Indeterminate Forms and L'Hôpital's Rule(4.5)

Expected Skills.
At the end of this section, students should be able to:

- explain in words what an indeterminate form is,
- explain what L'Hôpital's rule is, when we can use it, and what kinds of limits we can compute with it,
- correctly use L'Hôpital's rule to compute limits.

Pre-Class Activity (ch4-applications-1-lhospital-1-pc)**.** In the pre-class activity we have the students look at limits that give the expressions "$0/0''$, "∞/∞", " $0 \cdot \infty$", and ''∞^0". We then ask them if we can factor out and simplify these expression as we did at the beginning of the course (which we cannot do for these limits). We thus want them to realize that one needs another technique to compute these limits.

Worksheet (ch4-applications-1-lhospital-2-ws)**.** The worksheet starts by stating L'Hôpital's rule and its hypotheses. We first have the students use it on three slightly different cases. One the second page, we then look at limits that first need to be re-written to apply L'Hôpital's rule. This activity is well suited for a jigsaw (see more detail on the worksheet). When then have the students "test"' the hypotheses of L'Hôpital's rule by giving to them a misuse of the rule (the key message of this part is: "verify the hypotheses when using the rule"). Finally, we ask the students to describe in words what L'Hôpital's rule means in geometrical terms (in other words, to make sense of the equation given by the rule).

Supplemental Activity (ch4-applications-1-lhospital-3-sup-indeterminate forms)**.** This activity has students engage in a jigsaw discussion where half the class determines cases where an indeterminate form approaches a particular limit. The students then present their findings to the other half of the class.

Here we will explore here several "types" of limits.

1. First, we want to compute the limit $\lim_{x\to 3} \dfrac{x-3}{x^2-9}$.
   (a) What happens when you try to plug in $x = 3$?

   (b) Then, what "algebraic manipulation" can we do to compute such a limit ? What answer do we get? (check back section 2.2 of the textbook if you are not sure anymore.

2. Let us now try to compute the following limits:
   (a) $\lim_{x\to 0} \dfrac{e^x - 1}{\sin x}$,
   (b) $\lim_{x\to \infty} \dfrac{x}{e^x}$,
   (c) $\lim_{x\to 0^+} x \ln x$,
   (d) $\lim_{x\to \infty} x^{1/x}$

For each of them, what do you get if you try to directly plug in the values? What kind of expression do you get in each case?
Can you use a similar technique to the one used in i) to compute these limits?

In class we will see a technique to "easily" compute such limits.

L'Hopital's rule
This rule can be applied to compute limits when:

1. the limit is written as a quotient of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$,
2. $f(x)$ and $g(x)$ are differentiable on a open interval I that contains the point a and $g'(x) \neq 0$ on I except possibly at a,
3. the limit $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists.

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

L'Hôpital's rule can be applied several times as long as the quotient is $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and the other conditions hold as well (of course).
It also holds for one-sided limits.

Compute the following limits:

a) $\lim_{x \to 0} \frac{e^x - 1}{\tan x}$ *gives "0/0"*

b) $\lim_{x \to \infty} \frac{x}{e^x}$ *gives "∞/∞"*

c) $\lim_{x \to \infty} \frac{x^2}{e^x}$ *gives "∞/∞", apply the rule twice*

Let us now consider the following limits. What are their "types"?
What do we need to do to apply L'Hôpital's rule to compute these limits?

d) $\lim\limits_{x \to 0^+} x \ln x$ *gives* " $0 \cdot \infty$ " *(pre-class activity)*

For this activity, on can typically do a "jigsaw"; i.e. students work (in group or alone) on one of these cases. One then forms teams where each student has worked on a different limit and where this person explain to the others how to solve the limit they have worked on.

e) $\lim\limits_{x \to \infty} x^{1/x}$ *gives* " ∞^0 " *(pre-class activity)*

f) $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x$ *gives* " 1^∞ "

g) A student wants to compute the limit $\lim_{x\to\infty} \frac{x+\sin x}{x}$.
To do so, he uses L'Hôpital's rule and gets:

$$\begin{aligned}
&\lim_{x\to\infty} \frac{x+\sin x}{x} \ (= \text{“}\frac{\infty}{\infty}\text{”, use L'Hôpital's rule)} \\
&= \lim_{x\to\infty} \frac{1+\cos x}{1} \\
&= \lim_{x\to\infty} 1+\cos x.
\end{aligned}$$

As $\cos x$ oscillates when x goes to infinity, he concludes that the limit $\lim_{x\to\infty} \frac{x+\sin x}{x}$ does not exist.

Do you agree with this reasoning? If so, explain why you think it is correct. If not, explain where there is a flaw.

The goal here is the to have the students test the hypotheses of the theorem.

The rule only applies when the limit $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ *exists. Since it is not the case here, we cannot use it to compute this limit.*
A good exercise is: how do we compute this limit?

h) Explain in words (and without mathematical symbols) what L'Hôpital's rule means in geometrical terms.

L'Hopital's rule
This rule can be applied to compute limits when:

1. the limit is written as a quotient of the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$,
2. $f(x)$ and $g(x)$ are differentiable on a open interval I that contains the point a and $g'(x) \neq 0$ on I except possibly at a,
3. the limit $\displaystyle\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists.

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

L'Hôpital's rule can be applied several times as long as the quotient is $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ and the other conditions hold as well (of course).
It also holds for one-sided limits.

Compute the following limits:

a) $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{\tan x}$

b) $\displaystyle\lim_{x\to \infty} \frac{x}{e^x}$

c) $\displaystyle\lim_{x\to \infty} \frac{x^2}{e^x}$

Let us now consider the following limits. What are their "types"?
What do we need to do to apply L'Hôpital's rule to compute these limits?

d) $\lim_{x \to 0^+} x \ln x$

e) $\lim_{x \to \infty} x^{1/x}$

f) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$

g) A student wants to compute the limit $\lim\limits_{x\to\infty} \dfrac{x+\sin x}{x}$.
To do so, he uses L'Hôpital's rule and gets:

$$\begin{aligned}&\lim_{x\to\infty} \frac{x+\sin x}{x} \ (= \text{“}\frac{\infty}{\infty}\text{”, use L'Hôpital's rule)}\\ &= \lim_{x\to\infty} \frac{1+\cos x}{1}\\ &= \lim_{x\to\infty} 1+\cos x.\end{aligned}$$

As $\cos x$ oscillates when x goes to infinity, he concludes that the limit $\lim\limits_{x\to\infty} \dfrac{x+\sin x}{x}$ does not exist.

Do you agree with this reasoning? If so, explain why you think it is correct. If not, explain where there is a flaw.

h) Explain in words (and without mathematical symbols) what L'Hôpital's rule means in geometrical terms.

Working with other members of Team Pi, find examples of functions $f(x)$ and $g(x)$ such that the indeterminate form has a particular resolution. You will need to present your findings to a member of Team Tau.

a) Suppose $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$. Find a f and g such that the value of $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$.

b) Suppose $f(x)$ and $g(x)$ have a vertical asymptote at $x = a$. Find a f and g such that $\lim_{x\to a} \frac{f(x)}{g(x)}$ diverges to infinity.

c) Suppose $\lim_{x\to a} g(x) = 0$ and $f(x)$ has a vertical asymptote at $x = a$. Find a f and g such that $\lim_{x\to a} f(x)g(x) = 0$.

d) Suppose $\lim_{x\to a} f(x) = 1$ and $g(x)$ diverges to positive infinity. Find a f and g such that $\lim_{x\to a} f(x)g(x) = 0$.

Working with other members of Team Tau, find examples of functions $f(x)$ and $g(x)$ such that the indeterminate form has a particular resolution. You will need to present your findings to a member of Team Pi.

a) Suppose $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$. Find a f and g such that $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ diverges to infinity.

b) Suppose $f(x)$ and $g(x)$ have a vertical asymptote at $x = a$. Find a f and g such that the value of $\lim_{x\to a} \dfrac{f(x)}{g(x)} = 0$.

c) Suppose $\lim_{x\to a} g(x) = 0$ and $f(x)$ has a vertical asymptote at $x = a$. Find a f and g such that $\lim_{x\to a} f(x)g(x) = 1$.

d) Suppose $\lim_{x\to a} f(x) = 1$ and $g(x)$ diverges to positive infinity. Find a f and g such that $\lim_{x\to a} f(x)g(x)$ diverges to infinity.

Extreme Values of Functions (4.1)

Expected Skills.
At the end of this section, students will be able to:

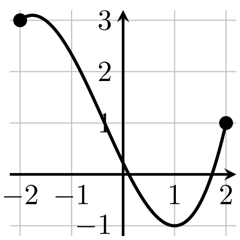
- define the notions of local/absolute min and max, and critical point,
- explain the extreme value theorem (in particular its hypotheses) and exhibit "counter-examples", i.e. functions that don't have an absolute min or max,
- find the absolute min and max of a continuous function on a closed interval [a, b].

Pre-Class Activity (ch4-applications-2-evt-1-pc)**.** The goal of the pre-class activity is to have the students think about the conditions that will assure that a function has an (absolute) min and max. To do this, we have them analyze functions and ask them about their domains, continuity or discontinuity and the existence of minimum/maximum.

Worksheet (ch4-applications-2-evt-2-ws)**.** We first continue the reflection initiated in the pre-class activity by having the students draw functions with specific properties. We then asked them about the conditions for a function to have a (global) minimum and maximum, i.e. we ask for the hypotheses of the Extreme Value Theorem.
In exercise 3, we have the students distinguish between the *hypotheses* and the *conclusion* of a theorem.
We then ask the students what would be a procedure to find the global min and max. It is also a good place to talk about local min and max.
Finally, exercises 6 and 7 are an application of the Extreme Value Theorem (and some modeling for 7).

The goal of this activity is to inquire about the following question: when can we assure that a function will have an “absolute” maximum and minimum? What conditions must be met?
These is an important question because if we look the for minimum or maximum (e.g. we want to minimize the cost of production of an item) it would be better if we new beforehand that such a minimum or maximum exists!

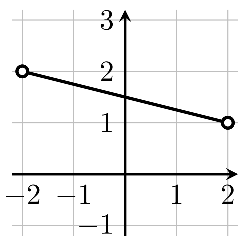
1. For each of the functions graphed below, determine:
   - the domain of the function,
   - the continuity or discontinuity of the function (and if discontinuous, at what point(s) is it so),
   - for what value(s) of x does the function attain a maximum or minimum and the value of the function at that/these point(s).

a) 

Domain:

Point(s) of discontinuity:

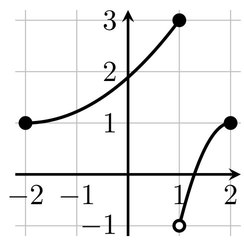
Min and Max:

b) 

Domain:

Point(s) of discontinuity:

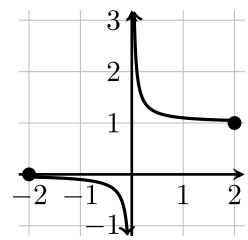
Min and Max:

c) 

Domain:

Point(s) of discontinuity:

Min and Max:

d) 

Domain:

Point(s) of discontinuity:

Min and Max:

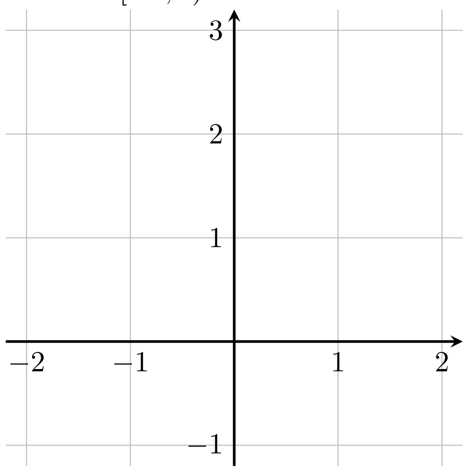
2. Which one(s) of the above function(s) have both a minimum and maximum?
   What are the properties of this/these function(s)? When can we assure that a function will have absolute min and max?

Definition

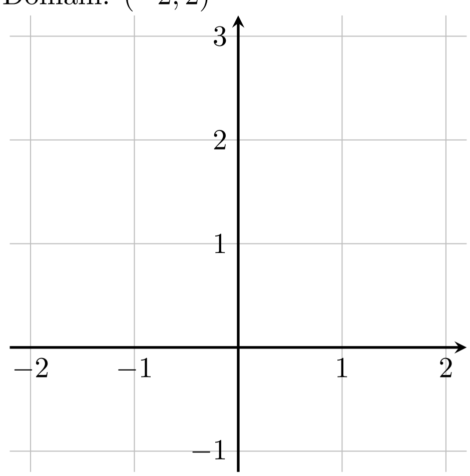
A function f has an **absolute maximum** (also known as a **global maximum**) at $x = c$ if $f(c)$ is the highest value of f anywhere; more precisely, f has an absolute maximum at $x = c$ if $f(c) \geq f(x)$ for all x in the domain of f. An absolute minimum is defined similarly.

1. If possible, create graphs of functions satisfying each description:

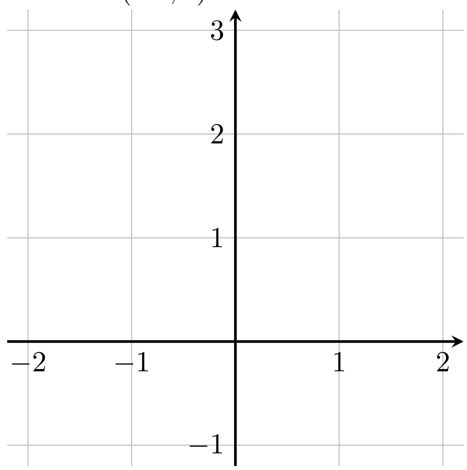
   (a) A continuous function with an absolute maximum of 3 and no absolute minimum.
   Domain: $[-2, 2)$

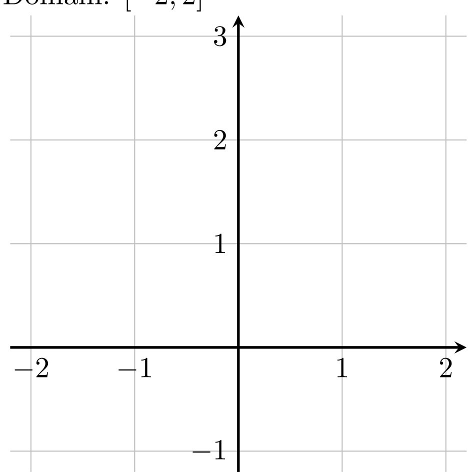
   (b) A continuous function with an absolute maximum of 3 and an absolute minimum of -1.
   Domain: $(-2, 2)$

   (c) A continuous function with no absolute maximum and no absolute minimum.
   Domain: $(-2, 2)$

   (d) A continuous function with no absolute maximum and no absolute minimum.
   Domain: $[-2, 2]$

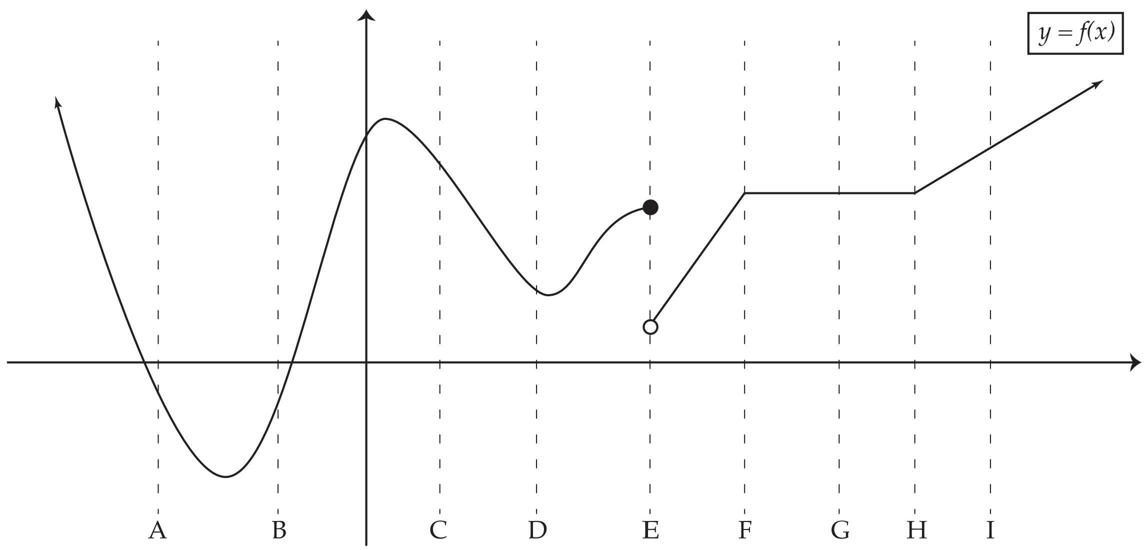
   

2. Based on the previous exercise and on the pre-class activity, what conditions must a function $f(x)$ satisfy to guarantee that it has an absolute maximum and an absolute minimum?

   Can have a "debate" here, or a poll (people writing down their answer on a piece of paper, or voting with Pingo)

   Introduce the Extreme Value Theorem and explain why it is interesting.

3. A portion of the graph of the function $f(x)$ is shown in the figure below.



   For each of the questions below, circle **ALL** of the available correct answers.

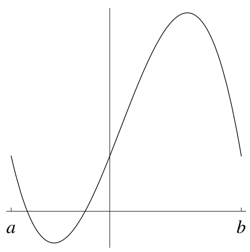
   (a) On which intervals does $f(x)$ satisfy the hypotheses of the **Extreme Value Theorem**?

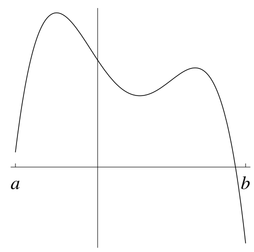
   $[A, C]$ $[A, F]$ $[B, E]$ $[D, F]$ $(G, I]$ None

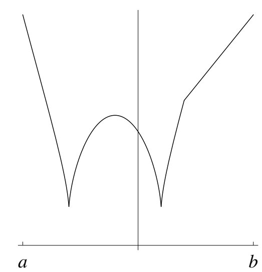
   (b) On which intervals does $f(x)$ satisfy the conclusion of the **Extreme Value Theorem**?

   $[A, C]$ $[A, F]$ $[B, E]$ $[D, F]$ $(G, I]$ None

4. We now know a continuous function defined on a closed interval will have an absolute maximum and an absolute minimum. Often, our goal is to find the absolute minimum or maximum (if it exists) of a function on a given interval.
   Look at the graphs below and think about a strategy. **How would you go about identifying absolute maximum and absolute minimum on $[a, b]$?**







*Define **critical points** as points where $f'(x) = 0$ or what $f'(x)$ doesn't exist. Have someone explain why we need to include $f'(x)$ doesn't exist.*

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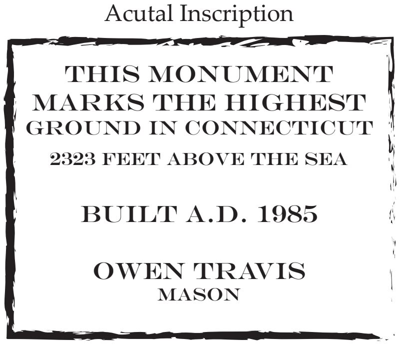
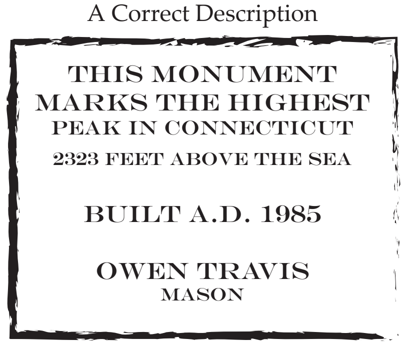


A Correct Description



Point to remember:

Always check boundary points!! This is also a good point to discuss the difference between local and global min/max).

5. Suppose $f(x)$ is a continuous function defined on a closed interval. Is every critical point an absolute maximum or an absolute minimum? Give examples.

   In addition to the peak in Connecticut, have students come up with other functions or graphs.

6. (a) Does $f(r) = 2\pi r^2 + \frac{256\pi}{r}$ have an absolute maximum and absolute minimum on $[1, 8]$? If so, where? (Give the values of r at which the absolute minimum and absolute maximum occur.)

   For this question and the following one, underline the Extreme Value Theorem and what it says.

   (b) Does f have an absolute minimum and absolute maximum on $(0, \infty)$? If so, where?

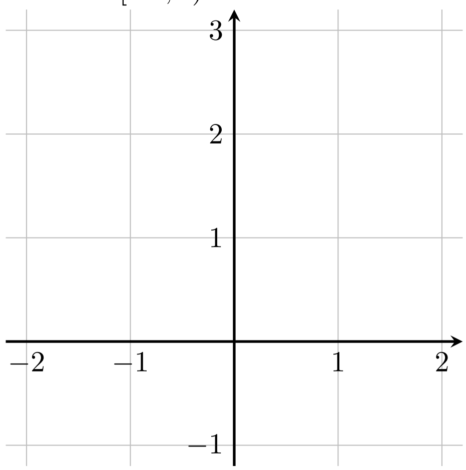
7. A soda company wants its aluminum soft drink cans to have a volume of 128π cubic centimeters. The company's factory can only manufacture cans that are at least 2 cm tall and have a radius of at least 1 cm. In order to conserve resources, the company wants to minimize the amount of aluminum needed for a single can. What dimensions should they make their cans?

   It turns out that the function one gets is the same as the function studied in part (a). So this question is really about modeling.
   Based on time, decide if you want to do it.

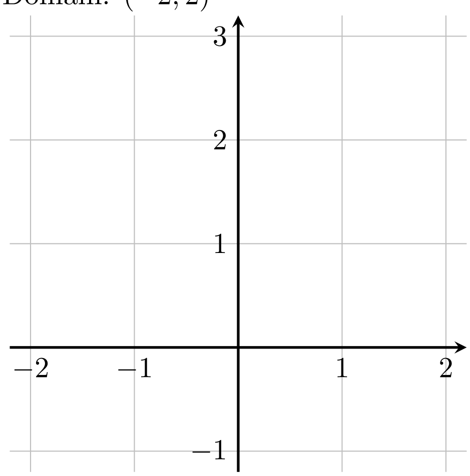
Definition
A function f has an **absolute maximum** (also known as a **global maximum**) at $x = c$ if $f(c)$ is the highest value of f anywhere; more precisely, f has an absolute maximum at $x = c$ if $f(c) \geq f(x)$ for all x in the domain of f. An absolute minimum is defined similarly.

1. If possible, create graphs of functions satisfying each description:

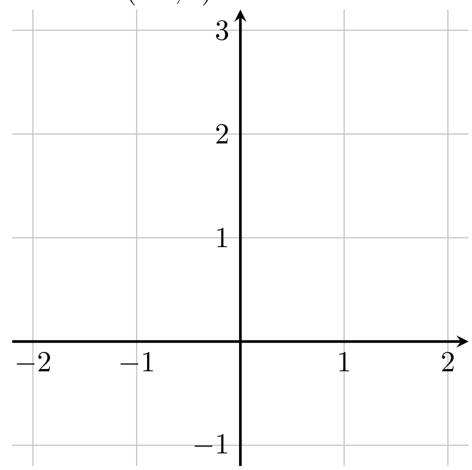
   (a) A continuous function with an absolute maximum of 3 and no absolute minimum.
   Domain: $[-2, 2)$



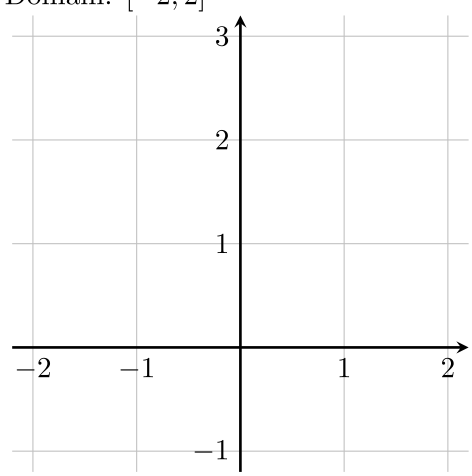
   (b) A continuous function with an absolute maximum of 3 and an absolute minimum of -1.
   Domain: $(-2, 2)$



   (c) A continuous function with no absolute maximum and no absolute minimum.
   Domain: $(-2, 2)$

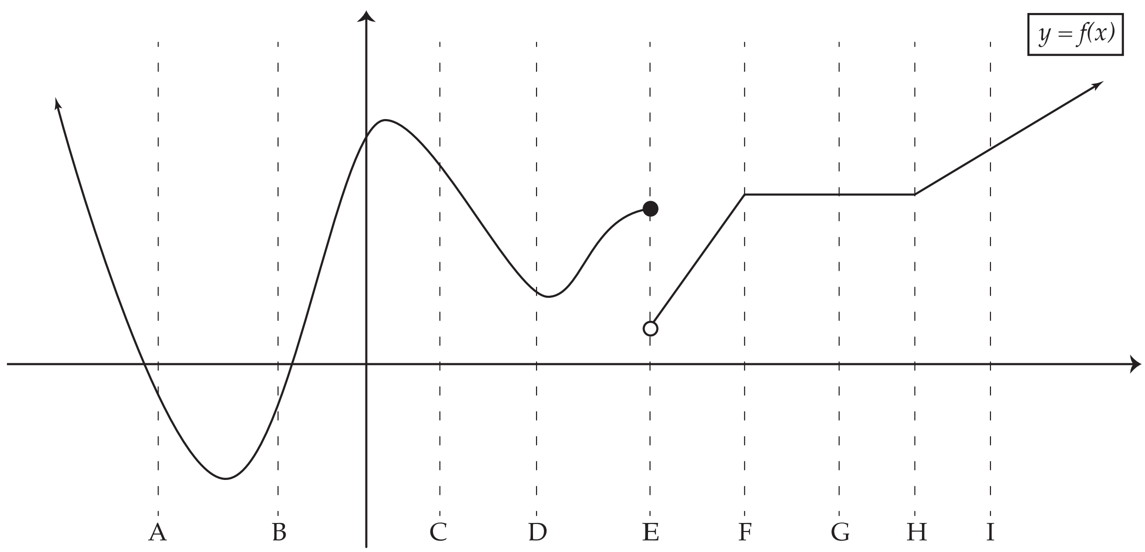


   (d) A continuous function with no absolute maximum and no absolute minimum.
   Domain: $[-2, 2]$



2. Based on the previous exercise and on the pre-class activity, what conditions must a function $f(x)$ satisfy to guarantee that it has an absolute maximum and an absolute minimum?

3. A portion of the graph of the function $f(x)$ is shown in the figure below.



For each of the questions below, circle **ALL** of the available correct answers.

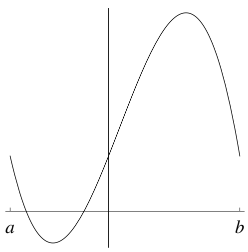
(a) On which intervals does $f(x)$ satisfy the hypotheses of the **Extreme Value Theorem**?

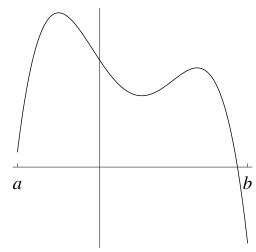
$[A, C]$ $[A, F]$ $[B, E]$ $[D, F]$ (G, I) NONE

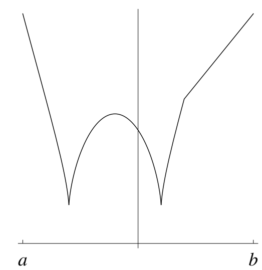
(b) On which intervals does $f(x)$ satisfy the conclusion of the **Extreme Value Theorem**?

$[A, C]$ $[A, F]$ $[B, E]$ $[D, F]$ (G, I) NONE

4. We now know a continuous function defined on a closed interval will have an absolute maximum and an absolute minimum. Often, our goal is to find the absolute minimum or maximum (if it exists) of a function on a given interval.
   Look at the graphs below and think about a strategy. **How would you go about identifying absolute maximum and absolute minimum on** $[a, b]$?







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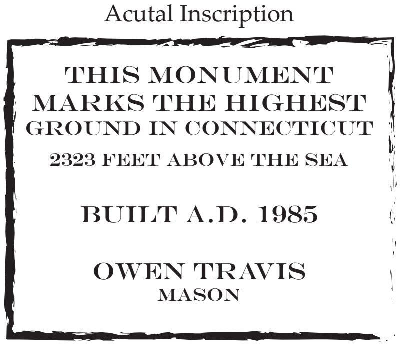
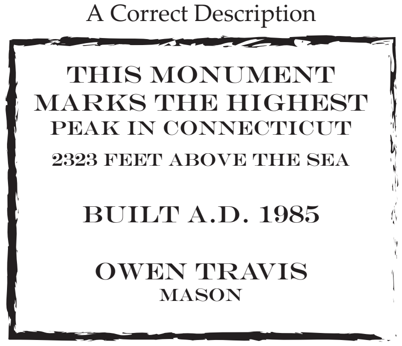


A Correct Description



Point to remember:

5. Suppose $f(x)$ is a continuous function defined on a closed interval. Is every critical point an absolute maximum or an absolute minimum? Give examples.

6. (a) Does $f(r) = 2\pi r^2 + \frac{256\pi}{r}$ have an absolute maximum and absolute minimum on $[1, 8]$? If so, where? (Give the values of r at which the absolute minimum and absolute maximum occur.)

   (b) Does f have an absolute minimum and absolute maximum on $(0, \infty)$? If so, where?

7. A soda company wants its aluminum soft drink cans to have a volume of 128π cubic centimeters. The company's factory can only manufacture cans that are at least 2 cm tall and have a radius of at least 1 cm. In order to conserve resources, the company wants to minimize the amount of aluminum needed for a single can. What dimensions should they make their cans?

The Mean Value Theorem (4.2)

Expected Skills.
At the end of this section, students should be able to:

- explain in words the Mean Value Theorem as well as its corollaries,
- explain the importance of the Mean Value Theorem,
- use the Mean Value Theorem to prove properties of a function based on information about its derivative.

Pre-Class Activity (ch4-applications-3-mvt-1-pc)**.** The goal of the pre-class activity is to have the students think about and "discover" Rolle's Theorem. They are in particular asked to think about the conditions necessary for the conclusion of the theorem to hold.

Worksheet (ch4-applications-3-mvt-2-ws)**.** In the class activity we start by using what has been done in the pre-class activity and state Rolle's Theorem. We then go on to state the Mean Value Theorem. To this end one could ask the students, "now, what happens if the two end points are not at the same level?".
Then, we look at a concrete example of how the theorem is used (point-to-point speed cameras).
Afer that we have the students come up with functions satisfying specific properties. The goal here is to have them think about Corollaries 1 and 2 (p. 232).

Note that one could add in the worksheet questions that are currently in the homework for that section.
In addition, note that the last exercise is very similar (but yet different) from the exercises on the pre-class activity for the next section, which is the first derivative test.

Supplemental Activity (ch4-applications-3-mvt-3-sup-theorems)**.** In this activity, the students probe the differences between the hypotheses and conclusions of Rolle's Theorem, the Mean Value Theorem, and the Extreme Value Theorem. It is suggested that this activity occur after all three theorems have been addressed in class, so that the whole class can be split into three groups to work on a single theorem before presenting their findings back to the class. It is also suggested that instructors leverage student-generated examples for Rolle's Theorem and ask students to check if hypotheses of the Extreme Value Theorem are held as well, etc.

You decide to climb Mount Marcy, which is New York State's highest peak. You park your car at the Adirondak Loj parking lot, hike up to the top of the mountain and then walk back down following the same trail.
Let $f(x)$ be your elevation in meters as a function of how many kilometers you have walked from the parking lot where you left your car.

1. Sketch this function knowing that the parking lot's elevation is 607m and that the peak is at 1,629m.

2. How is the tangent line at the point that represents the peak of the mountain (knowing that the top of this mountain is kind of "flat"). What assumption do we need to make to answer this question?

3. Is this true that all functions have a point with such a tangent line? Give the graphs of two examples and two counter-examples.

4. By looking at your examples, determine what conditions are necessary for such a point to exist.

5. Under these conditions, may we have several such points or do we have at most one?

1. What are Rolle's Theorem and the Mean Value Theorem (MVT)?
   What are their assumptions? How are they related to one another?

2. Here is a concrete application of the Mean Value Theorem.
   In the state of New South Wales, in Australia, the police uses "point-to-point" speed cameras. These cameras evaluate the speed of a given car by measuring how long it takes the car to go from point A to point B (hence the name).

   (a) What "kind" of speed is measured by such speed cameras? What are the advantages and disadvantages of such a system (in terms of measuring speed) compared to a "normal" speed camera?

   The goal of this activity to help the students determine what the mean value theorem implies in concrete terms. It should be used once the theorem has been presented to the students .

   (b) Imagine that using such a point-to-point speed camera, the police fines a driver for driving at 100 kph on a portion of the highway limited at 90 kph. The driver appeals the decision using the following argument. While its average speed was measured as being 100 kph, nothing proves that there was any given point where he drove at that speed. Yet, the law says that one is to be fined if one is observed going over the speed limit *at a given point*.
   The District Attorney calls you as an expert witness as she hopes you can help her show that the fine should be upheld. With your knowledge of calculus, how can you help her?

3. (a) For each of the following cases, draw a sketch for $f(x)$ (and where needed of $g(x)$):

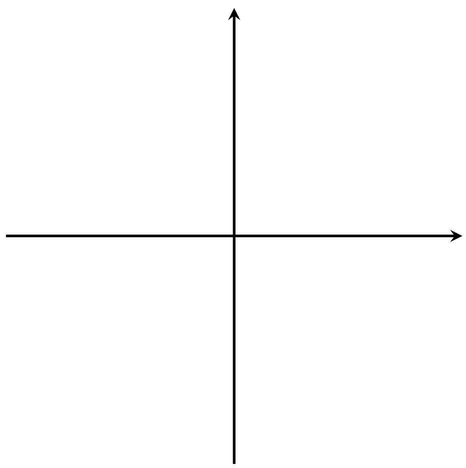
   i) $f(x)$ with $f'(x) = 0$,

   ii) $f(x)$ and $g(x)$ with $f'(x) = 1/2 = g'(x)$,

   iii) $f(x)$ and $g(x)$ with $f'(x) = g'(x)$,

   iv) $f(x)$ with $f'(x) > 0$,

i)



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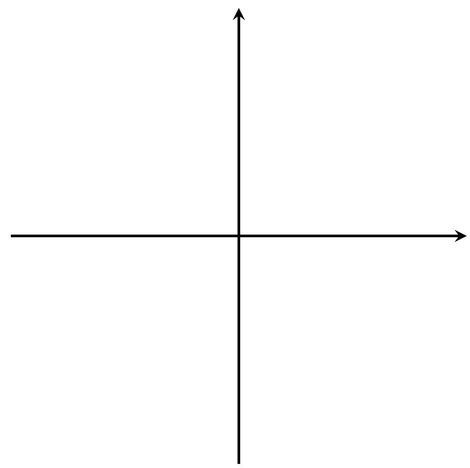

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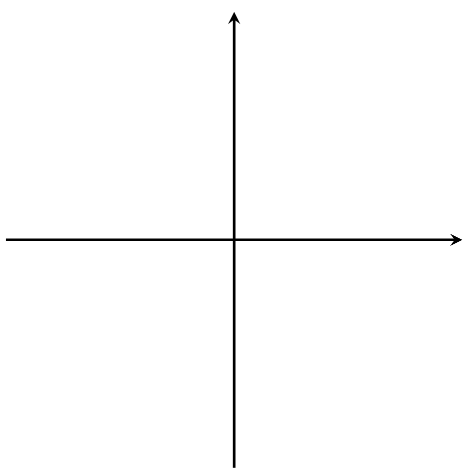

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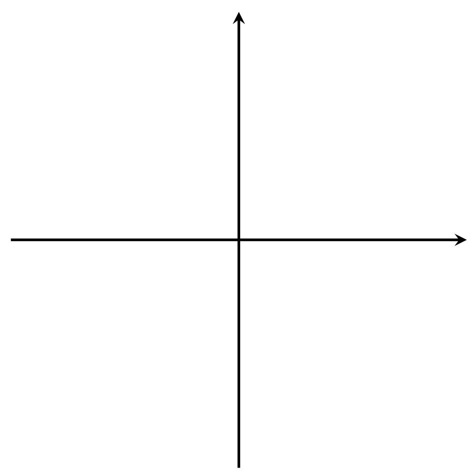


The goal of this activity is to introduce the students to corollaries 1 and 2 by having them see what these mean in graphical terms

(b) For each graph, justify your sketch using the Mean Value Theorem.
For each case, there are actually infinitely many functions that satisfy the given conditions. Thus identify what the important feature(s) of your sketch is/are.

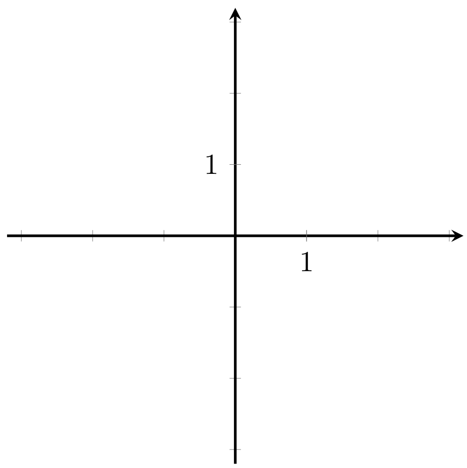
One can also only ask justifications for i) and iv): derivative is 0 implies constant function and positive derivative implies increasing function).

One could add 1 or 2 questions related to the homework on that topic.

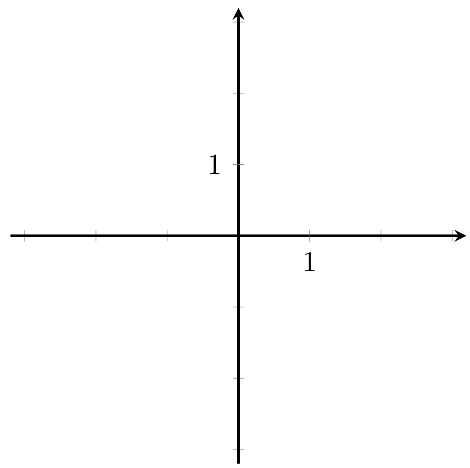
4. For each of the following cases, draw a sketch for $f(x)$ where $f(x)$ is defined everywhere and where:
   a) $f'(x) < 0$ on $(-3,-1)$, $f'(-1) = 0$, and $f'(x) < 0$ again on $(-1,3)$,
   b) $f'(x) < 0$ on $(-3,-1)$, $f'(-1) = 0$, and $f'(x) > 0$ on $(-1,3)$,
   c) $f'(x) < 0$ on $(-3,-1)$, $f'(-1)$ does not exist, and $f'(x) < 0$ on $(-1,3)$,
   d) $f(x)$ that is decreasing on $(-3,-1)$, increasing on $(-1,1)$, neither increasing nor decreasing on $(1,2)$ and increasing on $(2,3)$.

   For each of the preceding cases, what "kind" of point is $f(-1)$?

a)



c)



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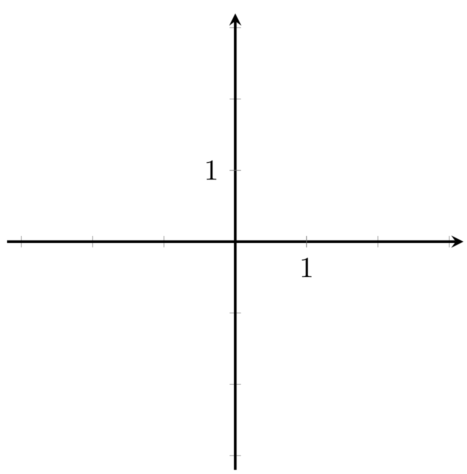

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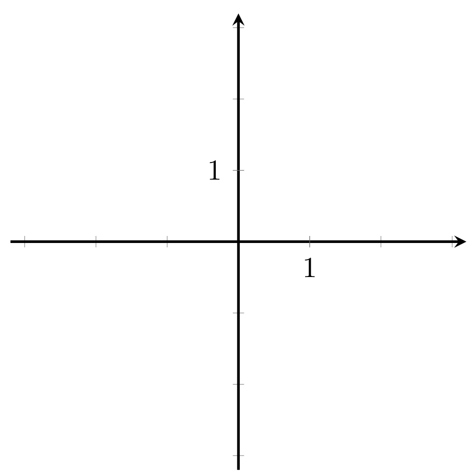

Next time we will see how to use this concretely.

1. What are Rolle's Theorem and the Mean Value Theorem (MVT)?
What are their assumptions? How are they related to one another?

2. Here is a concrete application of the Mean Value Theorem.
In the state of New South Wales, in Australia, the police uses "point-to-point" speed cameras. These cameras evaluate the speed of a given car by measuring how long it takes the car to go from point A to point B (hence the name).

   (a) What "kind" of speed is measured by such speed cameras? What are the advantages and disadvantages of such a system (in terms of measuring speed) compared to a "normal" speed camera?

   .

   (b) Imagine that using such a point-to-point speed camera, the police fines a driver for driving at 100 kph on a portion of the highway limited at 90 kph. The driver appeals the decision using the following argument. While its average speed was measured as being 100 kph, nothing proves that there was any given point where he drove at that speed. Yet, the law says that one is to be fined if one is observed going over the speed limit *at a given point*.
   The District Attorney calls you as an expert witness as she hopes you can help her show that the fine should be upheld. With your knowledge of calculus, how can you help her?

3. (a) For each of the following cases, draw a sketch for $f(x)$ (and where needed of $g(x)$):

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   ii) $f(x)$ and $g(x)$ with $f'(x) = 1/2 = g'(x)$,

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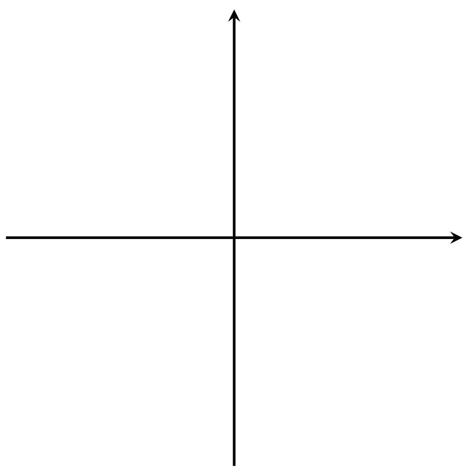
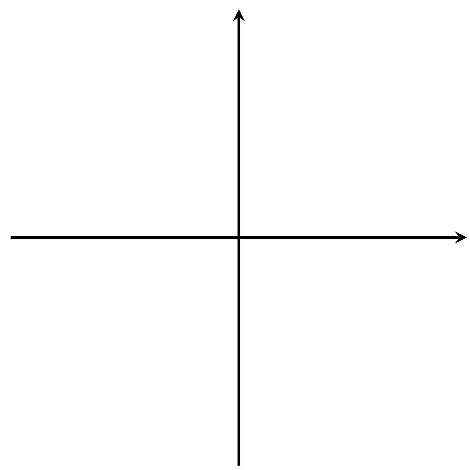
   iv) $f(x)$ with $f'(x) > 0$,

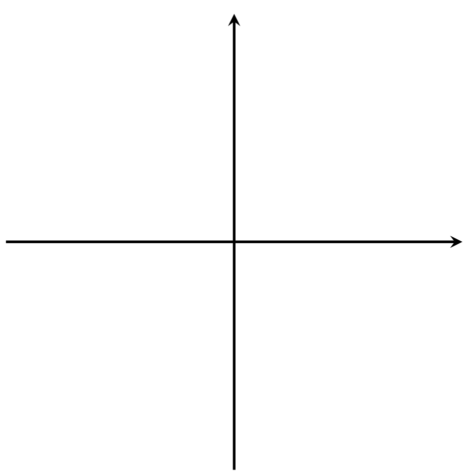
i)



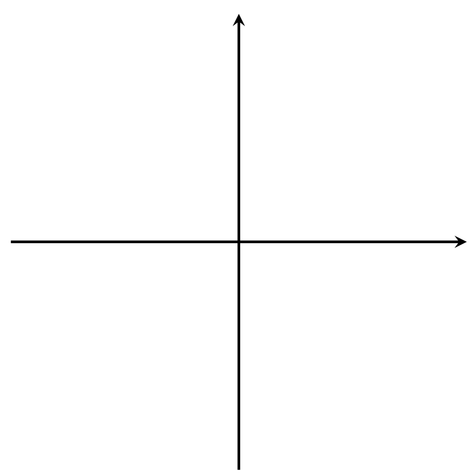
iii)



ii)



iv)



(b) For each graph, justify your sketch using the Mean Value Theorem.
For each case, there are actually infinitely many functions that satisfy the given conditions. Thus identify what the important feature(s) of your sketch is/are.

4. For each of the following cases, draw a sketch for $f(x)$ where $f(x)$ is defined everywhere and where:

   a) $f'(x) < 0$ on $(-3, -1)$, $f'(-1) = 0$, and $f'(x) < 0$ again on $(-1, 3)$,

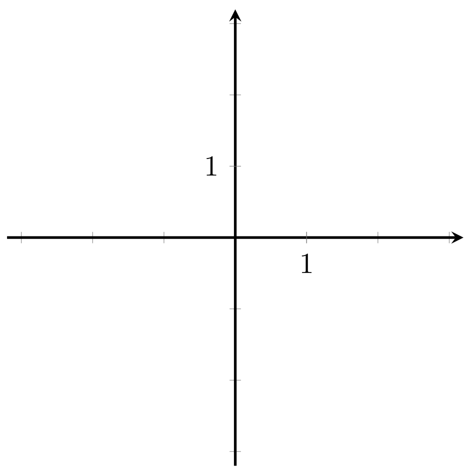
   b) $f'(x) < 0$ on $(-3, -1)$, $f'(-1) = 0$, and $f'(x) > 0$ on $(-1, 3)$,

   c) $f'(x) < 0$ on $(-3, -1)$, $f'(-1)$ does not exist, and $f'(x) < 0$ on $(-1, 3)$,

   d) $f(x)$ that is decreasing on $(-3, -1)$, increasing on $(-1, 1)$, neither increasing nor decreasing on $(1, 2)$ and increasing on $(2, 3)$.

   For each of the preceding cases, what "kind" of point is $f(-1)$?

a)



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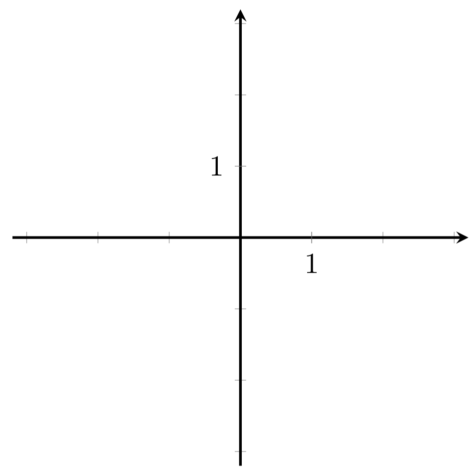

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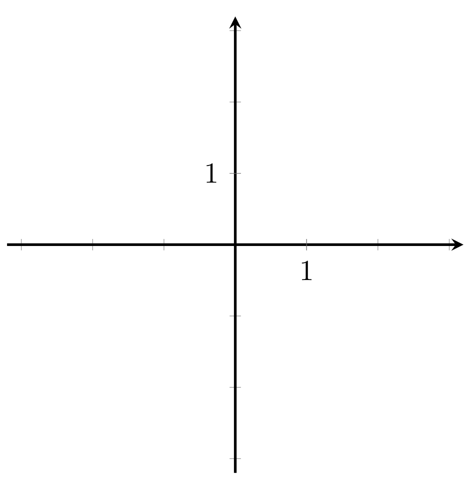

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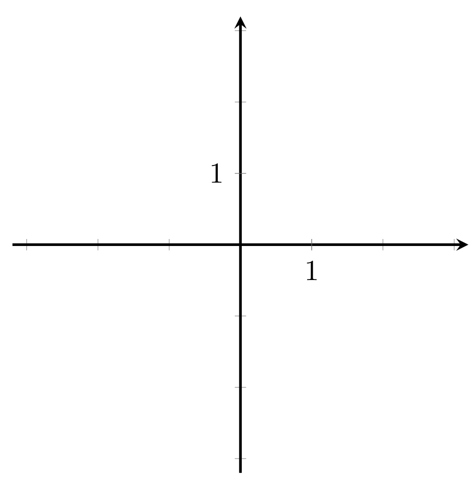


Next time we will see how to use this concretely.

Theorem (Extreme Value Theorem)**.** *If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a.b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$.*

a) Draw a graph of a function that satisfies the hypotheses of the Extreme Value Theorem.

b) Draw a graph of a function that satisfies the results of the Extreme Value Theorem.

c) Draw a graph of a function that satisfies the results of the Extreme Value Theorem but not it's hypotheses.

Theorem. *Rolle's Theorem Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.*

a) Draw a graph of a function that satisfies the hypotheses of Rolle's Theorem.

b) Draw a graph of a function that satisfies the results of Rolle's Theorem.

c) Draw a graph of a function that satisfies the results of Rolle's Theorem but not it's hypotheses.

Theorem. *Mean Value Theorem Suppose $y = f(x)$ is continuous over a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

a) Draw a graph of a function that satisfies the hypotheses of the Mean Value Theorem

b) Draw a graph of a function that satisfies the results of the Mean Value Theorem

c) Draw a graph of a function that satisfies the results of the Mean Value Theorem but not it's hypotheses.

The First Derivative Test (4.3)

Expected Skills.
At the end of this section, students should be able to:

- use the first derivative test to determine the nature of an extremum.

Pre-Class Activity (ch4-applications-4-firstderivtest-1-pc)**.** The pre-class activity asks students to graph functions which have certain properties. The goal here is to prepare the discussion on the first derivative test, i.e. how do we determine the nature of critical point by looking at its first derivative.

Worksheet (ch4-applications-4-firstderivtest-2-ws)**.** The first activity on the worksheet is similar to the one in the pre-class activity. The difference is that here we use the terminology of local min/max.
The second exercise is a more "traditional" application of the First Derivative Test where we ask the students to find the local min/max of a function.

The general question we want to solve is the following: once we have found the critical points of a function, how can we determine the nature of these critical points using the first derivative?

For each of the following cases, draw a sketch for $f(x)$:

a) $f(x)$ with $f'(x) < 0$,

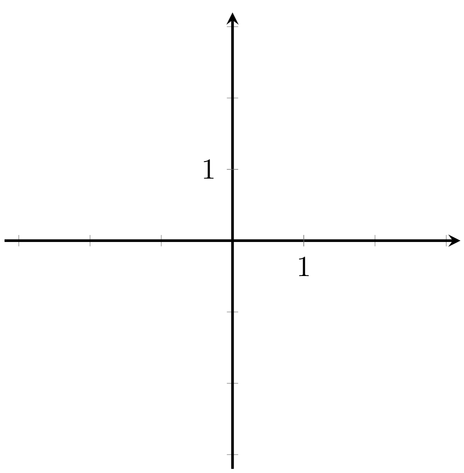
b) $f(x)$ with $f'(x) < 0$ on $(-3, -1)$, $f'(-1) = 0$, and $f'(x) < 0$ again on $(-1, 3)$,

c) $f(x)$ with $f'(x) < 0$ on $(-3, -1)$, $f'(-1) = 0$, and $f'(x) < 0$ on $(-1, 3)$,

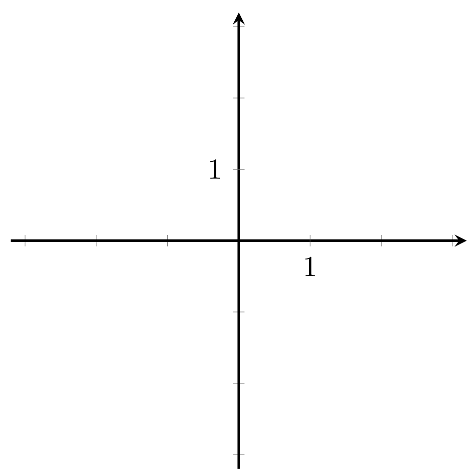
d) $f(x)$ with $f'(x) < 0$ on $(-3, -1)$, $f'(-1)$ does not exist, and $f'(x) < 0$ on $(-1, 3)$,

For each of the preceding cases, what "kind" of point is $f(-1)$.

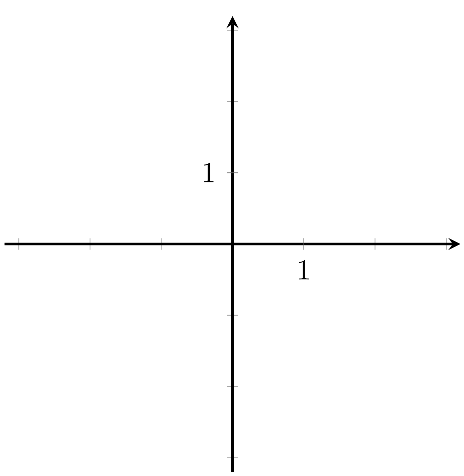
a)



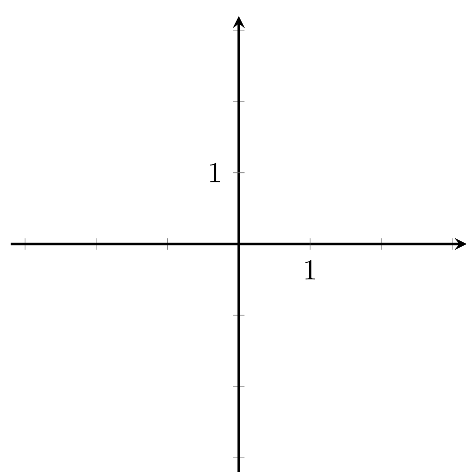
c)



b)



d)



1. Last time we saw that as a consequence of the Mean Value Theorem, for a differentiable function $f(x)$:

   if $f'(x) > 0$ then the functions is ...

   if $f'(x) < 0$ then the functions is ...

   Based on this observation, we can formulate the **First Derivative Test** (p. 239 in Thomas):
   If c is a critical point of a differentiable function and :

   - if $f'(x)$ changes from negative to positive at c, then c is
   - if $f'(x)$ changes from positive to negative at c, then c is
   - if $f'(x)$ does not change sign at c, then c is

2. Let us see an application of the First Derivative Test
   (a) Consider the function $f(x) = x^3 - 5x^2 + 8x - 4$.
      i. What are the critical points?

      ii. On what intervals is $f(x)$ increasing or decreasing?

      Discuss here why the intervals should be open and not closed.

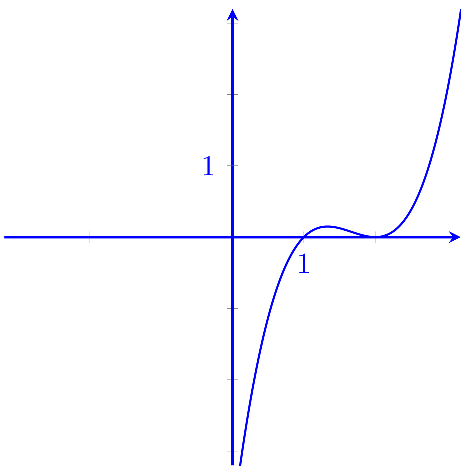
      iii. What are the local maxima and minima of $f(x)$ (if they exist)? Give their coordinates.

iv. What are the global maxima and minima of $f(x)$ (if they exist)? Give their coordinates.

(b) Same questions but for $f(x)$ defined on the interval $[-3, 3]$.
What changes and what remains the same?

We now have global min and max.

Look in the TeX file for the next case and add it if you find it interesting.



1. Last time we saw that as a consequence of the Mean Value Theorem, for a differentiable function $f(x)$:

   if $f'(x) > 0$ then the functions is ...

   if $f'(x) < 0$ then the functions is ...

   Based on this observation, we can formulate the **First Derivative Test** (p. 239 in Thomas):
   If c is a critical point of a differentiable function and :

   - if $f'(x)$ changes from negative to positive at c, then c is
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2. Let us see an application of the First Derivative Test

   (a) Consider the function $f(x) = x^3 - 5x^2 + 8x - 4$.

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(b) Same questions but for $f(x)$ defined on the interval $[-3, 3]$.
What changes and what remains the same?

Concavity and Curve Sketching (4.4)

Expected Skills.
At the end of this section, students should be able to:

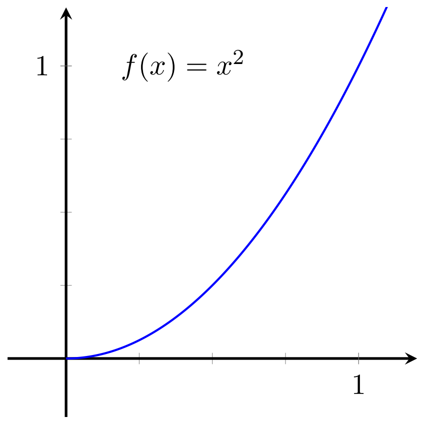
- explain the difference between concave up and concave down,
- use the second derivative of a function to determine:
  - on what interval(s) a curve is concave up, respectively concave down,
  - where the inflection points are,
  - the nature of a local extremum,
- qualitatively sketch the graph of a function using the information provided by the first and second derivatives,
- given the algebraic expression of a function as well as its graph (e.g. using a graphing software), qualitatively verify that the curve corresponds to the given function.

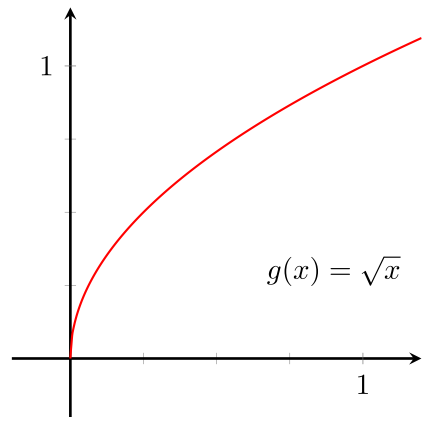
Pre-Class Activity (ch4-applications-5-concavity-1-pc)**.** The goal of this pre-class activity is to have the students *see* the difference between concave up and concave down.We also want to convey the message that looking at the second derivative gives us new information compared to looking only at the first derivative.

Worksheet (ch4-applications-5-concavity-2-ws)**.** In the class activity we start with the definition of the concavity. We can base ourself on the pre-class activity. We then ask the students to think about the second derivative test.
In the second exercise we first ask students to basically identify an inflection point (without telling them it is called an inflection point). When then introduce the notion of inflection point. Then, we have the students identify inflections point of various functions. Finally, we have the students look at "what happens" to the second derivative when the function has an inflection point; in particular, we underline that just looking for $f''(x) = 0$ or $f''(x)$ does not exist is not enough.
In the third exercise we have the students identify the steps for curves sketching and then have them sketch various functions.

In this activity we look at two increasing functions. We will see how we can describe the shape of their graphs more precisely by looking at the second derivative.

Consider the functions $f(x) = x^2$ and $g(x) = \sqrt{x}$ on the interval $(0, 1)$.





1. Are these functions increasing or decreasing? Verify it by computing the derivatives $f'(x)$ and $g'(x)$.

2. Draw the tangent lines (by hand) for $x = 0.25, 0.5, 1.5$.
   In these two examples, where are the tangent lines situated with respect to the functions?

3. How do the slopes of the tangent lines change as x increases?

4. Compute the second derivatives $f''(x)$ and $g''(x)$. What are their signs on $(0, 1)$?

5. What can we say about the relationship between the shape of a function and the sign of the second derivative?

Definition
A differentiable function $f(x)$ is *concave up* on an interval (a, b):

when the derivative is increasing (or equivalently when the tangent lines are below the graph of the function)

and *concave down* on (a, b) when:

when the derivative is decreasing (or equivalently when the tangent lines are above the graph of the function)

1. How can we test the concavity (up or down) of a function $f(x)$ that is twice differentiable?

   Introduce the second derivative test (p. 243 in Thomas).

2. (a) Consider the function $f(x) = x^3 + 3x^2 - 1$. On what intervals is the function concave up, respectively concave down?

   At the end of this example, introduce the notion of inflection point.

   Definition
   A point $(a, f(a))$ on a function is an *inflection point* if

   What was the inflection point in the previous example?

(b) Compute the inflection points for the following functions:

ii. $f(x) = x^3$ *$f''(0) = 0$ and $x = 0$ is an inflection point*

iii. $f(x) = \sqrt[3]{x}$ *$f''(0)$ doesn't exist and $x = 0$ is an inflection point*

iv. $f(x) = x^4$ *$f''(0) = 0$ but $x = 0$ is not an inflection point*

(c) Look at the four functions of part 2. For each function, compute the value of the second derivative at the inflection point. What can we conclude from this?

For $f(x) = x^3 + 3x^2 - 1$: inflection point at $x = -1$ and $f''(-1) = 0$.

One really needs to check that $f''(x)$ changes its sign. Just having $f''(a) = 0$ or $f''(x0$ does not exist at $x = a$ is not enough.

3. We now have all we need to sketch functions (or to check that the graph given by a graphing software is correct). List the pieces of information one needs to do a "good" sketch ? *adapted from Thomas p. 248*

    1. *Identify the domain of the function $f(x)$*
    2. *Compute $f'(x)$ and $f''(x)$)*
    3. *Find the critical points and the behavior of the function at these points*
    4. *Determine the intervals where the function is increasing/decreasing*
    5. *Determine the points of inflection and the concavity of the function*
    6. *Identify the possible asymptotes (if any)*
       Could add: plot the points identified in the previous steps

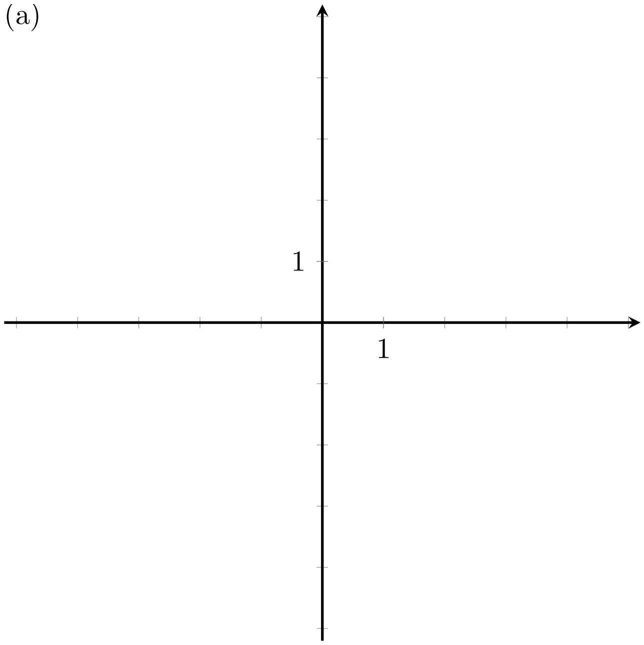
4. Using the procedure described above, sketch the following functions on the following page:

    (a) $f(x) = x\sqrt{9-x^2}$ *underline here that an inflection point is not necessarily a critical point!*

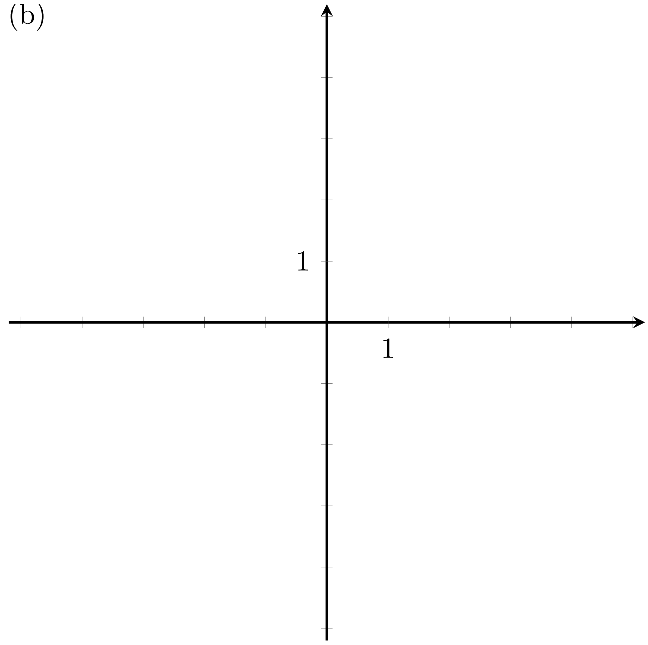
    (b) $f(x) = \dfrac{x^2-3}{x^2-4}$ *has vertical and horizontal asymptotes*

    (c) $f(x) = \sqrt{|x|}$ *$f''(x)$ doesn't exists at $x = 0$,this doesn't mean necessarily there is a change in concavity*

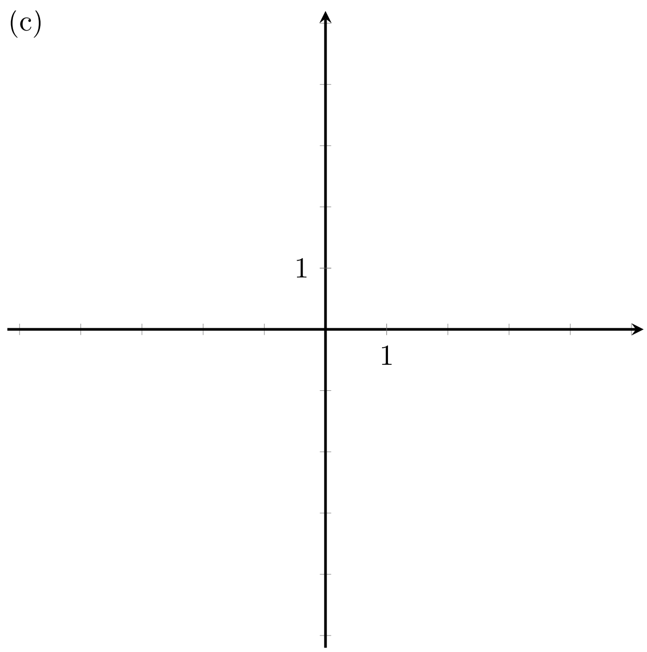
(a)



(b)



(c)



Definition
A differentiable function $f(x)$ is *concave up* on an interval (a, b):

and *concave down* on (a, b) when:

1. How can we test the concavity (up or down) of a function $f(x)$ that is twice differentiable?

2. (a) Consider the function $f(x) = x^3 + 3x^2 - 1$. On what intervals is the function concave up, respectively concave down?

Definition
A point $(a, f(a))$ on a function is an *inflection point* if

(b) Compute the inflection points for the following functions:

ii. $f(x) = x^3$

iii. $f(x) = \sqrt[3]{x}$

iv. $f(x) = x^4$

(c) Look at the four functions of part 2. For each function, compute the value of the second derivative at the inflection point. What can we conclude from this?

3. We now have all we need to sketch functions (or to check that the graph given by a graphing software is correct). List the pieces of information one needs to do a "good" sketch ?

    1.

    2.

    3.

    4.

    5.

    6.

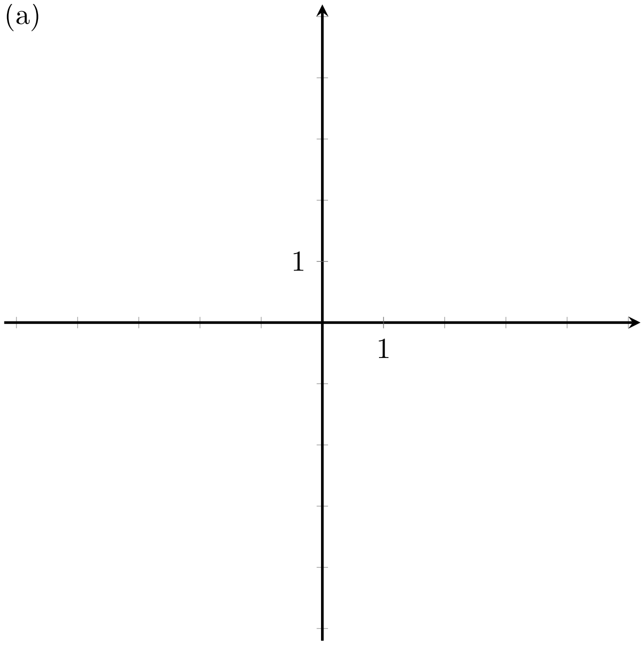
4. Using the procedure described above, sketch the following functions on the following page:

    (a) $f(x) = x\sqrt{9-x^2}$

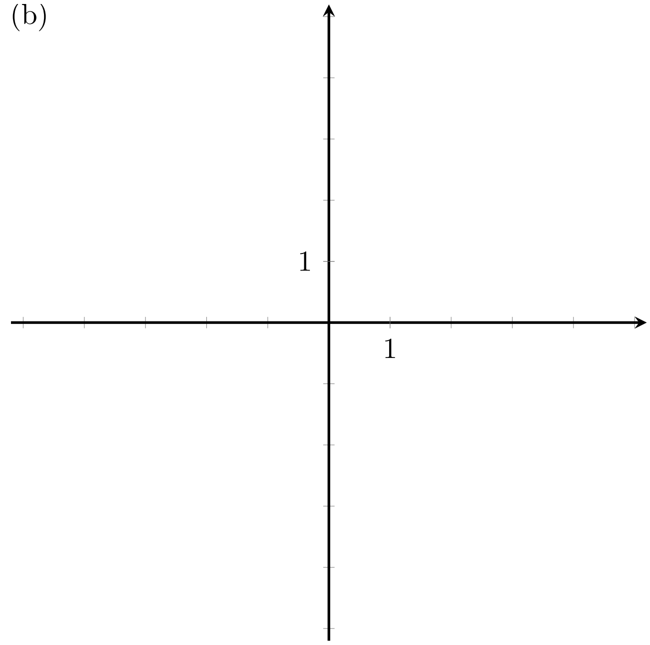
    (b) $f(x) = \dfrac{x^2-3}{x^2-4}$

    (c) $f(x) = \sqrt{|x|}$

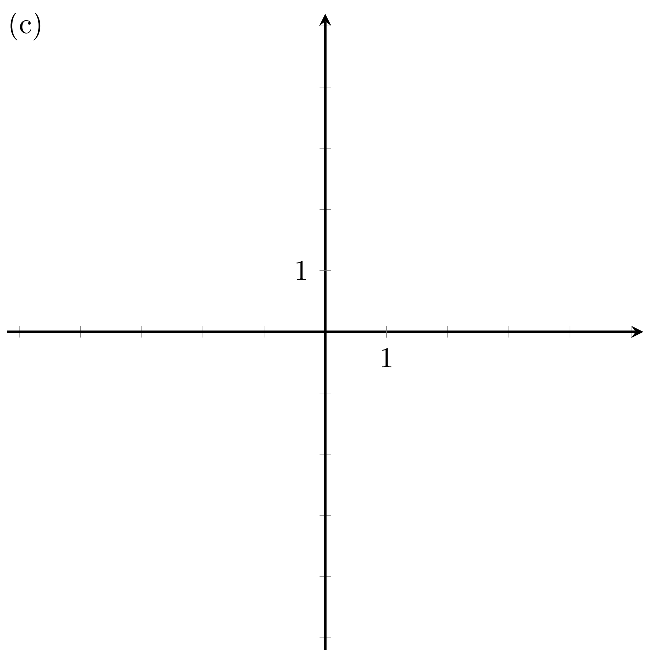
(a)



(b)



(c)



Let us consider the function $f(x) = x^2 + \frac{2}{x}$. Let us follow the curve sketching procedure to sketch this function.

1. What is the domain of $f(x)$?

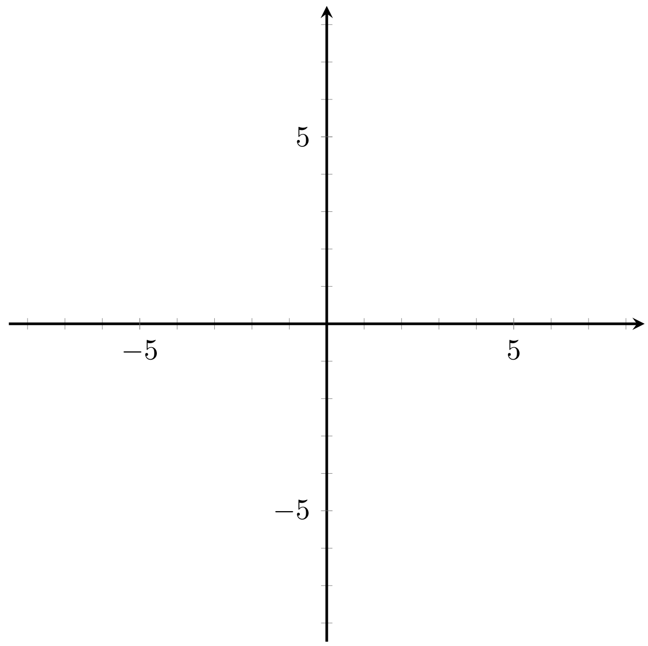
2. Compute $f'(x)$ and $f''(x)$.

3. Find the critical points of $f(x)$.

4. Find the intervals on which the functions is increasing/decreasing. Determine if the critical points are local min/max or neither.

5. Determine the intervals where the functions is concave up/dow and where concavity changes.

6. Identify the asymptotes (if any).


5
−5
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5

Optimization (4.6)

Expected Skills.
At the end of this section, students will be able to:

- build an appropriate mathematical model for word problems. This includes:
  - assign variables to appropriate quantities,
  - identify which numerical information is relevant and/or needed,
  - relate the variables using appropriate equations taking into account the numerical information provided,
- solve word problems using the differentiation techniques seen earlier in the term and determine the optimal solution,
- for a given problem, clearly explain with words, mathematical symbols and equations their reasoning, in particular, what is known, what we are looking for and the steps of the procedure to solve the question.

Pre-Class Activity (ch4-applications-6-optimization-1-pc)**.** The pre-class activity guides the students through a typical optimization exercise. The idea is that since the problem is broken down they will be able to solve it. We then ask them to identify the steps to solve such problems.

Worksheet (ch4-applications-6-optimization-2-ws)**.** The in-class activity consist of having the students solve optimization exercises of different types.
A good way of teaching these would probably be to first do one example on the board, then have the students work on the subsequent exercises (or if you feel your class is strong enough to have the students work directly on the exercises since they have had the pre-class activity). One can have the students work alone, in groups or do a “jigsaw”.

Minimizing perimeter

At the end of winter, a farmer needs to install a fence around the field in which her cows are going to spend the summer. In order for the cows to have enough space, the field should have a surface of 4 km^2. Out of simplicity, the field has a rectangular shape. In order to save time and money when installing the fence, the farmer wants her field to have the smallest possible perimeter.

1. What are we trying to optimize (in that case minimize)?

2. Draw a sketch of the situation and assign variables to the various lengths.

3. Write down the equation of the perimeter of the field.

   Then, using that the area of the field is 4 km^2, find a relation between the two variables. Use this relation to rewrite the perimeter of the field as a function of a single variable.

4. Find the minimum value for the perimeter (using differentiation).

Identify the steps you used to solve this exercise.

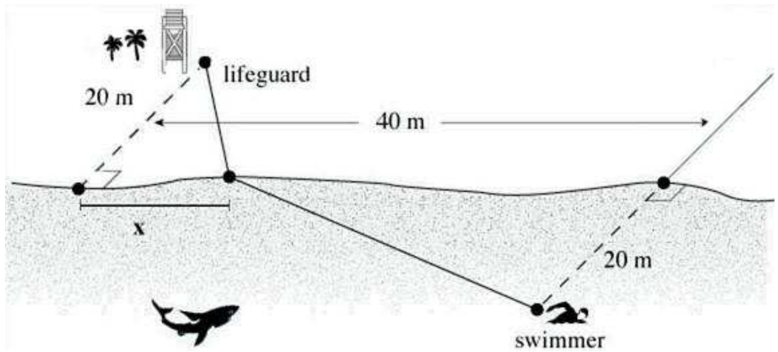
Steps for Solving Optimization Exercises

1. **Identify the quantity to be optimized**.
2. **Draw a picture** representing the situation. Label any part that is relevant.
3. **Introduce variables**. List every relation in the picture and in the problem as an equation or expression, and identify the unknown variables.
4. **Write an equation for the quantity you want to optimize**. Use the relations from the previous step to turn it into a function of a single variable. (This may require considerable manipulation.)
5. **Solve the Exercise**. Determine the domain of your function. Use the first and second derivative tests to identify and classify the critical points. Check critical points and endpoints to find the optimal value.

1. **Soda Can**
   You have to design a new can for a soda company. It has to have a volume of 330 ml (or 330 cm^3). The base and top cost 0.2 dollar per square centimeter to manufacture whereas the side costs 0.05 dollar per square centimeter to manufacture. What are the radius and height that minimize the cost of production?

   One could (as a second step) also ask: what if the top and bottom parts cost 5 times more than the side. (this is basically the same questions, just a bit more abstract).

2. **The Fastest Route**
Sophie, a lifeguard, sees a swimmer in trouble some distance down the beach and wants to reach him in minimum time. On land, Sophie can run at speed $v_L = 6$ meters per second, while in the water she can swim at speed $v_W = 2$ meters per second.



(a) Sketch the situation with a bird-eye view.

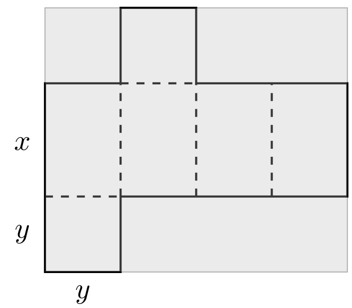
(b) Write a formula for the time required for Sophie to reach the swimmer if she enters the water at a point located x units down the beach from her station. Then compute the derivative of this function and explain how you can find the minimum. You don't need to actually find a value for x.

It turns out the computations to actually find x are complicated, hence we only ask the students to write down the equations need to solve the problem.

3. **U.S. Postal Service Package**
   The U.S. Postal Service only accept packages for standard shipment whose "combined length and girth" has a maximum of 108 inches. The girth is the "distance around" or put more simply, the addition of twice the height and twice the width of the package. If we have a package with a square end, i.e. for which the height equals the width, what are the dimensions that will maximize the volume?

   If students are confused, draw sketch similar to the one of ex. 20a, p. 271 in Thomas.

4. **Minimizing Cardboard**
Small boxes are sometimes made by folding cardboard shape like this one.



In our case, we want build a box of 1m^3. If we cut this shape out of a rectangular piece of cardboard (the grey-shaded area), what dimensions of the box will minimize the total area of the piece of cardboard?

5. **Best Picture of the Statue of Liberty**
   You are on a boat cruising around the Statue of Liberty in New York city. You want to take the best possible picture. To do so, you should maximize the angle under which you see the Statue (i.e. the angle formed by: the top of the torch, your camera and the feet of the statue).
   You know that the statue is 92-meter high and stands on a base that is 42-meter high. How far from the base should you be?

Steps for Solving Optimization Exercises

1. **Identify the quantity to be optimized**.
2. **Draw a picture** representing the situation. Label any part that is relevant.
3. **Introduce variables**. List every relation in the picture and in the problem as an equation or expression, and identify the unknown variables.
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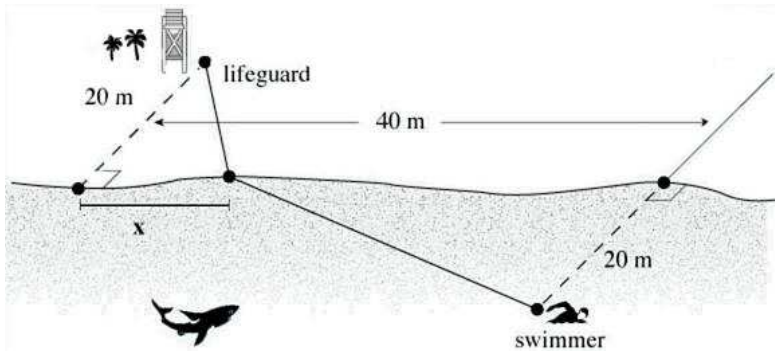

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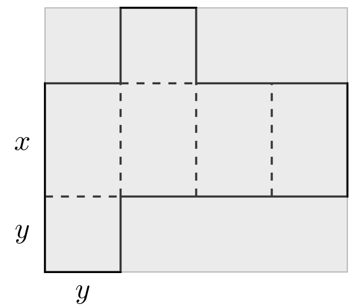
4. **Minimizing Cardboard**
Small boxes are sometimes made by folding cardboard shape like this one.



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   You know that the statue is 92-meter high and stands on a base that is 42-meter high. How far from the base should you be?

Antiderivatives (4.8)

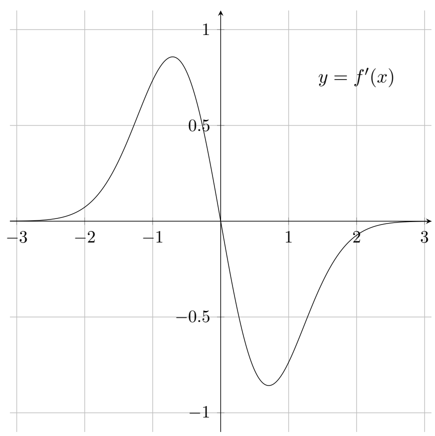
Expected Skills.
At the end of this section, students should be able to:

- compute the antiderivative of "simple" functions with or without initial value,
- verify that a function is the antiderivative of another function.

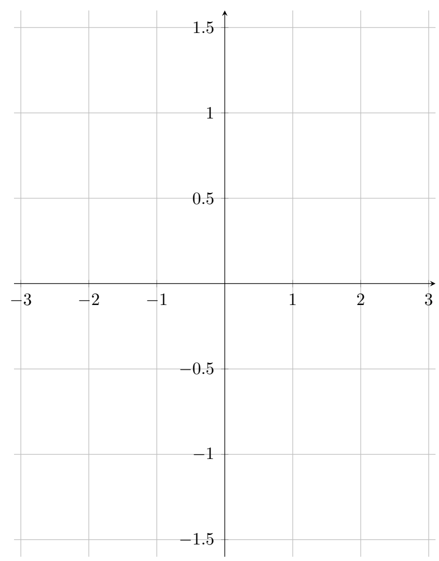
Worksheet (ch4-applications-7-antiderivatives-1-ws)**.** First we ask students to start with the graph of a function and draw the graph of its antiderivative.
Then we have students "compute" the antiderivatives of various functions (including some that students won't know to compute or that are "impossible").
Finally, we have them compute antiderivatives with initial condiitons.

1. Given the two derivatives below, sketch the two original functions.

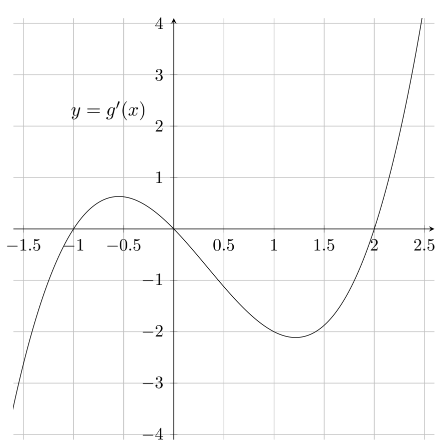
$f'(x)$ on the interval $[-2, 2]$



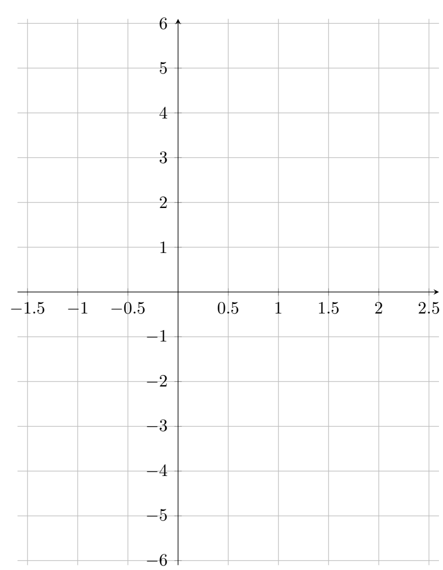
Sketch $f(x)$ on the interval $[-2, 2]$



$g'(x)$ on the interval $[-1.5, 2.5]$



Sketch $g(x)$ on the interval $[-1.5, 2.5]$



2. Could you give another possible antiderivative in either of the two examples? How many antiderivatives could a function have? How are they related?

3. Suppose that $F'(x) = f(x)$, $G'(x) = g(x)$ and c is a constant. Fill in the following table, where you can!

Function	**Particular antiderivative**
x^n $(n \neq -1)$	
$\frac{1}{x}$	
e^x	
e^{x^2}	
$\cos(x)$	
$\sin(x)$	
$\tan(x)$	
$\frac{1}{\cos^2(x)}$	
$c \cdot f(x)$	
$f(x) + g(x)$	
$F(x) \cdot g(x) + f(x) \cdot G(x)$	
$f(x) \cdot g(x)$	
$\frac{1}{1+x^2}$	
$\frac{x^2-2}{x^4+2x^2-3}$	

Theorem. *If F is an antiderivative of f on an interval $[a, b]$, then any other antiderivative of f on $[a, b]$ is of the form*

$$F(x) + C$$

where C is a constant.

In any particular scenario, the constant term can be determined if you specify one value of the antiderivative. In the following questions, please find the particular antiderivatives $f(x)$.

4. $f'(x) = \sqrt{x}(6 + 5x)$ where $f(1) = 10$.

5. $f'(x) = \frac{4}{\sqrt{1-x^2}}$ where $f(1/2) = 1$.

6. $f''(x) = 2 + \cos(x)$ where $f'(0) = 2$ and $f(0) = 3$.