L'Hopital's rule

This rule can be applied to compute limits when:

1. the limit is written as a quotient of the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$,
2. $f(x)$ and $g(x)$ are differentiable on a open interval I that contains the point a and $g'(x) \neq 0$ on I except possibly at a,
3. the limit $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$ exists.

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

L'Hôpital's rule can be applied several times as long as the quotient is $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ and the other conditions hold as well (of course).
It also holds for one-sided limits.

Compute the following limits:

a) $\lim\limits_{x\to 0} \dfrac{e^x - 1}{\tan x}$ *gives "0/0"*

b) $\lim\limits_{x\to \infty} \dfrac{x}{e^x}$ *gives "∞/∞"*

c) $\lim\limits_{x\to \infty} \dfrac{x^2}{e^x}$ *gives "∞/∞", apply the rule twice*

Let us now consider the following limits. What are their "types"?
What do we need to do to apply L'Hôpital's rule to compute these limits?

d) $\lim_{x \to 0^+} x \ln x$ *gives " $0 \cdot \infty$ " (pre-class activity)*

For this activity, on can typically do a "jigsaw"; i.e. students work (in group or alone) on one of these cases. One then forms teams where each student has worked on a different limit and where this person explain to the others how to solve the limit they have worked on.

e) $\lim_{x \to \infty} x^{1/x}$ *gives " ∞^0 " (pre-class activity)*

f) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$ *gives " 1^∞ "*

g) A student wants to compute the limit $\lim_{x\to\infty} \frac{x + \sin x}{x}$.
To do so, he uses L'Hôpital's rule and gets:

$$\begin{aligned}
&\lim_{x\to\infty} \frac{x + \sin x}{x} \text{ (= ``}\frac{\infty}{\infty}\text{'', use L'Hôpital's rule)} \\
&= \lim_{x\to\infty} \frac{1 + \cos x}{1} \\
&= \lim_{x\to\infty} 1 + \cos x.
\end{aligned}$$

As $\cos x$ oscillates when x goes to infinity, he concludes that the limit $\lim_{x\to\infty} \frac{x + \sin x}{x}$ does not exist.

Do you agree with this reasoning? If so, explain why you think it is correct. If not, explain where there is a flaw.

The goal here is the to have the students test the hypotheses of the theorem.

The rule only applies when the limit $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ *exists. Since it is not the case here, we cannot use it to compute this limit.*
A good exercise is: how do we compute this limit?

h) Explain in words (and without mathematical symbols) what L'Hôpital's rule means in geometrical terms.