Area and Estimating with Finite Sums (5.1)

Expected Skills.
At the end of this section, students will be able to:

- explain how the process of estimating an area with finite sums work and use it,
- use this process to estimate lengths, speeds and the average value of a function.

Pre-Class Activity (ch5-integrals-1-areas-1-pc)**.** In the pre-class activity we take an example from pharmacokinetics to have the students estimate the area under a curve. The goal here is threefold:

- having them realize that they need to compute the area under the curve,
- have them think about how the can overestimate or underestimate the area,
- how they can improve a given estimate.

Worksheet (ch5-integrals-1-areas-2-ws)**.** In the class activity, we build up from what we have done in the pre-class activity. Here we want to estimate the area under the line $x = y$. We purposefully take an easy function and an area that students can compute without calculus. The goal of this is that the can compare their answers with the exact value.
First, we ask them to find **as many ways as possible** to estimate the given area. It is important to underline that we want methods that could work for all types of functions, not only lines. It is also important that they should find **as many as possible** and not just 2 or 3. To "gamify" this part, one could form teams and see which teams comes up with the most ways. For each technique, it is important to ask them to determine if they are under- or over-estimating the area (and if the under/overestimation is inherent to the method or not).
Then we ask them to use the "traditional" left-handed, right-handed and midpoint rectangles.
In part 3, we ask students to do something similar but for a half-circle. The goal here is to reinforce what has just been done. The goal is also to push students to think about other ways to approximate an area, in particular using trapezoids.
Finally, in part 6 we have an "application" of the area under the curve by asking the students to compute the distance traveled by a particle for which we know the velocity.

Supplemental Activity (ch5-integrals-1-areas-3-sup-rectrules)**.** In the supplemental activity, students practice computing area approximations using various rectangle rules. It is suggested that this activity is prefaced by a discussion about rectangle rules and

- how an interval is subdivided,
- how each subinterval generates a rectangle, and
- how each rectangle contributes to the area estimate.

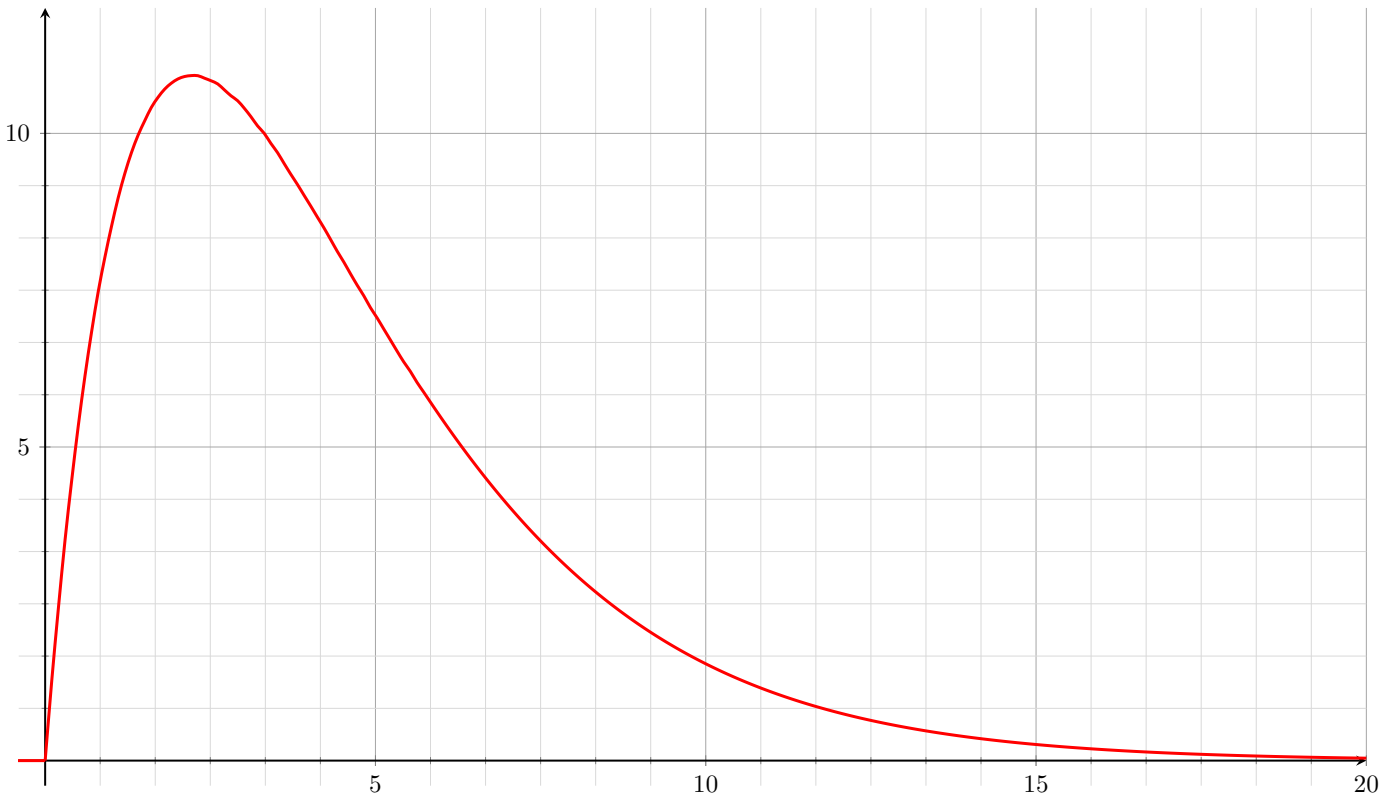
Given the variety of rectangular rules, the instructor should divide the class into groups with assigned rules. The groups should then complete the handout and bring back to the classroom their findings. The instructor

can then lead a debriefing discussion about patterns that students may see in their work and may challenge the students to think about when particular rules overestimate or underestimate areas.

Pharmacokinetics is the process whereby substances (like food and drugs) are ingested into the body and processed. When a nurse first administers a drug, the concentration of the drug in the blood stream is zero. As the drug moves around the body and is metabolized, the concentration of the drug increases. There comes a point when the concentration no longer increases and begins to decline. This is the period when the drug is fully distributed and metabolism is taking place. As time goes on, the drug concentration continues to decrease. The concentration can be modeled by the following piecewise defined function:

$$C(t) = \begin{cases} 0 & t < 0 \\ 133.3 \cdot \left(e^{-0.4t} - e^{-0.5t}\right) & t \geq 0, \end{cases}$$

where t is time measured in hours, and $C(t)$ is the concentration of a single dose in the body, measured in mg/L, t hours after ingestion. This is graphed below.



When administering chemotherapies to treat cancer, for example, doctors must ensure that there is enough drug present to kill the cancerous cells while not overdoing it and killing the patient. The doctor must limit the **total exposure to a drug**. One can then compute what medical researchers describe this as the *time-averaged concentration of drug circulating in the body*. The general question is how can we compute the total exposure to a given drug and then the time-averaged concentration of the drug.

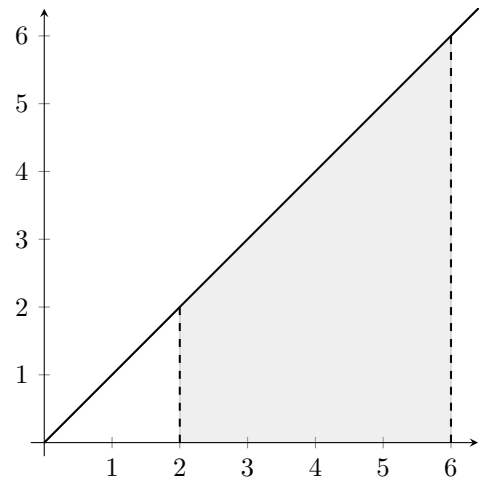
1. Using the graph above, how can we make an over-estimate of the total exposure?

2. Now, how would we make an under-estimate?

3. Given what we might think is a decent estimate, how could we refine our calculation? And then how could we refine even more our calculations?

Approximating Area Under a Curve

Let us start by estimating the area under a very simple "curve", namely the function $f(x) = x$. We want to approximate the area under this function between $x = 2$ and $x = 6$.
Let us underline that we want to find a **method that works for all kinds of functions and curves**, not only straight lines like this one (where we actually know how to compute the area).



1. Find as many ways as you can to estimate this area. In each case, determine if you are overestimating or under estimating the area, and how you could improve your estimate.

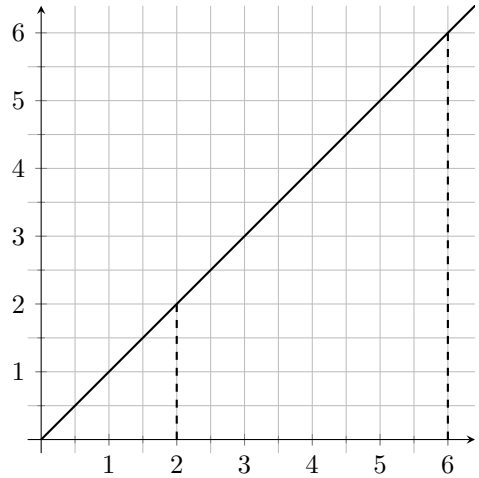
   One can think of left-handed rectangles, right-handed rectangles, mid-point rectangles, trapezoids, etc. Also underline that the intervals don't have to have the same length. Finally, as the length of the intervals decreases, the approximation improves.

   It is important here to ask the students to find ***as many ways as possible*** *so that those who know about integration (which probably the majority) don't stop with just left-handed and right-handed rectangles.*

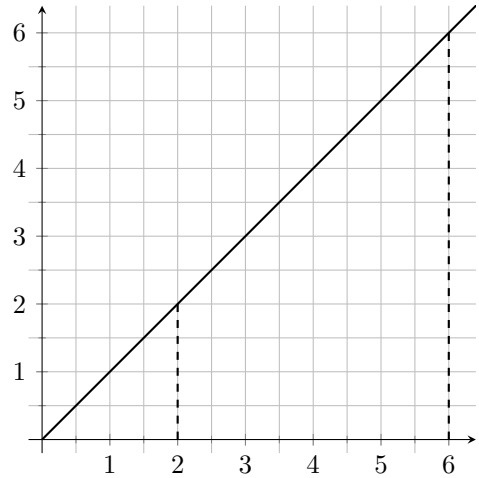
Three important ways of approximating an area under a curve are to use left-handed rectangles, right-handed rectangles, and mid-point rectangles.

2. On the graphs below, estimate the area under the function $f(x) = x$ between $x = 2$ and $x = 6$ using four rectangles with bases in the intervals [2, 3], [3, 4], [4, 5], [5, 6].
   In each case, draw the rectangles and compute the estimated area.

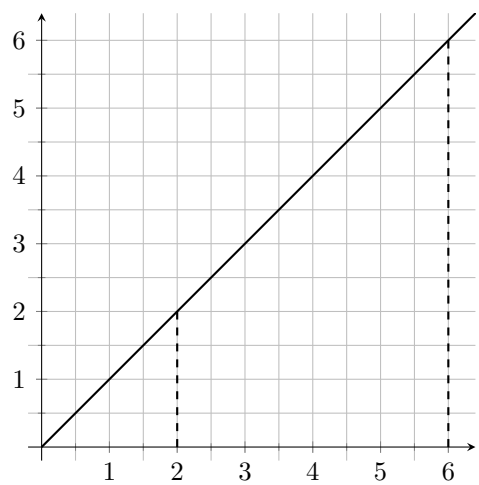
a) Left-handed rectangles



b) Right-handed rectangles

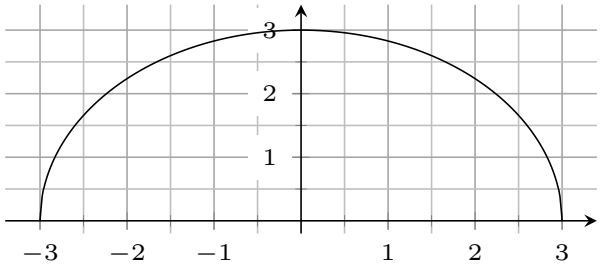


c) Midpoint rectangles

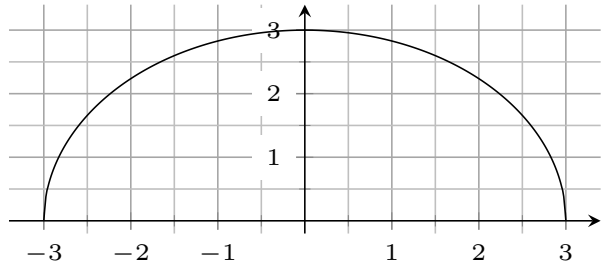


3. Let us now consider the half-circle whose equation is $f(x) = \sqrt{9 - x^2}$.
   We will compare the left-handed and right-handed rectangles. Before doing any computation, which one of these to sums do you think is going to be the biggest?

a) Left-handed rectangles



b) Right-handed rectangles



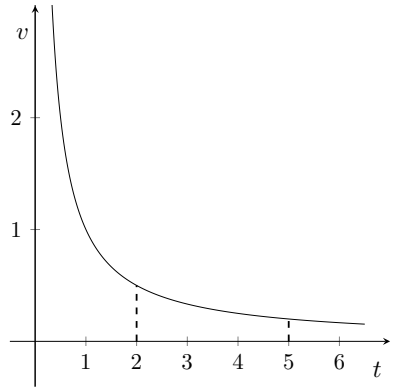
4. What do you get? How do you explain your result?

   Due to symmetry, the two sums are the same. Moreover, due to the shape of the circle, of the "rectangles" is flat and has an area equal to zero.

5. How could we improve the approximations for these areas? List as many ways as you can of how we could improve these approximation?

   One can use different shapes such as trapezoids (what would happen with the first function $f(x) = x$ if we did that?), or approximate the function with parabolas (Simpson's rule).
   One can also increase the number of rectangles (by increasing the number of subdivisions), and even take infinitely thin rectangles (i.e. take the limit).
   Students might need a bit of prompting.

6. The velocity of a particle is give by the functions $y = \dfrac{1}{x}$. Approximate the distance it has traveled between time $t = 2$s and $t = 5$s (use 3 rectangles for that).



Discuss here that computing the area under the curve correspond to computing the distance traveled.

Approximating Area Under a Curve

Let us start by estimating the area under a very simple "curve", namely the function $f(x) = x$. We want to approximate the area under this function between $x = 2$ and $x = 6$.
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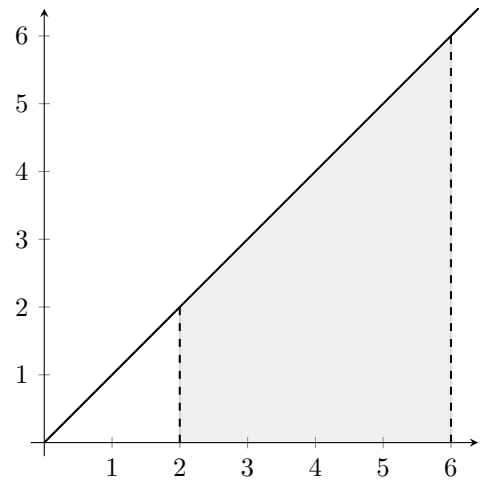
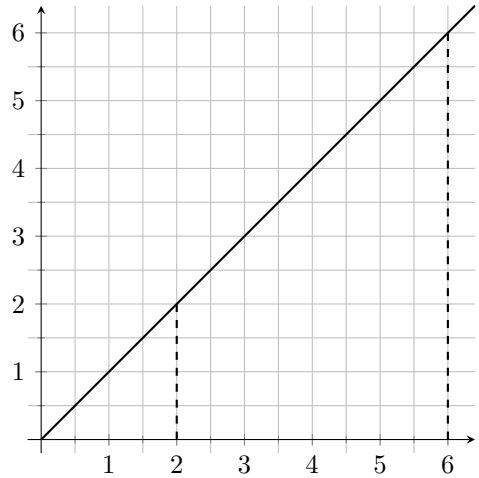


1. Find as many ways as you can to estimate this area. In each case, determine if you are overestimating or under estimating the area, and how you could improve your estimate.

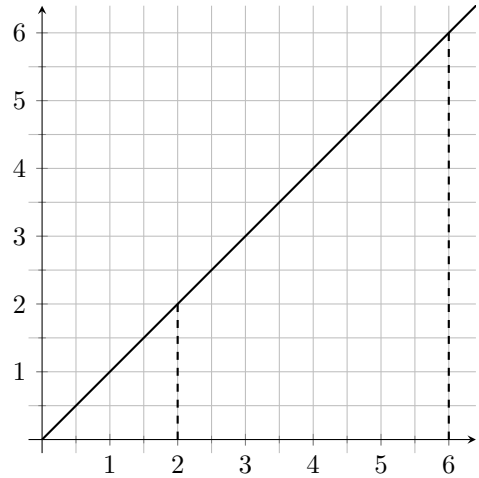
Three important ways of approximating an area under a curve are to use left-handed rectangles, right-handed rectangles, and mid-point rectangles.

2. On the graphs below, estimate the area under the function $f(x) = x$ between $x = 2$ and $x = 6$ using four rectangles with bases in the intervals [2, 3], [3, 4], [4, 5], [5, 6].
   In each case, draw the rectangles and compute the estimated area.

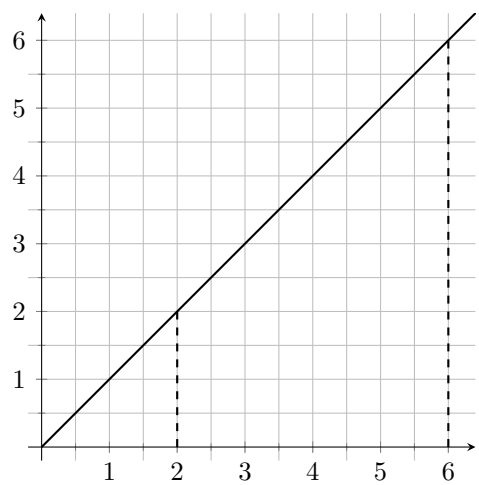
a) Left-handed rectangles



b) Right-handed rectangles

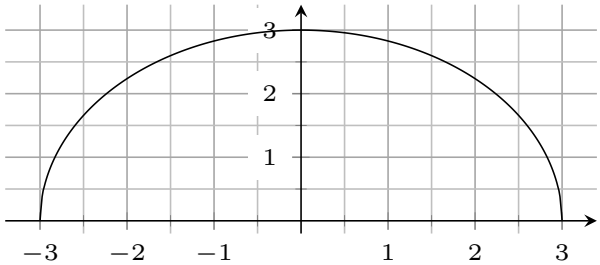


c) Midpoint rectangles

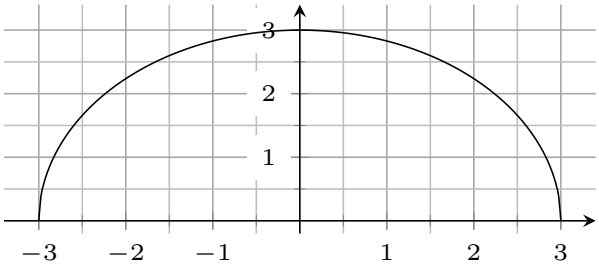


3. Let us now consider the half-circle whose equation is $f(x) = \sqrt{9 - x^2}$.
   We will compare the left-handed and right-handed rectangles. Before doing any computation, which one of these to sums do you think is going to be the biggest?

a) Left-handed rectangles



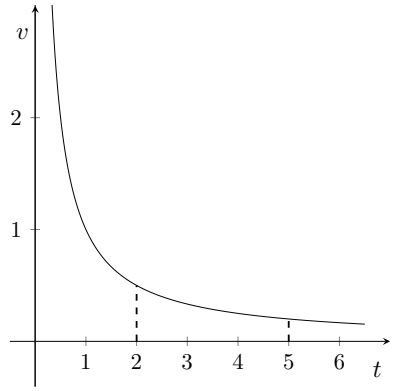
b) Right-handed rectangles



4. What do you get? How do you explain your result?

5. How could we improve the approximations for these areas? List as many ways as you can of how we could improve these approximation?

6. The velocity of a particle is give by the functions $y = \dfrac{1}{x}$. Approximate the distance it has traveled between time $t = 2$s and $t = 5$s (use 3 rectangles for that).



In this activity we will practice how to approximate areas under curves using different rectangle rules and using a different number of subintervals. Consider the function

$f(x) = 2x^3 - 9x^2 + 12x - 4$ over the interval $[0, 3]$:

1) In your own words and diagrams, describe your rectangle rule.
2) For the given function, approximate the area for $N = 1, 3, 6, 12, 24$ subintervals.
3) How would you write your findings using a summation?

Sigma Notation and Limits of Finite Sums (5.2)

Expected Skills.
At the end of this section, students will be able to:

- use the sigma notation for sums and compute sums,
- explain in words what the Riemann sums of a given function are.

Worksheet (ch5-integrals-2-sigma-1-ws)**.** In the class activity, students are introduced to sigma notation and explore various summations expressed using this notation. The goal is to have students recognize that finite sigma sums represent the terms in a summation using a common pattern and equal a constant value. Students then extend sigma notation to the context of estimating areas under graphs of functions. The goal is to have students develop and connect the graphical picture of area estimation using rectangles to the algebraic formulation of area estimates.

In calculus, we do a lot of adding. Since mathematicians tend to be lazy (or efficient, depending your point of view) there is a specific notation for sums, it is called the **sigma notation**.

To what sum does the following notation correspond to?

$$\sum_{k=1}^{1110} k$$

Here explain to the students the different "parts" of this notation: the dummy variable, the bounds.
Also, write down some "variations" of this sum (changing just one thing) to make sure everything is well understood

This notation is important because it is very often used in math!As it is often used in math. Our goal here is to be able to "go back and forth" between the notation with the dots and plus signs and the notation with the sigma. We will also compute some "simple" sums.

1. Rewrite the following sums using "+" signs and dots. Then compute these sums.

   (a) $\displaystyle\sum_{k=1}^{10}(k+1)$

   (b) $\displaystyle\sum_{k=7}^{10}(k+1)$ *Changing the lower bound*

   (c) $\displaystyle\sum_{k=1}^{5}(k+1)$ *Changing the upper bound*

   (d) $\displaystyle\sum_{n=10}^{10}(k+1)$ *Limit case where we don't have a sum anymore*

   Above are easy examples to have a soft start with this new notation.

2. Compute the following sums.

   (a) $\displaystyle\sum_{k=6}^{9}(3k+2)$

   (b) $\displaystyle\sum_{k=10}^{12}(k+1)^2$

   (c) $\displaystyle\sum_{k=1}^{5}k$

   (d) $\displaystyle\sum_{k=1}^{10}k$

   (e) $\displaystyle\sum_{k=1}^{100}k$ *Obviously, we want them to find the formula* $\frac{n(n+1)}{2}$

Space for exercises on p. 3 (if needed)

3. Let us consider the function $f(x) = \frac{1}{x}$. We want to estimate the area under this function between $x = 2$ and $x = 5$. Our goal is to make the connection between the sums below and various estimations of this area.
   Draw the graph of $f(x) = \frac{1}{x}$. Then for each of the following sums explain in words and pictures what they correspond to.

   (a) $\displaystyle\sum_{k=0}^{2} f(2+k) \cdot 1$

   (b) $\displaystyle\sum_{k=1}^{3} f(2+k) \cdot 1$

   (c) $\displaystyle\sum_{k=1}^{3} f\left(2 + k\frac{5-2}{3}\right) \cdot \left(\frac{5-2}{3}\right)$

   (d) $\displaystyle\sum_{k=1}^{3} f\left(1.5 + k\frac{5-2}{3}\right) \cdot \left(\frac{5-2}{3}\right)$

   This last exercise is an introduction to Rieman sums. It goes back to the last exercise of the previous worksheet.

(e) $\displaystyle\sum_{k=1}^{6} f\left(2 + k\frac{5-2}{6}\right) \cdot \left(\frac{5-2}{6}\right)$

(f) Write down the sum for an approximation of the area under $f(x) = 1/x$ between $x = 2$ and $x = 5$ using 10 right-handed rectangles.

4. Same question but for 1000 rectangles. Then, how can we write thus sum for "infinitely many" rectangles?

   This is how we are going to define the integral. But be careful, there are some subtleties here!

Challenge Exercises on Sums (only if you are **very** comfortable with the above material).

One can prove that $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. Use this formula as well as the formula for the sum $\displaystyle\sum_{k=0}^{n} k$ found above to compute the following sums.

i) $\displaystyle\sum_{k=1}^{21} (3+k) =$

ii) $\displaystyle\sum_{k=14}^{74} k(k+1) =$

In calculus, we do a lot of adding. Since mathematicians tend to be lazy (or efficient, depending your point of view) there is a specific notation for sums, it is called the **sigma notation**.

To what sum does the following notation correspond to?

$$\sum_{k=1}^{1110} k$$

This notation is important because it is very often used in math!As it is often used in math. Our goal here is to be able to "go back and forth" between the notation with the dots and plus signs and the notation with the sigma. We will also compute some "simple" sums.

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   (b) $\displaystyle\sum_{k=7}^{10}(k+1)$

   (c) $\displaystyle\sum_{k=1}^{5}(k+1)$

   (d) $\displaystyle\sum_{n=10}^{10}(k+1)$

2. Compute the following sums.

   (a) $\displaystyle\sum_{k=6}^{9}(3k+2)$

   (b) $\displaystyle\sum_{k=10}^{12}(k+1)^2$

   (c) $\displaystyle\sum_{k=1}^{5}k$

   (d) $\displaystyle\sum_{k=1}^{10}k$

   (e) $\displaystyle\sum_{k=1}^{100}k$

3. Let us consider the function $f(x) = \frac{1}{x}$. We want to estimate the area under this function between $x = 2$ and $x = 5$. Our goal is to make the connection between the sums below and various estimations of this area.
   Draw the graph of $f(x) = \frac{1}{x}$. Then for each of the following sums explain in words and pictures what they correspond to.

   (a) $\displaystyle\sum_{k=0}^{2} f(2+k) \cdot 1$

   (b) $\displaystyle\sum_{k=1}^{3} f(2+k) \cdot 1$

   (c) $\displaystyle\sum_{k=1}^{3} f\left(2 + k\frac{5-2}{3}\right) \cdot \left(\frac{5-2}{3}\right)$

   (d) $\displaystyle\sum_{k=1}^{3} f\left(1.5 + k\frac{5-2}{3}\right) \cdot \left(\frac{5-2}{3}\right)$

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(f) Write down the sum for an approximation of the area under $f(x) = 1/x$ between $x = 2$ and $x = 5$ using 10 right-handed rectangles.

4. Same question but for 1000 rectangles. Then, how can we write thus sum for "infinitely many" rectangles?

Challenge Exercises on Sums (only if you are **very** comfortable with the above material).
One can prove that $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. Use this formula as well as the formula for the sum $\displaystyle\sum_{k=0}^{n} k$ found above to compute the following sums.

i) $\displaystyle\sum_{k=1}^{21} (3+k) =$

ii) $\displaystyle\sum_{k=14}^{74} k(k+1) =$

The Definite Integral (5.3)

Expected Skills.
At the end of this section, students will be able to:

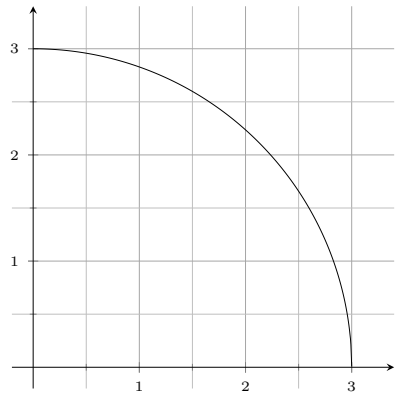
- explain in words what the definite integral is, both in terms of Riemann sums and of area under a curve,
- give examples of functions that are integrable and functions that are not,
- compute definite integrals using known areas and rules satisfied by definite integrals,
- justify the rules satisfied by definite integrals using area arguments.

Pre-Class Activity (ch5-integrals-3-definite-1-pc)**.** In the pre-class activity, students explore how we can extend area estimates by using successive approximations. Students take successive left-hand and right-hand approximations and are challenged to reconcile these different approximations to compute the area of a quarter circle.

Worksheet (ch5-integrals-3-definite-2-ws)**.** In the class activity, students work with definite integrals. The goal here is for students to view definite integrals from a graphical perspective, understand that definite integrals represent constant values, and to explain the various algebraic properties of definite integrals. These goals are primarily explore through the context of the function $f(x) = \sqrt{9 - x^2}$, a semi-circle.

The general goal of this pre-class activity is to answer the following question: how can we compute the area under a curve by using successive approximations.

Let us consider the quarter of a disc of radius 3. The function is given by $f(x) = \sqrt{9 - x^2}$ between $x = 0$ and $x = 3$.



1. Compute the actual area of this quarter of a circle.

2. Compute the approximation of this area using 3 right-handed rectangles. Remember that we have seen in class that in this case, the rectangle between 2 and 3 is actually "flat". (you need to use the formula of the function as well as a calculator).

3. Compute the approximation of this area using 6 right-handed rectangles. (you need to use the formula of the function as well as a calculator).

4. Which of the two approximations is the best? What would happen if we used 100 rectangles? Then, how could we improve even further this last approximation?

5. Since when taking right-handed rectangles the one that has $x = 3$ as a side (i.e. the rectangle that is the most on the right-hand side) is always flat, can we ever find the actual area under this curve by using approximations with rectangles?

6. Would we get different results if we had used left-handed (instead of right-handed) rectangles in parts 2 and 3?

7. Is there a way to "reconcile" the left-handed and right-handed approximations?

1. Last time we talked about approximating the area under a curve using sums of rectangles. Is there a way to use this idea to find the actual area under a curve (and not just an approximation)?

   Ask this question to the students and help them get to the formula: $\lim_{\Delta \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k$. *(p. 318 in the textbook)*
   Here we assume that Δ *is the length of the largest interval (it is thus positive).*
   Explain what each part of this formula ***means****:*

   - *the length of the intervals are going to zero and they DON'T have to have the same length,*
   - *we can take any value of the function within each interval,*
   - *the number of intervals is going to infinity.*

   Telate this to the pre-class activity: taking the limit (with intervals becoming infinitely small) enable us to "reconcile" the left-handed and right-handed rectangles approximations. This also explains how we can actually compute the area under a half-circle even though one rectangle is "always" flat.

2. Let us consider a function $f(x)$ on an interval $[a, b]$. **By definition**, the integral of f from a to b, denoted by

$$\int_a^b f(x)dx$$

   is ... *the area under the function between* a *and* b.

   Explain what the different parts of this notation mean: upper and lower limits, function, dummy variable.

3. Compute the following integrals. Drawing the graphs of these functions will probably help you.

   a) $\displaystyle\int_{-3}^{3} \sqrt{9 - x^2}\, dx$

   Underline here that we are computing the area under this curve and NOT looking for the antiderivative!

   b) $\displaystyle\int_{-2}^{2} (2 - |x|)\, dx$

4. Let us now look at one important property of integrals.

   a) Compute the integral $\int_{-3}^{3} -\sqrt{9-x^2}\,dx$. What do you conclude from this example?

   Integrals give the singed area.

   b) Then, what is $\int_{-1}^{1} x\,dx$

5. Let us see one more property of integrals.

   a) Compute the integrals $\int_{-3}^{0} \sqrt{9-x^2}\,dx$ and $\int_{0}^{3} \sqrt{9-x^2}\,dx$.

   b) How are they related to $\int_{-3}^{3} \sqrt{9-x^2}\,dx$? What property of integrals is this an example of?

   Here mention that $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$. *Also mention that* $\int_a^a f(x)dx = 0$.

   c) Compute $\int_{0}^{0} \sqrt{9-x^2}\,dx$.

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   a) Compute the integrals $\int_{-3}^{0} \sqrt{9-x^2}\,dx$ and $\int_{0}^{3} \sqrt{9-x^2}\,dx$.

   b) How are they related to $\int_{-3}^{3} \sqrt{9-x^2}\,dx$? What property of integrals is this an example of?

   c) Compute $\int_{0}^{0} \sqrt{9-x^2}\,dx$.

The Fundamental Theorem of Calculus (5.4)

Expected Skills.
At the end of this section, students will be able to:

- explain in words the two parts of the fundamental theorem of calculus and why it is important,

- mathematically state the two parts of the fundamental theorem of calculus,
- use the fundamental theorem of calculus to compute definite integrals.

Worksheet (ch5-integrals-4-ftc-1-ws)**.** In the class activity, students are introduced to the antiderivative of a function as an area accumulation function and the first part of the Fundamental Theorem of Calculus. The goal is for students to

1. explain in words and graphically the antiderivative of a function
2. explain in words the connection between antiderivatives and integrals

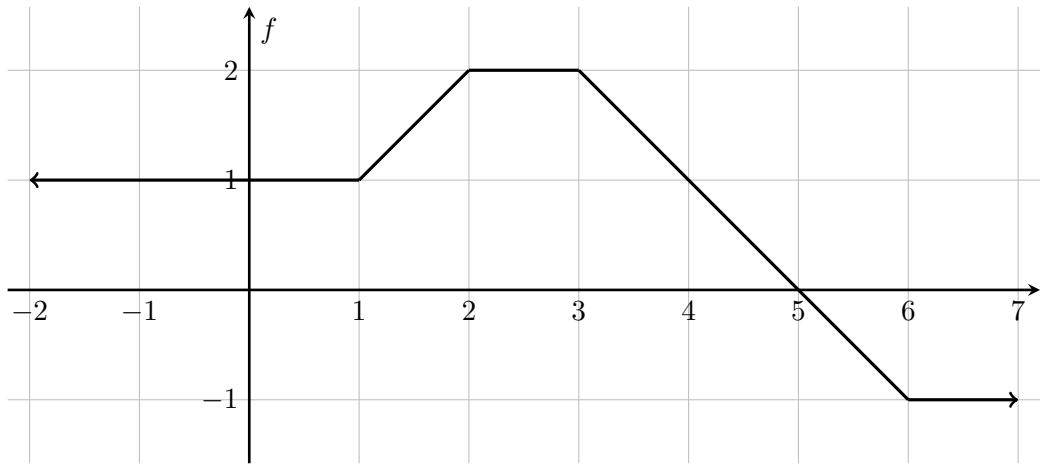
Worksheet (ch5-integrals-4-ftc-2-ws)**.** In the class activity, students are introduced to the second part of the fundamental theorem of calculus.

At this point, there are two general questions we would like to answer:

i) Does every continuous function have an antiderivative?

ii) Are the notions of integration and differentiation related and if so, how?
The answer to the first question will lead to to the answer of the second.

1. **Area Accumulation Functions**
Consider the function $f(x)$ whose graph is given below.



(a) Define $F(x) = \int_0^x f(t)\,dt$. What does this function represent?

(b) Evaluate the following:

$F(0) =$ *0* $\qquad$ $F(2) =$ *2.5*
$F(1) =$ *1* $\qquad$ $F(-1) =$ *-1*

(c) Shade in and find the area represented by $F(3) - F(1)$.

It is the area under the curve between $x = 1$ and $x = 3$, an area of 3.5.

(d) Find a formula for $F(x)$ between $x = 0$ and $x = 1$

x

(e) Give two values at which $F(x) = 0$. (Hint: assume the graph continues to the right.)

$x = 0$ *and* $x = 12$

(f) Which is larger: $F(3)$ or $F(4)$? Explain.

$F(4)$ is larger because $F(x)$ accumulates area and all the area between $x = 3$ and $x = 4$ counts as positive since $f(x)$ is positive there.

(g) Which is larger: $F(5)$ or $F(6)$? Explain.

$F(5)$ is larger because the area accumulated between $x = 5$ and $x = 6$ counts negatively, since $f(x)$ is negative there.

(h) Give the open intervals on which $F(x)$ is increasing and decreasing. Explain.

$F(x)$ is increasing from $x = 0$ to $x = 5$. On this interval the area is all positive, so as it accumulates, the value of $F(x)$ must increase. To the left of $x = 0$, $F(x)$ is also increasing because the value of the integrals becomes less negative as the value of x moves to the right.
Then it is decreasing on $(5, \infty)$. Underline that actually $f(5) = 0$.

2. Let us consider the function $Z(x) = \int_0^x 2\,dt$.

(a) Sketch a graph of $f(t) = 2$ for $t \geq 0$.

(b) Use geometry to fill in the chart below and generalize it for an arbitrary value of x.

x	0	1	2	3	...	x
$Z(x)$						

(c) Using the values in your table, sketch a graph of Z. What is the slope? The y-intercept?

(d) How would we change the definition of $Z(x)$ given above so that the slope would be 3?

3. **Fundamental Theorem of Calculus (Part 1)**

Let $f(t)$ be continuous function on $[a, b]$. Let $F(x) = \int_a^x f(t)dt$.

(a) Sketch the quantities $F(x)$ and $F(x + h)$ where h is a small positive number. To make it simple, take a function $f(t) > 0$.

The idea is to give of motivation of the FTC. Cf. p. 331 in Thomas

(b) Approximate $F(x + h) - F(x)$ by the area of a rectangle.

(c) This motivates the Fundamental Theorem of Calculus.

Give the first part of the Theorem p. 332.
This answers our questions at the beginning of the worksheet.

4. For the following questions, use the First Fundamental Theorem of Calculus to find $F'(x)$.

   (a) $F(x) = \int_1^x \sqrt[4]{t}\, dt$

   (b) $F(x) = \int_0^x \sec^2 t\, dt$

   (c) $F'(x)$ for $F(x) = \int_3^{5x^2+8} \cos(t)\, dt$ (Don't forget about the chain rule!)

   (d) $F'(x)$ for $F(x) = \int_{\sin(x)}^{x^2} \frac{1}{t^3}\, dt$ (Can you break this up somehow?)

5. A particle is moving along the x-axis. The position of the particle at time t seconds is given by $x(t) = (t-1)(t-3)^2$, $0 \le t \le 5$. Find the total distance (in meters) the particle travels in 5 seconds.

   Do this application of the FTC only if you have time.

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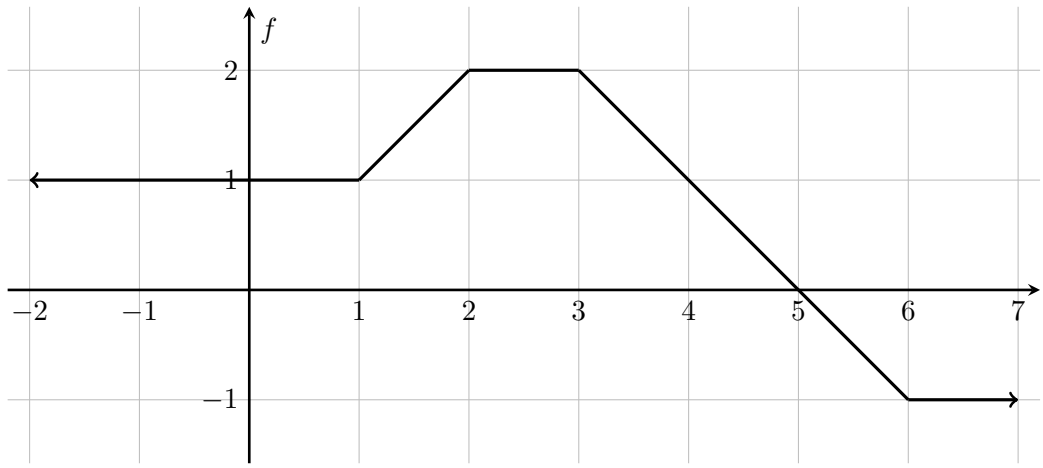

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The Fundamental Theorem of Calculus – Part 2

If $f(x)$ is continuous on $[a, b]$ and $F(x)$ is any antiderivative on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

1. How do we justify the second part of the Fundamental Theorem of Calculus using the first part of the theorem?

   Here the idea is to connect the two parts of the theorem by having the students be able to give the general idea of this part of the proof.

2. Compute the following definite integrals using the second part of the FTC.

   (a) $\displaystyle\int_0^{\pi} e^x + \cos x - 1\ dx$

   (b) $\displaystyle\int_0^{\pi/4} \frac{1}{\cos^2 x} + \pi - x\ dx$

(c) $\displaystyle\int_1^4 \sqrt{x} + \frac{1}{x}\ dx$

(d) $\displaystyle\int_1^{16} x^3 + \sin x + \frac{1}{x^3}\ dx$

(e) $\displaystyle\int_{-4}^{4} |x|\ dx$

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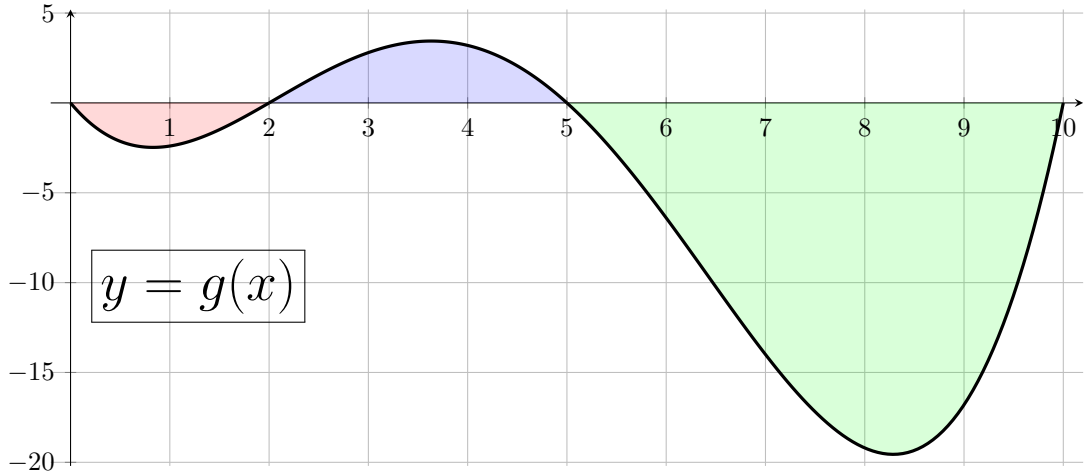
3. Consider the graph of the function $y = g(x)$ given below.



(a) What is the relationship between $\int_0^{10} g(x)\,dx$ and the shaded areas in the figure?

(b) What is the relationship between $\int_0^{10} |g(x)|\,dx$ and the shaded areas in the figure?

4. The temperature of a room $T(t)$ in °F at time t is given by

$$T(t) = 85 - 3\sqrt{25 - t} \quad \text{for } 0 \leq t \leq 25.$$

   (a) Find the room's temperature when $t = 0$, $t = 16$ and $t = 25$.

   (b) Find the room's average temperature for $0 \leq t \leq 25$.

5. Archimedes discovered that the area under a parabolic arch is two-thirds the base times the height. Sketch the parabolic arch $y = h - (4h/b^2)x^2$ for $-b/2 \leq x \leq b/2$, assuming that h and b are positive. Then use calculus to find the area of the region enclosed between the arch and the x-axis.

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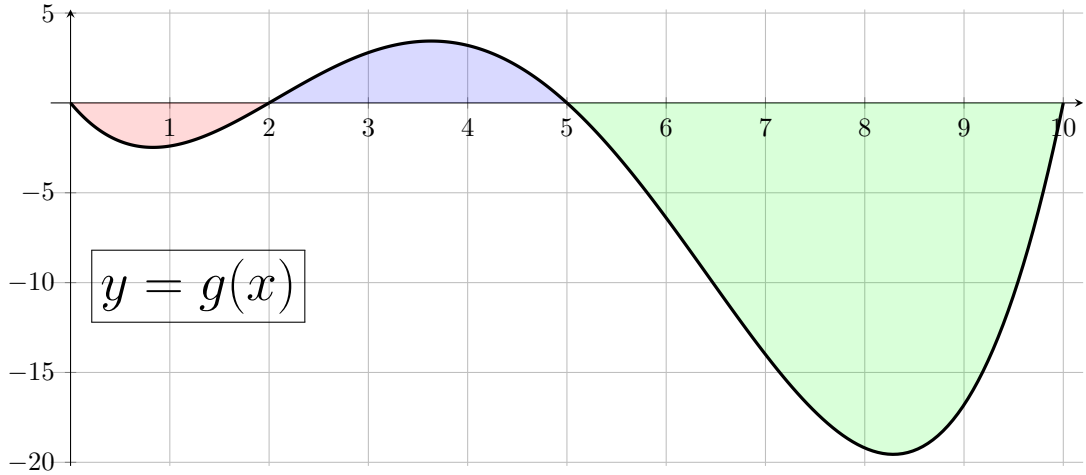

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