be differentiable.

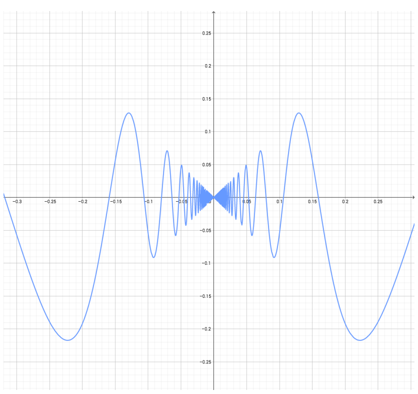
1. 

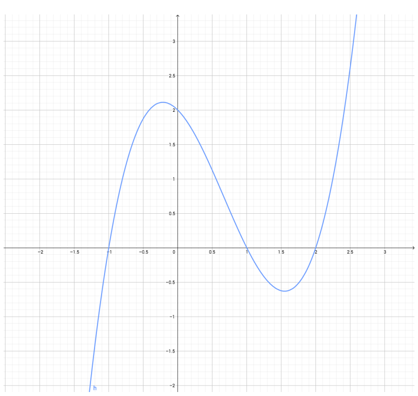
2. 

3. 

4. 

5. 


6. 

7. 

We have seen that the definition of the derivative at a point $x$ is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$

For the cases 1, 2, 3, and 6 above, explain what happens in terms of this limit.

# Differentiation Rules (3.3)

**Expected Skills.**
At the end of this section, students will be able to:

- correctly use the differentiation rules presented in the section (derivative of a constant, power rule, constant multiple rule, sum rule, natural exponential rule, product rule, quotient rule),
- compute the equation of the tangent line at a given point using these rules.

**Pre-Class Activity** (ch3-derivatives-2-rules-1-pc)**.** The goal of the pre-class activity is to have the students compute a derivative using the definition of the derivative and seeing how long or "painful" it can be. This is thus conceived as a motivation for having differentiation rules.

Alternatively, one can ask the students to prove one of the differentiation rule (still using the definition of the derivative): the rule for $f(x) - g(x)$ seems to be a good compromise as it is not too difficult to prove and it is not presented directly in the textbook. This other activity would serve the same purpose but would focus a bit more on how we "come up" (i.e. prove) with differentiation rules.
*If you don't use it as a pre-class activity, then it could be good to prove one rule in class.*

The second question simply ask the students to list the rules presented in the textbook to make sure they have all of them when they come to class and can use them directly to solve exercises.

**Worksheet** (ch3-derivatives-2-rules-2-ws)**.** The goal of the activity is to have the students use differentiation rules on more and more "difficult" cases.
We then ask them to compare using the quotient rule with "simple" fractions. The goal is to have them realize that in such situations it is easier to simplify the fraction or write the variable with a negative exponent rather than using the quotient rule.
The last part focuses on having the students being able to justify why $(f(x)g(x))' \neq f'(x)g'(x)$.

1. Using the definition of the derivative, compute the derivative of $f(x) = \dfrac{x}{2x+1}$, i.e., compute the limit $\displaystyle\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$.

2. List all the differentiation rules presented in section 3.3 of the textbook.

3. Compute the derivative of $f(x) = \dfrac{x}{2x+1}$ using differentiation rules.
   Which method is easier compared to computing the derivative using the definition (question 1)?

1. Using the differentiation rules, compute the derivatives of the following functions:

   (a) $f(x) = 3x^4 - 2x^3 + 2x - 5$

   *In this first part we ask students to use the differentiation rules for cases of increasing complexity. The goal in this part is "just" to have them use the rules.*
   *Adding one or two challenge exercises can be good here as some students will already know this and will finish these exercises very quickly.*

   (b) $f(x) = ax^3 + bx^2 + cx + d$ where $a, b, c$ and $d$ are constants

   (c) $f(x) = \dfrac{x-2}{x+2}$

   (d) $f(t) = \dfrac{3t+1}{t^2+t-2}$

   (e) $f(x) = 2x^3e^x$

2. Compute the derivatives of the following functions using each time two different methods, once using the quotient rule and once without using the quotient rule. In each case, which method is easier?
*Here we want the students to compare ways of computing derivatives.*

   (a) $f(t) = \dfrac{t^3 + 5t^2 - 2t}{t}$ and $g(t) = \dfrac{t}{t^3 + 5t^2 - 2t}$

   (b) $f(x) = \dfrac{3}{x^4}$ and $g(x) = \dfrac{x^2 + x + 1}{\sqrt{x}}$

   (c) Based on the calculation you have done, when is it easier *not* to use the quotient rule?

3. The goal of this exercise is to see why $(f(x)g(x))' \neq f'(x)g'(x)$.

   (a) A friend of yours claims (contrary to what the textbook says) that the product rule *is* $(f(x)g(x))' = f'(x)g'(x)$. You want to show him that his claim is wrong. If you compute the derivative of $f(x) = x^2$ using your friend's differentiation rule, what do you get?

   *the next parts of this exercise are probably not "worth" the time they take. One could just stop here and still get the message across.*

   (b) Why is this answer incorrect? Give a graphical argument.

   *Give some directions to the students here. We are looking for something similar to the figure on p. 139.*

   (c) Find as many other arguments you could give to your friend to show him his claim is wrong (e.g. by computing the derivative of $x^2$ using other differentiation rules and see that the answers don't match with the answer obtained above).

   *E.g. one can use the definition of the derivative, or with the power rule. There are probably also other ways of showing that the above computation is wrong.*

(d) Let $p(x)$ be $p(x) = f(x)g(x)$ with $f(x)$ and $g(x)$ differentiable.
Write down the limit definition of $p'(x)$. Explain in words what this limit represent.

(e) Let us consider a graphical representation of the situation. Let $f(x)$ be the width of a rectangle and $g(x)$ its height. Let us suppose that both $f(x)$ and $g(x)$ are positive and growing with time. How is this picture you have drawn connected to the limit you have written above? (You can also compare to what you would get using $f'(x)g'(x)$).

1. Using the differentiation rules, compute the derivatives of the following functions:

   (a) $f(x) = 3x^4 - 2x^3 + 2x - 5$

   (b) $f(x) = ax^3 + bx^2 + cx + d$ where $a, b, c$ and $d$ are constants

   (c) $f(x) = \dfrac{x-2}{x+2}$

   (d) $f(t) = \dfrac{3t+1}{t^2+t-2}$

   (e) $f(x) = 2x^3e^x$

2. Compute the derivatives of the following functions using each time two different methods, once using the quotient rule and once without using the quotient rule. In each case, which method is easier?

   (a) $f(t) = \dfrac{t^3 + 5t^2 - 2t}{t}$ and $g(t) = \dfrac{t}{t^3 + 5t^2 - 2t}$

   (b) $f(x) = \dfrac{3}{x^4}$ and $g(x) = \dfrac{x^2 + x + 1}{\sqrt{x}}$

   (c) Based on the calculation you have done, when is it easier *not* to use the quotient rule?

3. The goal of this exercise is to see why $(f(x)g(x))' \neq f'(x)g'(x)$.

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   (b) Why is this answer incorrect? Give a graphical argument.

   (c) Find as many other arguments you could give to your friend to show him his claim is wrong (e.g. by computing the derivative of $x^2$ using other differentiation rules and see that the answers don't match with the answer obtained above).

(d) Let $p(x)$ be $p(x) = f(x)g(x)$ with $f(x)$ and $g(x)$ differentiable.
Write down the limit definition of $p'(x)$. Explain in words what this limit represent.

(e) Let us consider a graphical representation of the situation. Let $f(x)$ be the width of a rectangle and $g(x)$ its height. Let us suppose that both $f(x)$ and $g(x)$ are positive and growing with time. How is this picture you have drawn connected to the limit you have written above? (You can also compare to what you would get using $f'(x)g'(x)$).

# Derivatives of Trigonometric Functions (3.5)

**Expected Skills.**
At the end of this section, students will be able to:

- compute the derivative of trigonometric functions (sin x, cos x, tan x).

*Note: We will focus only on the derivatives of* $\sin x, \cos x$ *and* $\tan x$*. Indeed, the derivatives of* $\sec x, \csc x$ *and* $\cot x$ *can be found from there. We thus don't talk about it.*

**Pre-Class Activity** (ch3-derivatives-3-trig-1-pc)**.** The goal of this pre-class activity is to have the students compute $\lim_{h\to 0} \frac{\cos h - 1}{h}$ as it is used to compute the derivatives of $\sin x$ and $\cos x$ (cf. class activity below).

**Worksheet** (ch3-derivatives-3-trig-2-ws)**.** The first part of the activity is to have the students make an "eudcated" guess about the derivative of $\sin x$.
In the second part, we actually prove the formula and in the third part have the students compute the derivative of $\tan x$.
The fourth part is composed of "mechanical" exercises using these derivatives. The fifth part is a simple application.

**Supplemental Activity.** The goal of this activity is to have the students derive the derivative of the sine and cosine functions using the squeeze theorem and other derivative rules. It is suggested that the instructor use one as a student exercise and the other as a board problem. This activity would also be useful to begin discussions about derivatives of other trigonometric functions.

The goal of this exercise is to compute the limit $\lim_{h\to 0}\frac{\cos h - 1}{h}$ that we will need in class to compute the derivatives of trigonometric function. Moreover, since limits are so fundamental, It is a good opportunity to review them.

1. What is $\lim_{h\to 0}\frac{\sin h}{h}$ ? (We have computed it in a previous section and it is an important limit to know).

2. One of the trigonometric identities for the half-angle is $\sin^2(t/2) = \frac{1}{2}(1-\cos t)$ or equivalently, $\cos t = 1 - 2\sin^2(t/2)$.
   How could this identity be used to compute the limit $\lim_{h\to 0}\frac{\cos h - 1}{h}$? Explain in words.

3. We now want to compute the limit $\lim_{h\to 0}\frac{-\sin^2(h/2)}{(h/2)}$. Compute this limit using part 1 and limit laws. What limit laws have you used?

1. The goal of this first part is to make a first "educated guess" of what the derivative of $f(x) = \sin x$ is.

   (a) Draw the graph of $\sin x$ on the interval $[-2\pi, 2\pi]$ (do that on page 38, leave some room below your graph to draw another graph).

   (b) On the graph you have drawn, for what point(s) can you determine the slope of the tangent line without any doubt?

   *Where the tangent line is horizontal.*

   (c) What do you think is the value of the derivative at $x = 0$?
   Confirm your guess by computing $f'(0)$ (i.e. the derivative of $\sin x$ at $x = 0$) using the definition of the derivative.

   (d) Using the information above as well as the shape of the graph of $\sin x$, draw, below the graph of $\sin x$, the graph of $f'(x) = (\sin x)'$.

2. Let us now prove (or disprove) that your guess is correct by computing the actual derivative of $\sin x$.

   (a) Writing the $\lim_{h\to 0}$ definition of the derivative, write down the definition of $f'(x) = (\sin x)'$.

   (b) Let us compute this limit.
   To that end, we will need the trigonometric identity $\sin(u+v) = \sin u \cos v + \cos u \sin v$. We will moreover need the limits $\lim\limits_{h\to 0} \dfrac{\sin h}{h}$ and $\lim\limits_{h\to 0} \dfrac{\cos h - 1}{h}$.

   *If enough time we can ask which limits laws have been used and why we can use them.*

   (c) So, the derivative of $\sin x$ is [1]:

   $$\boxed{(\sin x)' = \qquad\qquad}$$

[1] The goal of this exercise was to have you compute the derivative of the $\sin x$ using the definition. While you need to know what the derivative of $\sin x$ is, you don't need by heart how to compute it, i.e. you don' t need to learn by heart the above computations.

3. Similarly to what we have done above, one can prove that $(\cos x)' = -\sin x$.
   Using these two derivatives as well as the definition of $\tan x$, compute the derivative $(\tan x)'$.

4. Compute the derivatives of the following functions.

   i) $f(t) = \sin t \cos t$

   ii) $g(x) = \sin x + e^x \cos x$

   iii) $h(z) = 2 \tan z + \dfrac{3z}{\cos z}$

   iv) $f(x) = \sqrt{x} + \dfrac{\cos x}{\sin x} + 4$

5. Does the graph of the function $f(t) = t + \cos t$ have a horizontal tangent line on the interval $[0, 2\pi]$? What about $g(t) = 2t + \cos t$?

Draw your graphs below.

1. The goal of this first part is to make a first "educated guess" of what the derivative of $f(x) = \sin x$ is.

   (a) Draw the graph of $\sin x$ on the interval $[-2\pi, 2\pi]$ (do that on page 38, leave some room below your graph to draw another graph).

   (b) On the graph you have drawn, for what point(s) can you determine the slope of the tangent line without any doubt?

   (c) What do you think is the value of the derivative at $x = 0$?
   Confirm your guess by computing $f'(0)$ (i.e. the derivative of $\sin x$ at $x = 0$) using the definition of the derivative.

   (d) Using the information above as well as the shape of the graph of $\sin x$, draw, below the graph of $\sin x$, the graph of $f'(x) = (\sin x)'$.

2. Let us now prove (or disprove) that your guess is correct by computing the actual derivative of $\sin x$.

   (a) Writing the $\lim_{h\to 0}$ definition of the derivative, write down the definition of $f'(x) = (\sin x)'$.

   (b) Let us compute this limit.
   To that end, we will need the trigonometric identity $\sin(u+v) = \sin u \cos v + \cos u \sin v$. We will moreover need the limits $\displaystyle\lim_{h\to 0} \frac{\sin h}{h}$ and $\displaystyle\lim_{h\to 0} \frac{\cos h - 1}{h}$.

   (c) So, the derivative of $\sin x$ is [2]: $\boxed{(\sin x)' = \quad}$

[2] The goal of this exercise was to have you compute the derivative of the $\sin x$ using the definition. While you need to know what the derivative of $\sin x$ is, you don't need by heart how to compute it, i.e. you don' t need to learn by heart the above computations.

3. Similarly to what we have done above, one can prove that $(\cos x)' = -\sin x$.
   Using these two derivatives as well as the definition of $\tan x$, compute the derivative $(\tan x)'$.

4. Compute the derivatives of the following functions.

   i) $f(t) = \sin t \cos t$

   ii) $g(x) = \sin x + e^x \cos x$

   iii) $h(z) = 2\tan z + \dfrac{3z}{\cos z}$

   iv) $f(x) = \sqrt{x} + \dfrac{\cos x}{\sin x} + 4$

5. Does the graph of the function $f(t) = t + \cos t$ have a horizontal tangent line on the interval $[0, 2\pi]$? What about $g(t) = 2t + \cos t$?

Draw your graphs below.

1) The goal of this first part is to make a first "educated guess" of what the derivative of $f(x) = \cos x$ is.

   (a) Draw the graph of $\cos x$ on the interval $[-2\pi, 2\pi]$ (do that on page 40, leave some room below your graph to draw another graph).

   (b) On the graph you have drawn, for what point(s) can you determine the slope of the tangent line without any doubt?

   (c) What do you think is the value of the derivative at $x = \pi/2$? Confirm your guess by computing $f'(\pi/2)$ (i.e. the derivative of $\cos x$ at $x = \pi/2$) using the definition of the derivative.

(d) Using the information above as well as the shape of the graph of $\cos x$, draw the graph of $f'(x) = (\sin x)'$.

Draw your graphs below.

2) Let us now prove (or disprove) that your guess is correct by computing the actual derivative of $\cos x$.

   (a) Writing the $\lim_{h\to 0}$ definition of the derivative, write down the definition of $f'(x) = (\cos x)'$.

   (b) Let us compute this limit.
   To that end, we will need the trigonometric identity $\cos(u+v) = \cos u \cos v - \sin u \sin v$. We will moreover need the limits $\lim\limits_{h\to 0} \dfrac{\sin h}{h}$ and $\lim\limits_{h\to 0} \dfrac{\cos h - 1}{h}$.

   (c) So, the derivative of $\cos x$ is [3]:

   $$\boxed{(\cos x)' = \qquad}$$

[3]The goal of this exercise was to have you compute the derivative of the $\cos x$ using the definition. While you need to know what the derivative of $\cos x$ is, you don't need by heart how to compute it, i.e. you don' t need to learn by heart the above computations.

# The Chain Rule (3.6)

**Expected Skills.**
At the end of this section, students will be able to:

- recognize when the chain rule is needed,
- appropriately apply the chain rule to compute derivatives of functions.

**Pre-Class Activity** (ch3-derivatives-4-chain-1-pc)**.** The goal here is twofold. First, to connect to the previous differentiation rules we have seen and point out that they are not sufficient for composed functions. Second, to have the students work on composing functions as preparation for the chain rule.

**Worksheet** (ch3-derivatives-4-chain-2-ws)**.** In this activity we have the students use the chain rule on cases with increasing difficulty. We also use some contrasting cases to help the students identify when the chain rule is used from when it is not.
The goal of the second part is to have the students focus on the chain rule itself and not so much on formulas. The goal is also to have students be able to *read* information on a graph.
The last exercise to have the students think about we implicitly use the chain rule "all the time" but we don't see it because we don't write the "$x' = 1$" part.
The challenge exercise is only for students who have finished everything else and are bored.

If there is enough time, one could add an exercise on the composition of functions (e.g. ask the student the compose the functions of the first exercise. In this part, one could also add the students to compose functions (e.g. compose the functions of this exercise).
One could also add an exercise where the values of the functions and derivatives are given at certain points and then we ask the students to compute the derivative of composition of functions (cf. GranValley: [p. 132][gv17])?

**Supplemental Activity** (ch3-derivatives-4-chain-3-sup-chain)**.** This supplemental activity challenges students to provide justifications for why common student misconceptions of the chain rule are incorrect. The activity also challenges the students to explain the product rule with their own words. It is suggested that students within their small groups split up the misconceptions and report back their individual findings to their group.

1. For each of the following functions, explain which differentiation rule(s) apply.
   (You don't need to actually compute the derivatives).

   (a) $f(x) = 2x^4 - 3\cos x$,

   (b) $g(t) = 5te^t$,

   (c) $h(z) = \dfrac{z^2}{\sin z}$

2. Let us now look at the function $f(x) = \sin(x^2)$. What differentiation rule(s) apply here?
   What do you think is $f'(x)$? Check your answer by graphing $f(x)$ and $f'(x)$ on GeoGebra or Desmos.
   Was your answer correct?

3. Same question with for $f(t) = \sin(4t)$? (This time you have to look more closely at the graph).

4. What is going on in these examples? Why "simply" applying directly the rules we have learned so far doesn't work? What feature makes the whole thing fail?

In class we will use composition of functions. It is thus important to understand how it works.

If $f(x) = \sin x$ and $g(x) = 3x + 1$, then $f \circ g(x) = f(g(x))$ is given by $x \xrightarrow{g} 3x + 1 \xrightarrow{f} \sin(3x + 1)$, i.e. $f(g(x)) = \sin(3x + 1)$.
Conversely, given the function $h(x) = e^{\sqrt{x}}$, we can decompose it as $x \xrightarrow{g} \sqrt{x} \xrightarrow{f} e^{\sqrt{x}}$. Here we thus have $h(x) = f(g(x)) = e^{\sqrt{x}}$ with $f(x) = e^x$ and $g(x) = \sqrt{x}$.

Fill in the blanks with the appropriate functions:

$$\begin{array}{lll}
\text{if } f(x) = 1/x, & g(x) = 3x^2 + x + 4, & \text{then } f(g(x)) = \ldots\ldots\ldots, \\
\text{if } f(x) = \ldots\ldots\ldots, & g(x) = \sin x, & \text{then } f(g(x)) = \cos(\sin x)), \\
\text{if } f(x) = e^x, & g(x) = \ldots\ldots\ldots, & \text{then } f(g(x)) = e^{2x^3}, \\
\text{if } f(x) = \ldots\ldots\ldots, & g(x) = 2x^2 + 3, & \text{then } f(g(x)) = \sqrt{2x^2 + 3}, \\
\text{if } f(x) = \tan x, & g(x) = \ldots\ldots\ldots, & \text{then } f(g(x)) = \tan(x^3 + x + 6).
\end{array}$$

i) Compute the derivatives of the following functions (using the appropriate rules). Here you do not need to simplify your answer.

(a) $f(t) = \cos(t^2)$

(b) $g(x) = \sqrt{2x^3 + 4x + 2}$

(c) $h(z) = 2e^{z^2+4z} + 5z + 3$

(d) $k(x) = \sin(x) \cdot (x^2 + 5x)^{100}$

(e) $f(x) = \sin^2(x) \cdot (x^2 + 5x)^{100}$

(f) $g(t) = \dfrac{e^t + 1}{\sin(t^4)}$

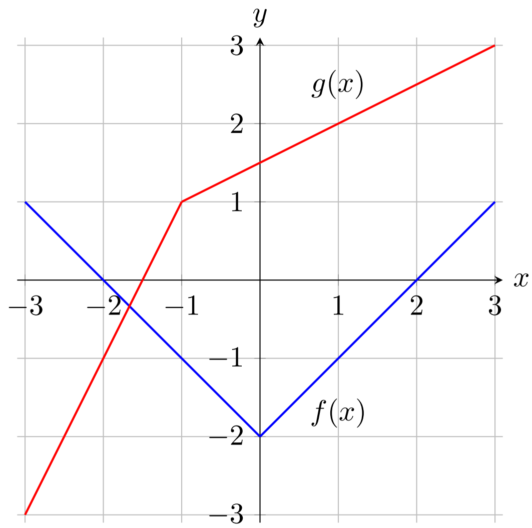
(g) $h(t) = \dfrac{e^{t^3+t} + 1}{\sin(t^4)}$

(h) $k(z) = e^{\cos(z^2)}$

(i) $f(x) = \tan^3(\sqrt{x^5 + 2})$

*For this exercise, once the students have made their computations, one can also ask them to indicate for each answer which part of the answer correspond to the derivative of what function.*

ii) Consider the two functions $f(x)$ and $g(x)$ below.




(a) Let $h(x) = f(g(x))$. Find: $h'(-2)$, $h'(0)$ and $h'(1)$.

(b) Let $h(x) = g(f(x))$. Find: $h'(-2)$, $h'(0)$ and $h'(1)$.

(c) Let $h(x) = f(f(x))$. Find: $h'(-2)$, $h'(0)$ and $h'(1)$.

(d) Let $h(x) = (f(x))^2$. Find: $h'(-2)$, $h'(0)$ and $h'(1)$.

iii) While revising your course with a friend, this one claims "The chain actually applies all the time!". What do you think of this claim? To what extent is it correct, or incorrect?

*The idea is to have the students realize that the derivative of $x$ is 1 and that we don't write it. In that sense, the chain rule is "always applied".*

**Challenge exercises**

A. Use the chain rule to find the derivative of $\dfrac{1}{g(x)}$.

B. Use part A., the chain rule and the product rule to prove the quotient rule.

i) Compute the derivatives of the following functions (using the appropriate rules). Here you do not need to simplify your answer.

(a) $f(t) = \cos(t^2)$

(b) $g(x) = \sqrt{2x^3 + 4x + 2}$

(c) $h(z) = 2e^{z^2+4z} + 5z + 3$

(d) $k(x) = \sin(x) \cdot (x^2 + 5x)^{100}$

(e) $f(x) = \sin^2(x) \cdot (x^2 + 5x)^{100}$

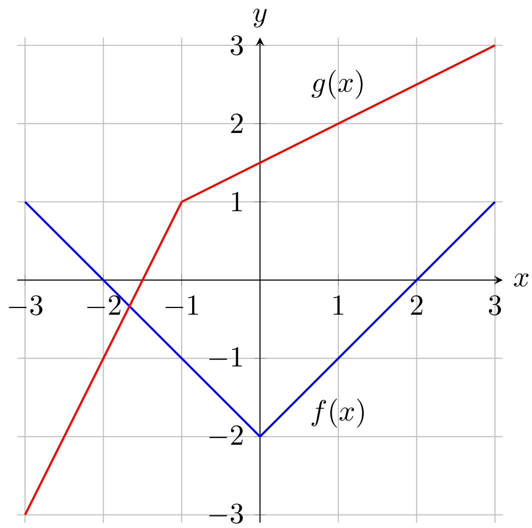
(f) $g(t) = \dfrac{e^t + 1}{\sin(t^4)}$

(g) $h(t) = \dfrac{e^{t^3+t} + 1}{\sin(t^4)}$

(h) $k(z) = e^{\cos(z^2)}$

(i) $f(x) = \tan^3\left(\sqrt{x^5 + 2}\right)$

ii) Consider the two functions $f(x)$ and $g(x)$ below.




(a) Let $h(x) = f(g(x))$. Find: $h'(-2)$, $h'(0)$ and $h'(1)$.

(b) Let $h(x) = g(f(x))$. Find: $h'(-2)$, $h'(0)$ and $h'(1)$.

(c) Let $h(x) = f(f(x))$. Find: $h'(-2)$, $h'(0)$ and $h'(1)$.

(d) Let $h(x) = (f(x))^2$. Find: $h'(-2)$, $h'(0)$ and $h'(1)$.

iii) While revising your course with a friend, this one claims "The chain actually applies all the time!". What do you think of this claim? To what extent is it correct, or incorrect?

**Challenge exercises**

A. Use the chain rule to find the derivative of $\dfrac{1}{g(x)}$.

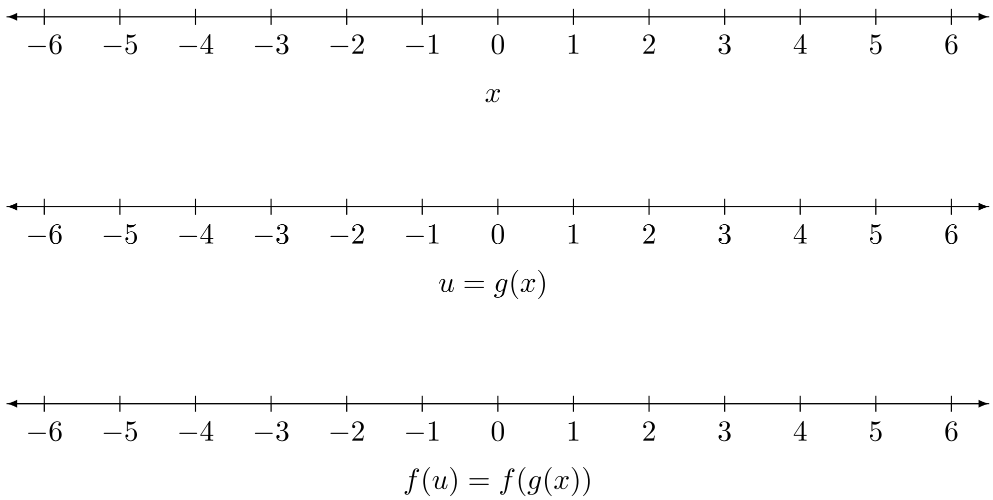
B. Use part A., the chain rule and the product rule to prove the quotient rule.

The goal of this exercise is to see why $f(g(x))' = f'(g(x))g'(x)$.

1) A friend of yours claims that the chain rule is $f(g(x))' = f'(x)g'(x)$. Another friend claims that the chain rule is $f(g(x))' = f'(g'(x))$. Compute the derivative of $f(x) = x^6$ using your each friend's differentiation rule, what do you get?

2) Why are these alternative answers incorrect? Find a few other functions that can serve as counterexamples to your friends' claims.

3) Let us consider a graphical representation of the situation using three number lines. How can you use these number lines to explain the chain rule?




4) Let $p(x)$ be $p(x) = f(x)g(x)$ with $f(x)$ and $g(x)$ differentiable. Write down the limit definition of $p'(x)$. Explain in words what this limit represent. (Hint: Try using a rectangle to help you!)

# Implicit Differentiation (3.7)

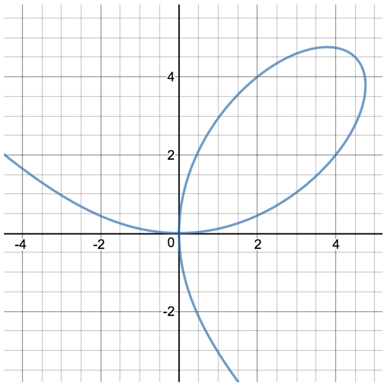
**Expected Skills.**
At the end of this section, students will be able to:

- explain the interest of using implicitly defined functions,
- explain how to implicitly differentiate functions and when it applies,
- recognize when implicit differentiation applies and use it correctly to differentiate implicitly defined functions,
- use this process to compute the equations of tangent or normal lines to a given curve.

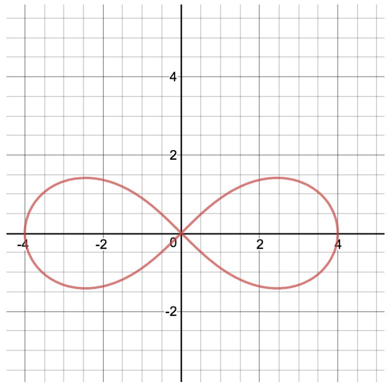
**Pre-Class Activity** (ch3-derivatives-5-implicit-1-pc)**.** The goal of the pre-class activity is twofold. First, to provide motivation of why defining curves or functions implicitly is interesting. Second, to have the students understand the idea that when using implicitly defined function, $y$ is actually a function of $x$ and therefore the chain rule applies (i.e. the derivative of $y$ is $y'$).

**Worksheet** (ch3-derivatives-5-implicit-2-ws)**.** In class, one should start making the connection with the pre-class activity by asking about the interest of defining functions implicitly.
One can then move to computing derivatives. The class activities focus on having the students make such computations (computing derivatives and equations of tangent lines).

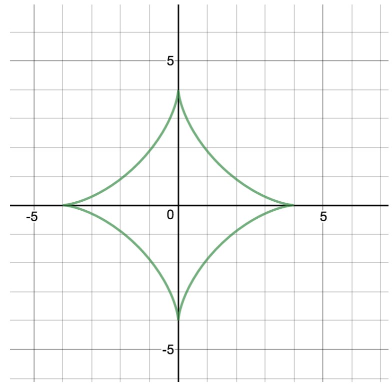
i) For each of the curves presented below, how many functions do we need to describe them and why? Moreover, can we describe any of them with a single function?




(a) Folium of Descartes
$x^3 + y^3 = 9xy$




(b) Lemniscate of Bernoulli
$x^4 + 2x^2y^2 + y^4 = 16(x^2 - y^2)$




(c) Astroid
$x^{2/3} + y^{2/3} = 4^{2/3}$

ii) Using the chain rule (that we studied last time), what is the derivative of the function $(3\sin x + 4x)^2$?

iii) Let us suppose we now decide to set $y = y(x) = 3\sin x + 4x$. Rewrite the above function and its derivative using only $y$ and $y'$.

iv) Compute the derivative of $\cos(x^3 + 2x + 5)$.

v) Let us set $y = y(x) = x^3 + 2x + 5$. Rewrite the above function and its derivative using $y$ and $y'$ but not $x$.

1. Compute the derivative $y'$ of $x^3 - xy + y^3 = 1$.

2. Compute the derivative $y'$ of $(xy + 7)^2 = 2y$.

3. Compute the equation of the tangent line to the curve $2xy + \pi \sin y = 2\pi$ at $(1, \pi/2)$.

4. Consider the folium of Descartes whose equation is $x^3 + y^3 = 9xy$. Find $y'$.
   Then explain the steps to find the point(s) at which the tangent line is horizontal.

   *It this is too easy, one can then ask: find the coordinates of these points* .

5. Compute the slope of the tangent line of the lemniscate $x^4 + 2x^2y^2 + y^4 = 16(x^2 - y^2)$ at the point $(4, 0)$.

   *It students are done, one can ask to compute extra derivatives (or tangent lines), for example for the curves:* $e^{2x} = \sin(x + 3y)$ *and* $(x^2 + y^2)^2 = 16x^2 - 16y^2$ *(which is also a lemniscate).*

1. Compute the derivative $y'$ of $x^3 - xy + y^3 = 1$.

2. Compute the derivative $y'$ of $(xy + 7)^2 = 2y$.

3. Compute the equation of the tangent line to the curve $2xy + \pi \sin y = 2\pi$ at $(1, \pi/2)$.

4. Consider the folium of Descartes whose equation is $x^3 + y^3 = 9xy$. Find $y'$.
   Then explain the steps to find the point(s) at which the tangent line is horizontal.

   .

5. Compute the slope of the tangent line of the lemniscate $x^4 + 2x^2y^2 + y^4 = 16(x^2 - y^2)$ at the point $(4, 0)$.

## Derivatives of Inverse Functions and Logarithms (3.8)

**Expected Skills.**
At the end of this section, students will be able to:

- compute the derivative of the inverse of a function,
- compute the derivatives of $\ln(x)$, $a^x$, $\log_a(x)$,
- explain in mathematical terms why the derivative of the inverse function is the reciprocal of the derivative,

**Pre-Class Activity** (ch3-derivatives-6-inverse-1-pc)**.** The goal of the pre-class activity is threefold. First, present the general question/problem we want to solve. Second, make the link with prior knowledge on inverse functions. And third, have the students develop an intuition on what $(f^{-1}(x))'$ may be based on graphs (this intuition will then be confirmed with computations in class).

**Worksheet** (ch3-derivatives-6-inverse-2-ws)**.** This activity focuses first on proving the formula for the derivative of the inverse function and when it applies. We then have the students compute/find the formulae for $(\ln x)'$, $(a^x)'$ and $\log_a x$ (giving the last one). Finally, we focus on using these formulae to compute derivatives.

**Supplemental Activity** (ch3-derivatives-6-inverse-3-sup-applications)**.** This activity focuses on the core ideas of using function composition, the chain rule, and implicit differentiation to compute the derivative of the inverse of a function. Students are then asked to take these ideas and apply them to determine derivatives of inverse trigonometric functions as well as to compute the derivative of rational functions.

In class, we will inquire the following question: giving a function $f(x)$ and its derivative $f'(x)$, can we compute the derivative $(f^{-1}(x))'$ of the inverse function ? And if so, what is the relationship between the two derivatives $f'(x)$ and $(f^{-1}(x))'$? (This result will enable us to compute the derivatives of functions such as $\log_a(x), \ln x$ or $\arctan x$. )

The goal here is to develop an intuition of what the answers to the above questions may be.

1. Let us consider the function $f(x) = x^2 + 1$. What is its inverse function $f^{-1}(x)$?

2. Is it always true? Or more precisely, on what interval(s) is this true?

3. Draw the graph of $f(x)$. Then, draw the graph of $f^{-1}(x)$.
   How do you get the graph of the inverse $f^{-1}(x)$ from the graph of $f(x)$?

4. On the graph you have drawn above, choose a point $(x_0, f(x_0))$ on the graph of $f(x)$. How do you find the "corresponding" point on $f^{-1}(x)$? What are the coordinates of this point?

5. Draw the tangent lines at $(x_0, f(x_0))$ and $(f^{-1}(x_0), x_0)$. What do you notice about the slopes of these two tangent lines (you can approximately measure them)?

1. The goal of this first part is to compute the derivative of the inverse function $f^{-1}(x)$ of a function $f(x)$ (and this way prove the formula).

   (a) Write down the chain rule.

   *The goal of this exercise is to have the students prove the formula for $(f^{-1}(x))'$.*

   (b) Let $f(x)$ be a function and $f^{-1}(x)$ its inverse.
   What is $f(f^{-1}(x))$ equal to?

   (c) Take the equation from the preceding point and differentiate both sides with respect to $x$.
   What do you get? Have you used the chain rule? If so, where?

   (d) Use what you have found to determine the derivative $(f^{-1}(x))'$.

   (e) What do we assume when using the previous formula?

   *The existence of $f'(x)$ and $f'(x) \neq 0$.*

Let us now apply what we have just seen.

2. We want to compute the derivative of $f(x) = \ln x$. How can we use the preceding formula?

3. Let us now focus on the derivative of $f(x) = a^x$.
   How can we rewrite this function in terms of $e^x$ and $\ln x$?
   Then what is $(a^x)'$ ?

   *One should introduce here the formula for* $(\log_a)'$.

4. Compute the derivatives of:

   i) $f(x) = \ln(\sin x)$

   *This exercise and the next one are a good opportunity to talk about the "limitations" of the formula of the derivative. I.e. when is the* $\ln$ *defined.*

   ii) $g(x) = \dfrac{1}{\ln 3x}$

   iii) $h(t) = 3^{t^2}$

   iv) $f(z) = \log_5 e^z$

v) $g(t) = \ln(e^{3t} \sin^2 t)$

vi) $h(x) = \log_2(2^x e^2)$

vii) What is wrong with the following statement?
The derivative of $f(x) = \ln(\ln x))$ is $f'(x) = \frac{1}{x} \ln x + \ln x \frac{1}{x} = \frac{2 \ln x}{x}$.

*One could also ask: how many points should a student who gives this answer get?*

1. The goal of this first part is to compute the derivative of the inverse function $f^{-1}(x)$ of a function $f(x)$ (and this way prove the formula).

   (a) Write down the chain rule.

   (b) Let $f(x)$ be a function and $f^{-1}(x)$ its inverse.
   What is $f(f^{-1}(x))$ equal to?

   (c) Take the equation from the preceding point and differentiate both sides with respect to $x$.
   What do you get? Have you used the chain rule? If so, where?

   (d) Use what you have found to determine the derivative $(f^{-1}(x))'$.

   (e) What do we assume when using the previous formula?

Let us now apply what we have just seen.

2. We want to compute the derivative of $f(x) = \ln x$. How can we use the preceding formula?

3. Let us now focus on the derivative of $f(x) = a^x$.
How can we rewrite this function in terms of $e^x$ and $\ln x$?
Then what is $(a^x)'$ ?

4. Compute the derivatives of:

   i) $f(x) = \ln(\sin x)$

   ii) $g(x) = \dfrac{1}{\ln 3x}$

   iii) $h(t) = 3^{t^2}$

   iv) $f(z) = \log_5 e^z$

v) $g(t) = \ln(e^{3t} \sin^2 t)$

vi) $h(x) = \log_2(2^x e^2)$

vii) What is wrong with the following statement?
The derivative of $f(x) = \ln(\ln x))$ is $f'(x) = \dfrac{1}{x} \ln x + \ln x \dfrac{1}{x} = \dfrac{2 \ln x}{x}$.

This activity will focus on special differentiation techniques that arise from various combinations from function composition, the chain rule, and implicit differentiation.

1) How does function composition, the chain rule, and implicit differentiation relate to the derivative of the inverse of a function?

2) Let $f(x) = e^{x^3}$. Compute the inverse function $f^{-1}$ and it's derivative of the inverse $f^{-1}$. Verify your derivative by computing the derivative in a different way.

3) Recall that the canonical inverse function for sine, cosine, and tangent functions are arcsine, arccosine, and arctangent.

   a) Without using the derivative of the inverse function, compute the derivative $(\arctan x)'$. Express your derivative without any trigonometric functions – a triangle diagram may be useful here.

   b) Using the same procedure, compute the derivative $(\arcsin x)'$.

   c) Using the same procedure, compute the derivative $(\arccos x)'$.

4) To compute the derivative of an inverse function, we used a function composition with its corresponding inverse. Note that as long as the outer function of the composition is differentiable, we can use the chain rule and implicit differentiation to ease the algebraic burden of computing the derivative of some functions. Logarithmic differentiation is one such technique using the logarithm as the outer function in the composition.

   a) Consider the function $f(x) = \frac{x^2+x+1}{2x^3-x^2+3x-8}$. Use the expression $\ln(f(x))$ to compute the derivative $f'(x)$.

b) Consider the quotient function $q(x) = \frac{f(x)}{g(x)}$. Compute the derivative $q'(x)$. What assumptions must you make to guarantee this derivative?

# Related Rates (3.10)

**Expected Skills.**
At the end of this section, students will be able to:

- build an appropriate mathematical model for word problems. This includes:
  - assign variables to appropriate quantities,
  - identify which numerical information is relevant and/or needed,
  - relate the variables using appropriate equations taking into account the numerical information provided,
- solve word problems using the differentiation techniques seen earlier in the term,
- for a given problem, clearly explain with words, mathematical symbols and equations their reasoning, in particular, what is known, what we are looking for and the steps of the procedure to solve the question.

*Note*
The hardest part for the students is the modeling part, not so much the computational part. It is therefore best to focus on that part.

This topic is well suited to have students actually solve problems in class.
It is probably best to do one or two worked-examples at the beginning of the lesson (or to have the students read them beforehand) and then have the students solve problems.

**Pre-Class Activity** (ch3-derivatives-7-relatedrates-1-pc)**.** The goal of the pre-class activity is to have the students draw a proper diagram and start to solve a concrete related rate problem. This will be used directly in class.
Hopefully, working on a concrete example will motivate them!

**Worksheet** (ch3-derivatives-7-relatedrates-2-ws)**.** For the activities, I would suggest the following format: for the first exercise, do it on the board while asking many questions to the students. Separate clearly the modeling part from the numerical application part. The idea is to give a worked example to the students.
At the end of the exercise, we ask the students to reflect on it and identify the various steps of the resolution (if enough time, this can be done as a think-pair-share).

For the following two exercises, the idea is to have the students work by themselves on them.
I suggest the following: give the exercises without the numerical application part. Split the class in two (to make it easy for the following part, one can do one row is A, the next one is B, the next one is A, etc.). Then have each half of the class work on one question where they have to find the equation that relates the rates of change (they can work in group within their team). Make sure people in each team students understand the exercise and can explain it. After a few minutes, each student pairs up with someone from the other team. They have to explain to each other the exercise and how to solve it. Once this is done, one can give them the numerical application part. This would be also a good place to discuss what information is needed to solve

the exercise. This type of activity is often called a jigsaw

The last exercise can be used as a challenge exercise.

At the end of the section or as a review exercise before the prelim/final, have the students look at a number of exercises in the book and "classify" them.
Or have first the students write down the equation that relates the variables for these problems and then group the problems together.
For this exercise (in either variant we need the students to have done some exercises like these before. Thus it should come at the "end".

Here is a video of the first launch of NASA's Space Shuttle from the Kennedy Space Center in Cape Canaveral, FL, in 1981: https://youtu.be/kdKltNx42AQ?t=3m50s. Watch a minute of it. As you can see the camera (which has a fixed location) follows the shuttle going up into the atmosphere. In doing so, the angle between the horizontal direction and the direction to which the camera is pointing increases as the shuttle goes higher and higher in the sky.
Our goal is to determine how fast this angle increases with respect to time (such an information would be very useful if, for example, one wants to automate the camera).

i) Draw a sketch of the situation (make it big enough, a least half a page).

ii) What information do we need in order to determine the variation of this angle? Make a list.

iii) Find an equation (or equations) that relates these pieces of information together?

1. **Space Shuttle** – *Have student work in group, then have a class discussion.*

   At the Kennedy Space Center, the VIP site for launch viewing (where the camera is located) is 3.9 miles away form pad 39A (where the shuttle is launched) [4].
   According to NASA [5], during the first stage ascent, the shuttle reaches the speed of 4,828 kph within 2 minutes or an average speed of 2,414 kph.

   Using this information, compute how fast the angle that the camera makes with the horizontal increases.

   - *Have students form groups of 2-4. Have them compare their sketches from the pre-class activity. Have them also compare their lists of the information one needs to solve this problem.*
   - *Quickly go around the room to see the sketches of the students.*
   - *Using students' input, either ask one/several student(s) to draw their sketches and/or equations on the board (you can do this while the students work in group and you go around the room) or use their input to draw a proper sketch on the board and give the equation relating the angle and the height of the shuttle. Then, take the derivative, the variation of the angle and the speed of the shuttle. This part constitute a sort of worked example for the students.*
   - *What simplification are we making here? – We assume the velocity of the space shuttle is constant.*

   Now that we have solved this exercise, identify the steps we have taken to solve it.
   *You can do a think-pair-share if time allows*

   *For the following two activities (on the following pages), you can create a jigsaw where each expert group work on its problem up to e) (before the numerical application). Then pair up students and give them the numerical applications part.*
   *Or just have students work in groups on the two problems (without more directions).*

[4] cf. http://www.launchphotography.com/Delta_4_Atlas_5_Falcon_9_Launch_Viewing.html
[5] cf. https://spaceflight.nasa.gov/shuttle/reference/basics/launch.html

2. **In-car Speed Camera**
An undercover police car equipped with an in-car speed camera is driving along a road toward an intersection. A car is driving on the perpendicular road passing directly in front of the police car. The speed camera measures the speed at which the car is driving away from the police car. We want to determine the actual speed of the car.

   a) Draw a sketch of the situation. What assumptions are we making here? *The roads are straight lines*

   b) Determine what the variables are. In terms of your variables, what are we trying to determine? And what does the speed camera measure?

   c) Find and equation that relates your variables.

   d) Taking the derivative, find an equation that relates the speed of the police car, the speed of the car and the speed measured by the speed camera.

   e) To determine the speed of the car, what information do we need?

   f) When the speed camera takes its measurement, the police car is 30 meters away from the intersection and drives at a speed of 20 kph. At that moment, the car is 40 meters away from the intersection. The speed measured by the speed camera is 50 kph. What is the speed of the car?

3. **Blowing a balloon** *Do take an actual balloon in class and start with a demonstration!*
You are preparing the birthday party of your 5-year old cousin. An essential element is of course to have balloons all around the room!
You use a pump (or your lungs), for which we know the output, to blow the balloons. We want to compute how fast the radius of a balloon increases as we blow it.

   a) Draw a sketch of the situation. What assumptions are we making here? *Balloons are perfect spheres*

   b) What are the variables and what are the constants?

   c) In terms of the variables you have defined, what are we looking for?

   d) To determine the speed at which the radius increases, what information do we need?

   e) Using the fact that the volume $V$ of a sphere is $V = \frac{4}{3}\pi r^3$, relate the variables by an equation. Then taking the derivative, find an equation that related the rate of change of the volume of a balloon and the rate of change of its radius.

   f) If the pump you are using (or you lungs!) has an output of $50\pi$ cm$^3$/s, how fast is the radius increasing when the radius of the balloon is 5 cm.

4. **Connecting Rods – Challenge Exercise**
Connecting rods are used in internal combustion engines (i.e. most of today's car engines) or formerly in steam engines. Their role is to transform a linear movement into a circular one. For an illustration or this mechanism see https://upload.wikimedia.org/wikipedia/commons/0/01/Slider_Crank_animation.svg and https://en.wikipedia.org/wiki/Crankshaft#/media/File:Cshaft.gif.[6]
Our goal for this exercise is to determine how the linear and circular velocities are related.

   a) Draw a sketch of the situation. What is constant and what is variable?

   b) Find an equation that relates the angular velocity of the crankpin (the part that has a circular motion) to the velocity of the piston (the part of the crankshaft that moves on the horizontal axis).

   You will probably need to use the law of sines $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma}$ and the law of cosines $c^2 = a^2 + b^2 - 2ab\cos\gamma$. [7]

   c) If the radius of the circle around which the crankpin is moving is 5cm and the length of the connecting rod 15cm, at what speed is the piston moving in $cm/s$ if the crankpin is moving at 2000 round per minute?

[6] For more on this, look at https://en.wikipedia.org/wiki/Crankshaft or https://en.wikipedia.org/wiki/Internal_combustion_engine.

[7] More detail on Wikipedia: https:en.wikipedia.org/wiki/Law_of_sines and https://en.wikipedia.org/wiki/Law_of_cosines.

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# Linearization (3.11)

**Expected Skills.**
At the end of this section, students will be able to:

- explain in words what the process of linearization consist of and why it is interesting,
- use the linear approximation of a function at a given point to compute an approximate value of the function,
- using the graph of a function explain if a linear approximation gives an underestimate or overestimate of the true value of the function,
- explain in general terms what the conditions are for the process to give a "reasonable" approximation.

**Pre-Class Activity** (ch3-derivatives-8-linearization-1-pc)**.** In the pre-class activity we ask the students to compute the tangent lines to $f(x) = \sqrt{x}$ at $x = 3$ and $x = 9$. We then ask them if they see a difference between these two cases (to introduce the idea that at some points, such as $x = 9$ the tangent line stays "closer" to the function than at $x = 3$). Then we have them think about how to compute the "exact" values of $\sqrt{2}$ and $\sqrt{10}$ and how the tangent lines can be useful to approximate such values.

**Worksheet** (ch3-derivatives-8-linearization-2-ws)**.** The idea here is to start by connecting with the questions in the pre-class activity and introduce the concept of linearization. We then do a worked example on the board and then ask the students identify the steps to compute an approximation. Then we have the students solve such an exercise by themselves (a think-pair-share would work well here).
Exercise 3 introduces the ideas that the approximation given by a linearization can be better or worse depending on the shape of the function around the point we use.
The last exercise looks at whether a linear approximation underestimates or overestimates the real value we are approximating. (this will come more precise later when we introduce concavity).

The goal of this activity is to see how we can *compute* (as opposed to only look at) an approximation of a given function around some points.

1. Using Desmos or Geogebra, draw the graph of the function $\sqrt{x}$.
   Compute the equation of the tangent line $y_1$ at $x = 1$. Then draw this line $y_1$ on the graph.

2. When you zoom in around $(1, 1)$, what do you notice about the function and the tangent line?
   A specific feature becomes more prominent as you zoom in.

3. Compute now the equation of the tangent line $y_9$ at $x = 9$ and draw it on the graph.

4. When you zoom in around $(9, 3)$, what do you notice about the function and its tangent line? What is similar and what is different from zooming in around $(1, 1)$?

5. Can you compute the exact value of the point on the tangent line $y_1$ for the $x = 2$. What is it?
   Same question for $y_9$ at $x = 10$.

6. What about computing the exact values of $\sqrt{2}$ and $\sqrt{10}$?

7. How can these tangent lines be useful for approximating the values of $\sqrt{2}$ and $\sqrt{10}$?

8. Finally, how can we compute the equation of the tangent line that touches the function $x = 6$. What works and what doesn't work? What do you conclude from that?

1. (a) Let us start by looking at $\sqrt{10}$. What is a fundamental difference between $\sqrt{9}$ and $\sqrt{10}$?

   *In one case we know the exact value: $\sqrt{9} = 3$ whereas in the other case we don't have an exact value.*

   (b) How could we use what you have done in the pre-class activity to approximate a function?

   *If nothing comes up, point out step 5., i.e. we know how to compute exact values on the tangent line.*

   *Then introduce the definition of the linearization (or have the students come up with the equation of the tangent line to $f(x)$ at $a$).*

   (c) Approximating $\sqrt{10}$. Keep some space on the right-hand side of the sheet.

   *The idea here is to do an example on the board, then give a few minutes to the students and to re-read it and write down the different steps involved. We can also them compare the steps with a neighbor.*

   *The steps are:*

   *i. Determine what the function is.*

   *ii. Identify a point $a$ that is "close" to the point we want to approximate and for which we know the exact value.*

   *iii. Compute the equation of the tangent line at $a$.*

   *iv. Use the equation of the tangent line to approximate the value of the function by determining the $y$ value of the tangent line at $x = a$.*

2. Approximate $\frac{1}{4.9}$ using an approximate linearization.
   Use the steps you have identified in the previous part.

   *For this activity, one can typically do a Think-Pair-Share: first the students do the exercise alone, then compare with their neighbor, finally the whole class discuss the solution.*

   *If time allows, do the same exercise for another approximation such as* $\ln(1.1)$ *or* $\sin(0.1)$.

3. (a) In the pre-class activity, you looked the tangent lines to $\sqrt{x}$ at both $x = 1$ and $x = 9$. What difference have you noticed?

   *The tangent line is "closer to the function at 9 than at 1. If no answer comes up, go directly to the computation part.*

   (b) Let us approximate $\sqrt{2}$ by using the linearization (i.e. the tangent line). What do you get?

   (c) Using a calculator, compute the error of this approximation (i.e. the difference between this approximation and what you get with your calculator). Also compute the error of the approximation of $\sqrt{10}$ we did before.

   (d) What do you notice? How can we explain this difference? What factors explain this difference?

   *This points out to the limitation of this method.*
   *The error can become quite big and thus the approximation "bad" depending on the shape of the function around the point on the function we use.*

4. We want to compute an approximation of $e^{0.1}$ and $e$ $(= e^1)$ using the tangent line.

   (a) Compute the appropriate linear approximation(s) $L(x)$. What is the function? Around what point(s) can we do that?

   (b) Using this approximation for $e^{0.1}$ and $e$, are we underestimating or overestimating the actual values of $e^{0.1}$ and $e$?

   *The idea is to look at whether the tangent line are below or above the function. It is a first suggestion about the idea of concavity.*

   (c) If we compare the approximations for $e^{0.1}$ and $e$, which one is closer to the actual values of $e^{0.1}$ and $e$? What are your arguments to support your answer?

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