International Journal of Mathematics
Vol. 19, No. 6 (2008) 699–746

World Scientific
www.worldscientific.com

THE BIG CHERN CLASSES AND THE CHERN CHARACTER

AJAY C. RAMADOSS

Department of Mathematics, University of Oklahoma
621 Elm Avenue, Norman, OK-73019
aramadoss@math.ou.edu

Received 26 January 2007

Let X be a smooth scheme over a field of characteristic 0. The Atiyah class of the tangent bundle T_X of X equips $T_X[-1]$ with the structure of a Lie algebra object in the derived category $\mathrm{D}^+(X)$ of bounded below complexes of $\mathcal{O}_X$ modules with coherent cohomology [6]. We lift this structure to that of a Lie algebra object $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ in the category of bounded below complexes of $\mathcal{O}_X$ modules in Theorem 2. The "almost free" Lie algebra $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ is equipped with Hochschild coboundary. There is a symmetrization map $I : \mathrm{Sym}^\bullet(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \to \mathrm{D}^\bullet_{\mathrm{poly}}(X)$ where $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$ is the complex of polydifferential operators with Hochschild coboundary. We prove a theorem (Theorem 1) that measures how I fails to commute with multiplication. Further, we show that $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$ is the universal enveloping algebra of $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ in $\mathrm{D}^+(X)$. This is used to interpret the Chern character of a vector bundle E on X as the "character of a representation" (Theorem 4). Theorems 4 and 1 are then exploited to give a formula for the big Chern classes in terms of the components of the Chern character.

Keywords: Complex of polydifferential operators; Hochschild–Kostant–Rosenberg quasi-isomorphism; Atiyah class; big Chern classes; Chern character; derived category; lie algebra; universal enveloping algebra.

Mathematics Subject Classification 2000: 14F99,16E40,18G99

1. Introduction

1.1. *Outline of main results*

This paper is a result of an effort to understand the works of Markarian [3] and Caldararu [10]. Another goal was to see whether the works of Markarian [3] and Caldararu [10] lead to an explicit formula relating the big Chern classes to the Chern character.

We begin by outlining the main results in this paper. Let X be a smooth scheme over a field of characteristic 0. Let $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$ denote the complex of poly-differential operators on X, with Hochschild co-boundary. We denote the sheaf of differential operators on X by $\mathrm{D}^1_{\mathrm{poly}}(X)$. Let $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ denote the free Lie algebra generated over $\mathcal{O}_X$ by $\mathrm{D}^1_{\mathrm{poly}}(X)$ concentrated in degree 1, equipped with the Hochschild co-boundary. There is a symmetrization map

$$I : \oplus_k \mathrm{Sym}^k(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \to \mathrm{D}^\bullet_{\mathrm{poly}}(X).$$

I is an isomorphism of complexes of $\mathcal{O}_X$-modules. Let

$$m : \mathrm{D}^\bullet_{\mathrm{poly}}(X) \otimes \mathrm{D}^\bullet_{\mathrm{poly}}(X) \to \mathrm{D}^\bullet_{\mathrm{poly}}(X)$$

denote the multiplication on $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$ and let μ denote the natural product on $\oplus_k \mathrm{Sym}^k(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$. Let

$$\mathrm{ad} : \oplus_k \mathrm{Sym}^k(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \otimes L(\mathrm{D}^1_{\mathrm{poly}}(X)) \to \oplus_k \mathrm{Sym}^k(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$$

denote the right adjoint action of $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ on $\oplus_k \mathrm{Sym}^k(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$. In Sec. 5, we describe a map

$$\omega : \oplus_k \mathrm{Sym}^k(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \otimes L(\mathrm{D}^1_{\mathrm{poly}}(X)) \to \oplus_k \mathrm{Sym}^k(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \otimes L(\mathrm{D}^1_{\mathrm{poly}}(X))$$

such that $\mu \circ \omega = \mathrm{ad}$. Let $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$ denote the category of bounded below complexes of $\mathcal{O}_X$-modules. We then have the following theorem.

Theorem 1. *The following diagram commutes in* $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$

$$\begin{tikzcd}
D^\bullet_{\mathrm{poly}}(X) \otimes L(D^1_{\mathrm{poly}}(X)) \arrow[r, "m"] & D^\bullet_{\mathrm{poly}}(X) \\
\mathrm{Sym}^\bullet(L(D^1_{\mathrm{poly}}(X))) \otimes L(D^1_{\mathrm{poly}}(X)) \arrow[u, "I\otimes\mathrm{id}"] \arrow[r, "\mu\circ\frac{\omega}{1-e^{-\omega}}"] & \mathrm{Sym}^\bullet(L(D^1_{\mathrm{poly}}(X))) \arrow[u, "I"']
\end{tikzcd}$$

The next sections of this paper are devoted to a conceptual understanding of the above theorem and its corollaries. Let $\mathrm{D}^+(X)$ denote the derived category of bounded below complexes of $\mathcal{O}_X$-modules with coherent cohomology. Recall that the Atiyah class of a vector bundle E is an element in $\mathrm{Hom}_{\mathrm{D}^+(X)}(E \otimes T_X[-1], E)$. In particular, the Atiyah class α_{TX} of the tangent bundle of X is an element in

$$\mathrm{Hom}_{\mathrm{D}^+(X)}(T_X \otimes T_X[-1], T_X) = \mathrm{Hom}_{\mathrm{D}^+(X)}(T_X[-1] \otimes T_X[-1], T_X[-1]).$$

Let I_{HKR} denote the Hochschild–Kostant–Rosenberg quasi-isomorphism

$$\mathrm{I}_{HKR} : \oplus_i \wedge^i T_X[-i] \to \mathrm{D}^\bullet_{\mathrm{poly}}(X).$$

In Sec. 4.2, we define a map of complexes $\beta : T_X[-1] \to L(\mathrm{D}^1_{\mathrm{poly}}(X))$ such that

$$I \circ \mathrm{Sym}^\bullet \beta = \mathrm{I}_{HKR}$$

and show that $\beta : T_X[-1] \to L(\mathrm{D}^1_{\mathrm{poly}}(X))$ is a quasi-isomorphism. Moreover, as proven in Sec. 6,

Theorem 2. *The following diagram commutes in* $D^+(X)$

$$\begin{tikzcd}
T_X[-1] \otimes_{\mathcal{O}_X} T_X[-1] \arrow[r, "\beta\otimes\beta"] \arrow[d, "\alpha(T_X)"'] & L(D^1_{\mathrm{poly}}(X)) \otimes_{\mathcal{O}_X} L(D^1_{\mathrm{poly}}(X)) \arrow[d, "{[,]}"] \\
T_X[-1] \arrow[r, "\beta"] & L(D^1_{\mathrm{poly}}(X))
\end{tikzcd}$$

Note that all arrows in the diagram above except α_{T_X} are honest maps in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$. This theorem says that the natural Lie bracket on $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ realizes the Atiyah class of T_X as a map in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$. It follows from the fact that β is a quasi-isomorphism that the map ω yields a map

$$\bar{\omega} : \oplus_i \wedge^i T_X[-i] \otimes T_X[-1] \to \oplus_i \wedge^i T_X[-i] \otimes T_X[-1]$$

in $\mathrm{D}^+(X)$. By an abuse of notation, let μ also denote the natural product in $\oplus_i \wedge^i T_X[-i]$. An immediate consequence of Theorems 1 and 2 is the following corollary.

Corollary 1. *The following diagram commutes in $D^+(X)$*

$$\begin{tikzcd}
D^\bullet_{\mathrm{poly}}(X) \otimes D^\bullet_{\mathrm{poly}}(X) \arrow[r, "m"] & D^\bullet_{\mathrm{poly}}(X) \\
\oplus_i \wedge^i T_X[-i] \otimes T_X[-1] \arrow[u, "I_{HKR} \otimes I_{HKR}"] \arrow[r, "\mu \circ \frac{\bar{\omega}}{1-e^{-\bar{\omega}}}"] & \oplus_i \wedge^i T_X[-i] \arrow[u, "I_{HKR}"']
\end{tikzcd}$$

This is a result “dual” to [3, Theorem 1]. In another paper [16], we use this to prove the relative Riemann–Roch theorem, thereby completely explaining [3]. One can think of the Hochschild–Kostant–Rosenberg map as a symmetrization map from $\oplus_i \wedge^i T_X[-i]$ to $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$. Corollary 1 thus tells us that the error term measuring how this map fails to commute with multiplication is “$d(\exp^{-1})$-like”. In order to understand the classical situation of which Corollary 1 is an analog, we have the following theorem.

Theorem 3. *$D^\bullet_{\mathrm{poly}}(X)$ is the universal enveloping algebra of $T_X[-1]$ in $D^+(X)$. In other words, let A be an associative algebra in $D^+(X)$. If $f : T_X[-1] \to A$ is a morphism in $D^+(X)$ making the following diagram commute in $D^+(X)$,*

$$\begin{tikzcd}
T_X[-1] \otimes_{\mathcal{O}_X} T_X[-1] \arrow[r, "\alpha(T_X)"] \arrow[d, "f \otimes f"'] & T_X[-1] \arrow[d, "f"] \\
A \otimes_{\mathcal{O}_X} A \arrow[r, "{[,]}"] & A
\end{tikzcd}$$

then there exists a unique morphism $\bar{f} : D^\bullet_{\mathrm{poly}}(X) \to A$ of algebras in $D^+(X)$ so that the composite $\bar{f} \circ I_{HKR} = f$.

This makes the parallel between Theorem 1, Corollary 1 and their classical analogs more explicit. In fact, as explained in Sec. 5, the classical analog of Theorem 1 is a commutative diagram equivalent to the formula $d(\exp^{-1}) = \frac{\mathrm{ad}}{1-e^{-\mathrm{ad}}}$. Theorem 3 shows us that Corollary 1 is also analogous to the same classical result.

In the original version of this paper, Theorem 3 was proven first. Corollary 1 was then interpreted as the analog for the Lie algebra $T_X[-1]$ of its classical version, which followed from the formula $d(\exp^{-1}) = \frac{\mathrm{ad}}{1-e^{-\mathrm{ad}}}$. However, no explicit details of its proof were given. The same interpretation of an equivalent result has been offered by Markarian [14] in an arXiv preprint subsequent to the first arXiv version of this paper. Again, as in the original version of this paper, hardly any further details were offered. Theorem 1 removes these shortcomings. It is also stronger.

Theorem 3 is equivalent to a result claimed by Roberts [7]. It was however, not proven in printed or online literature available to the author when the first version of this paper was written. In a paper [11] that appeared on the arXiv after the first version of this paper was uploaded on the arXiv, Roberts and Willerton prove an equivalent result. Their proof is however, very different from the proof here. The proof here is more explicit in the sense that $\bar{f}$ as in Theorem 3 is directly constructed in our proof.

There are other, more serious applications of Theorem 3. If E is a vector bundle on X, the Atiyah class of E, $\alpha(E)$ is an element of

$$\mathrm{Hom}_{\mathrm{D}^+(X)}(E \otimes T_X[-1], E) = \mathrm{Hom}_{\mathrm{D}^+(X)}(T_X[-1], \mathcal{E}\mathrm{nd}(E)).$$

By [6], $\alpha(E)$ equips E with the structure of a module over the Lie algebra $T_X[-1]$ in $\mathrm{D}^+(X)$. Thus,

$$\alpha(E) : T_X[-1] \to \mathcal{E}\mathrm{nd}(E)$$

is a morphism of Lie algebras in $\mathrm{D}^+(X)$. By Theorem 3, there exists a morphism $\theta_E : \mathrm{D}^\bullet_{\mathrm{poly}}(X) \to \mathcal{E}\mathrm{nd}(E)$ of algebras in $\mathrm{D}^+(X)$ lifting $\alpha(E)$. We also have a map $\mathrm{tr} : \mathcal{E}\mathrm{nd}(E) \to \mathcal{O}_X$. Let $\varphi_E = \mathrm{tr} \circ \theta_E$. Let $p : \oplus_i \wedge^i T_X[-i] \to \oplus_n T_X^{\otimes n}[-n]$ be the symmetrization map. There is a map of complexes $\mathrm{J} : \oplus_n T_X^{\otimes n}[-n] \to \mathrm{D}^\bullet_{\mathrm{poly}}(X)$, such that $\mathrm{I}_{HKR} = \mathrm{J} \circ p$. Let $\tilde{\mathrm{t}_k}(E)$ denote $\alpha(E)^{\circ k} \in \mathrm{Hom}_{\mathrm{D}^+(X)}(T_X^{\otimes k}[-k], \mathcal{E}\mathrm{nd}(E))$. Let $\tilde{\mathrm{ch}_k}(E) = \tilde{\mathrm{t}_k}(E) \circ p$. Let $\mathrm{t}_k(E)$ denote the kth big Chern class of E and let $\mathrm{ch}_k(E)$ denote the kth component of the Chern character of E. The following easy consequence of Theorem 3 is stated as a theorem in its own right.

Theorem 4. (i) $\tilde{t_k}(E) = \theta_E \circ J$

(ii) $\tilde{\mathrm{ch}_k}(E) = \theta_E \circ I_{HKR}$

(iii) $\oplus_n t_n(E) = \varphi_E \circ J$

(iv) $\mathrm{ch}(E) = \varphi_E \circ I_{HKR}$.

Part (iv) of Theorem 4 interprets the Chern character of E as the "character of the representation E of $T_X[-1]$". We will comment on this aspect in greater detail in Sec. 8. Theorem 4 is similar to Theorem 4.5 of Caldararu [10]. However, that result does not lend itself to our interpretation of the Chern character as directly as Theorem 4 does. Further, even more interesting applications of Theorem 3 may be found in [7], but they are beyond the scope of this paper.

Note that we have a PROP $\mathrm{END}_{T[-1]}$ such that

$$\mathrm{END}_{T[-1]}(n, m) = \mathrm{Hom}_{\mathrm{D}^+(X)}(T_X^{\otimes n}[-n], T_X^{\otimes m}[-m]).$$

Let $\Psi \in \oplus_{m \leq n} \mathrm{END}_{T[-1]}(n, m)$ be the element of $\mathrm{END}_{T[-1]}$ given in Sec. 9. Let Ψ_{kl} denote the component of Ψ in $\mathrm{END}_{T[-1]}(k, l)$. Let $\pi : \oplus_n T_X^{\otimes n}[-n] \to \oplus_i \wedge^i T_X[-i]$ be the standard projection. Then,

Theorem 5. (i) $\tilde{t_k}(E) = \tilde{\mathrm{ch}}_k(E) \circ \pi + \sum_{l<k} \tilde{\mathrm{ch}_l}(E) \circ \pi \circ \Psi_{kl}$

(ii) $t_k(E) = \mathrm{ch}_k(E) \circ \pi + \sum_{l<k} \mathrm{ch}_l(E) \circ \pi \circ \Psi_{kl}$.

We remark here that Ψ has been described by an explicit, albeit lengthy formula in Sec. 9. This expresses the big Chern classes of a vector bundle on an arbitrary smooth scheme over a field of characteristic 0 in terms of the components of its Chern character. The existence of a formula similar to (ii) was proven in the author's thesis for vector bundles on projective varieties over a field of characteristic 0. The proof there was entirely different. It crucially required the existence of an ample line bundle on X. The proof there therefore did not generalize to arbitrary smooth schemes, smooth complex manifolds etc. unlike the proof here. Further, it was difficult to see the formula arising out of the trace applied to an almost identical formula akin to (i) of Theorem 5. Also, the explicit description of Ψ was not given in the author's thesis [8].

1.2. *Structure of this paper*

Sections 2 and 3 are introductory and describe the basic properties of $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$. In particular, Sec. 3 describes the Hopf-algebra structure of $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$ in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$.

Section 4 proves a key lemma (Lemma 1) stating that the symmetrization map $I : \oplus_k \mathrm{Sym}^k(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \to \mathrm{D}^\bullet_{\mathrm{poly}}(X)$ is an isomorphism in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$. This is done by first showing that it is a map of complexes of $\mathcal{O}_X$-modules, followed by showing that it is an isomorphism of graded $\mathcal{O}_X$-modules.

Section 4 also recalls the definition of the Hochschild–Kostant–Rosenberg (HKR) quasi-isomorphism I_{HKR} from Yekutieli and shows that there is a quasi-isomorphism $\beta : T_X[-1] \to L(\mathrm{D}^1_{\mathrm{poly}}(X))$ so that $I \circ \mathrm{Sym}^\bullet \beta = \mathrm{I}_{HKR}$. This is later used in Sec. 6.

Section 5 states and proves Theorem 1. Once again, this is done in stages. The first stage involves an explicit calculation showing that all maps involved commute with the relevant differentials. This enables us to "forget" the differentials in the complexes involved. We then only need to show that the diagram involved commutes in the category of graded $\mathcal{O}_X$-modules.

Let V be a vector space over a field K of characteristic 0. Let $L(V)$ be the (graded) free Lie algebra generated over K by V concentrated in degree 1. The universal enveloping algebra of $L(V)$ is the tensor algebra $T(V)$ of V. The second stage in proving Theorem 1 involves reducing the problem further to the problem of finding the error term that measures how the PBW map from $\mathrm{Sym}^\bullet(L(V))$ to $T(V)$ fails to commute with multiplication. This is carried out in the appendix at the end of this paper.

Section 5 also contains a lemma (Lemma 2) that furthers our understanding of Theorem 1. This states that $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$ is the universal enveloping algebra of $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$. This enables us to interpret I as a PBW-map from $\mathrm{Sym}^\bullet(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$ to $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$. Theorem 1 then says that the error term that measures the failure of I to commute with multiplication is "$d(\exp^{-1})$ like".

Section 6 recalls the definition of the Atiyah class of a perfect complex of $\mathcal{O}_X$-modules on X. Theorem 2, which states that the Lie bracket on $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ realizes the Atiyah class of T_X as a map of complexes of $\mathcal{O}_X$-modules, is also proven here by an explicit computation. Theorems 1 and 2 immediately imply Corollary 1, which is also stated in Sec. 6.

Section 7 is devoted to the proof of Theorem 3. We have attempted a careful and self-contained treatment of Theorem 3 in this section. Theorem 3 enables us to interpret the HKR-quasi-isomorphism as a PBW-map. Corollary 1 then says that the error term that measures the failure of the HKR map to commute with multiplication in $\mathrm{D}^+(X)$ is "$d(\exp^{-1})$ like".

Section 8 is used to state and prove Theorem 4. A spinoff of this result is a new conceptual proof of result already proven in the author's thesis (Corollary 6). This result states that the big Chern classes commute with Adams operations. On the other hand, one had the representation theoretic identity $\chi_{\psi^p E}(g) = \chi_E(g^p)$ for any element g of a group G and for any representation E of G. The parallel between the fact that the big Chern classes commute with Adams operations and the identity $\chi_{\psi^p E}(g) = \chi_E(g^p)$ is made transparent by this proof.

Section 9 is devoted to describing a formula for the element Ψ of the PROP $\mathrm{END}_{T[-1]}$ mentioned before stating Theorem 5, and then proving Theorem 5.

There is an appendix at the end of this paper. Let K be a field of characteristic 0. Let V be a graded vector space concentrated in degree 1. Let $L(V)$ be the (graded) free Lie algebra generated by V over K. Let $T(V)$ be the tensor algebra of V. The appendix devoted to a result (Theorem 6) about the error term measuring how the PBW map from $\mathrm{Sym}^\bullet(L(V))$ to $T(V)$ fails to commute with multiplication. It says that this error term is "$d(\exp^{-1})$ like". The precise statement and proof are in the Appendix. Even though we expect this to be standard, such a result is standard only for the case when V is concentrated in degree 0 as far as I know. This is the reason for this result to be included as a theorem in the Appendix to this paper.

1.3. *Notation*

Throughout this paper, $\mathrm{Ch}^+(\mathcal{O}_X-\mathrm{mod})$ will denote the category of bounded below complexes of $\mathcal{O}_X$-modules. $\mathrm{D}^+(X)$ will denote the bounded below derived category of complexes of $\mathcal{O}_X$-modules with coherent co-homology.

2. The Complete Hochschild Chain and Cochain Complexes

The purpose of this section is to recall definitions of and facts about the completed Hochschild chain and cochain complexes. Most of the material in this section is recalled from Yekutieli's paper [1]. The notation also follows the same source closely.

Throughout this article, we shall work with smooth schemes over fields of characteristic 0. Let X be a smooth separated scheme over a field K of characteristic 0. We have the (closed) diagonal embedding $\Delta : X \to X \times_K X$. Let $\mathcal{I}$ be the sheaf of ideals defining the diagonal in $X \times_K X$.

2.1. *The complete Bar complex and the complete Hochschild chain complex*

Let $\mathcal{O}_\Delta = \Delta_* \mathcal{O}_X$. On $X \times_K X$, $\mathcal{O}_\Delta$ has a free $\mathcal{O}_{X \times_K X}$-module resolution given by the Bar resolution:

$$B_n(X) = \mathcal{O}_X \otimes_K \cdots \otimes_K \mathcal{O}_X (n+2 \text{ times}).$$

The $\mathcal{O}_X \otimes_K \mathcal{O}_X$-module structure given by multiplication with the extreme factors. If $U = \operatorname{Spec} R$ is an open affine subscheme of X, then the differential $d : B_n(R) \to B_{n-1}(R)$ is given by the formula

$$\begin{aligned} d(a_0 \otimes \cdots \otimes a_{n+1}) &= a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_{n+1} - a_0 \otimes a_1 a_2 \otimes \cdots \otimes a_{n+1} \\ &\quad + \cdots + (-1)^n a_0 \otimes \cdots \otimes a_n a_{n+1} a_i \in R. \end{aligned}$$

Let $\mathcal{I}_n$ be the kernel of the multiplication map $B_n(X) \to \mathcal{O}_X$. Let $\widehat{B}_n := \lim {}_k B_n / {\mathcal{I}_n}^k$. Note that the differential d takes $\mathcal{I}_n$ into $\mathcal{I}_{n-1}$. The differential in $B_\bullet$ thus extends to a differential in the complete Bar complex $\widehat{B}_\bullet$. Note that the complete Bar complex comes equipped with the $\mathcal{I}$-adic topology.

Yekutieli ([1, Lemma 1.2]) shows that the complete Bar complex gives us a resolution of $\mathcal{O}_\Delta$ in terms of flat $\mathcal{O}_{X \times_K X}$-modules. A resolution of $\Delta^* \mathcal{O}_\Delta$ by flat $\mathcal{O}_X$-modules would thus be given by what is called the completed Hochschild chain complex of X. This complex $\widehat{C}_\bullet$ is defined by

$$\widehat{C}_\bullet = \Delta^{-1} \widehat{B}_\bullet \otimes_{\Delta^{-1} \mathcal{O}_{X \times_K X}} \mathcal{O}_X.$$

The complex $\widehat{C}_\bullet$ is called the complete Hochschild chain complex of X. It has a topology induced by that on $\widehat{B}_\bullet$.

Note that if $U = \operatorname{Spec} R$ is an open affine subscheme of X, and if $M(R)$ denotes $\Gamma(U, M)$ for any $\mathcal{O}_X$-module M, and I is the kernel of the multiplication map $R^{\otimes_K n+2} \to R$, we have

$$\begin{aligned} B_n(R) &= R^{\otimes_K n+2} \quad \widehat{B}_n(R) = \lim R^{\otimes_K n+2} / I^k \\ C_n(R) &= R^{\otimes_K n+1} \end{aligned}$$

and the differential $d : C_n(R) \to C_{n-1}(R)$ is given by

$$\begin{aligned} d(a_0 \otimes \cdots \otimes a_n) &= a_0 a_1 \otimes \cdots \otimes a_n - a_0 \otimes a_1 a_2 \cdots \otimes a_n \\ &\quad + \cdots + (-1)^n a_n a_0 \otimes \cdots \otimes a_{n-1} \\ \widehat{C}_n(R) &= \lim R^{\otimes_K n+2} / I^k \otimes_{R^{\otimes_k n+2}} C_n(R) \end{aligned}$$

and the differential on $\widehat{C}_\bullet$ is the one induced by d.

2.2. *Hochschild homology, Hochschild cohomology and the completed Hochschild cochain complex*

We begin with the following definitions.

Definition 1. The Hochschild homology of X is defined to be $\mathrm{RHom}(\mathcal{O}_X, \Delta^*\mathcal{O}_\Delta)$.

Definition 2. The complex of continuous Hochschild cochains on X is the complex

$$\mathrm{RHom}_{\mathcal{H}_X}(\Delta^*\mathcal{O}_\Delta, \mathcal{O}_X) = \mathrm{RHom}_{\mathcal{H}_{X\times_K X}}(\mathcal{O}_\Delta, \mathcal{O}_\Delta).$$

Fact 1. Yekutieli ([1, Theorem 0.3]) shows that the complex of continuous Hochschild co-chains is given by the complex $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{H}_{X\times_K X}}(\widehat{B}_\bullet(X), \mathcal{O}_\Delta)$. This is seen to be equal to the complex $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{H}_X}(\widehat{C}_\bullet(X), \mathcal{O}_X)$. Here $\mathcal{O}_\Delta$ and $\mathcal{O}_X$ are both given the discrete topology.

Fact 2. It is proven by Yekutieli ([1, Proposition 1.6]) that the complex of continuous Hochschild cochains on X is none other than the complex of polydifferential operators on X introduced by Kontsevich [2]. In other words, if $U = \mathrm{Spec}\, R$ is an open affine subscheme of X, and if ${C_{cd}}^n(X) = \mathrm{Hom}^{\mathrm{cont}}(\widehat{C}_n(X), \mathcal{O}_X)$, then

$${C_{cd}}^n(U) = \{f \in \mathrm{Hom}_K(R^{\otimes n}, R) | f \text{ is a differential operator in each factor}\}.$$

We shall henceforth denote ${C_{cd}}^n(X)$ by $\mathrm{D}^n_{\mathrm{poly}}(X)$.

Fact 3. We recall from Kontsevich ([2, Sec. 3.4.2]) that if $U = \mathrm{Spec}\, R$ and if $f \in {C_{cd}}^n(U)$, then the differential d of ${C_{cd}}^\bullet$ is given by

$$df(a_0 \otimes \cdots \otimes a_n) = a_0 f(a_1 \otimes \cdots \otimes a_n) + \sum_{i=1}^{i=n} (-1)^i f(\cdot\cdot \otimes a_{i-1}a_i \otimes \cdots \otimes a_n)$$
$$+ (-1)^{n+1} f(a_0 \otimes \cdots \otimes a_{n-1})a_n$$

for all $a_0, \ldots, a_n \in R$.

Remark. What we refer to here as the complex of polydifferential operators is a shifted version of what Kontsevich [2] refers to as the complex of polydifferential operators. Kontsevich's complex of polydifferential operators is, in our notation, $\mathrm{D}^\bullet_{\mathrm{poly}}(X)[1]$.

3. Hopf Algebra Structure on $\mathbf{D}^\bullet_{\mathbf{poly}}(X)$

In this section, we describe the operations that make $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$ a Hopf algebra in $\mathrm{Ch}^+(\mathcal{O}_X-\mathrm{mod})$. By this, we mean that $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$ has a multiplication m, a comultiplication Δ, a unit η and a counit ϵ, all of which are morphisms in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$.

Recall that $\mathrm{D}^1_{\mathrm{poly}}(X)$ is the sheaf of differential operators on X. Note that $\mathrm{D}^1_{\mathrm{poly}}(X)$ is a *left* $\mathcal{O}_X$ module. We have the following proposition

Proposition 1. *As $\mathcal{O}_X$ modules, $D^n_{\mathrm{poly}}(X)$ is isomorphic to $D^1_{\mathrm{poly}}(X)^{\otimes_{\mathcal{O}_X} n}$.*

Proof. This is something that can be checked locally. Consider an open affine subscheme U of X with local coordinates $\{x_1, \ldots, x_m\}$. Then, an element of $\mathrm{D}^n_{\mathrm{poly}}(U)$ is given by a map of the form

$$f_1 \otimes \cdots \otimes f_n \rightsquigarrow \sum_{(I_1,\ldots,I_n)} C_{(I_1,\ldots,I_n)}(x_1, \ldots, x_m)\partial_{I_1} f_1 \cdots \partial_{I_n} f_n.$$

Here the I_j's are multi-indices and ∂_{I_j} is the partial derivative corresponding to I_j. The above polydifferential operator maps to $\sum_{(I_1,\ldots,I_n)} C_{(I_1,\ldots,I_n)}(x_1, \ldots, x_m)\partial_{I_1} \otimes \cdots \otimes \partial_{I_n}$. This gives us a well defined map from $\mathrm{D}^n_{\mathrm{poly}}(U)$ to $\mathrm{D}^1_{\mathrm{poly}}(U)^{\otimes_{\mathcal{O}_U} n}$. On the other hand, we have a map from $\mathrm{D}^1_{\mathrm{poly}}(U)^{\otimes_{\mathcal{O}_U} n}$ to $\mathrm{D}^n_{\mathrm{poly}}(U)$ which takes $D_1 \otimes \cdots \otimes D_n$ to the polydifferential operator $f_1 \otimes \cdots \otimes f_n \rightsquigarrow D_1(f_1) \cdots D_n(f_n)$. These maps are clearly inverses of each other. □

We now describe the Hopf algebra structure on $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$.

Multiplication on $D^\bullet_{\mathrm{poly}}(X)$: Let $U = \operatorname{Spec} R$ be an affine open subscheme of X. Let $D_1 \in \mathrm{D}^k_{\mathrm{poly}}(U)$ and $D_2 \in \mathrm{D}^l_{\mathrm{poly}}(U)$. Then we can set

$$m(D_1, D_2)(a_1 \otimes \cdots \otimes a_{k+l}) = D_1(a_1 \otimes \cdots \otimes a_k)D_2(a_{k+1} \otimes \cdots \otimes a_{k+l})$$

for all $a_1, \ldots, a_{k+l} \in R$. This defines the multiplication m on $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$. Note that $m(D_1, D_2) = D_1 \otimes D_2$ after identifying $\mathrm{D}^n_{\mathrm{poly}}(X)$ with $\mathrm{D}^1_{\mathrm{poly}}(X)^{\otimes_{\mathcal{O}_X} n}$.

Comultiplication on $D^\bullet_{\mathrm{poly}}(X)$: On the other hand, if $f_1, \ldots, f_n$ are differential operators on an open subscheme U, then $f_1, \ldots, f_n \in \mathrm{D}^n_{\mathrm{poly}}(U)$ and we can set

$$\Delta(f_1, \ldots, f_n) = \sum_{p+q=n} \sum_{\sigma \text{ a (p,q)-shuffle}} \operatorname{sgn}(\sigma) f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(p)} \bigotimes f_{\sigma(p+1)} \otimes \cdots \otimes f_{\sigma(p+q)}.$$

This gives us a well defined map $\Delta : \mathrm{D}^\bullet_{\mathrm{poly}}(U) \to \mathrm{D}^\bullet_{\mathrm{poly}}(U) \otimes_{\mathcal{O}_X} \mathrm{D}^\bullet_{\mathrm{poly}}(U)$. This can be easily seen to commute with restrictions, thus giving us a map

$$\Delta : \mathrm{D}^\bullet_{\mathrm{poly}}(X) \to \mathrm{D}^\bullet_{\mathrm{poly}}(X) \otimes \mathrm{D}^\bullet_{\mathrm{poly}}(X),$$

the tensoring being over $\mathcal{O}_X$.

Unit for $D^\bullet_{\mathrm{poly}}(X)$: We have the obvious inclusion map $\eta : \mathcal{O}_X \to \mathrm{D}^\bullet_{\mathrm{poly}}(X)$. On $U = \operatorname{Spec} R$, this is just the inclusion $R \hookrightarrow \mathrm{D}^0_{\mathrm{poly}}(U) \hookrightarrow \mathrm{D}^\bullet_{\mathrm{poly}}(U)$.

Counit for $D^\bullet_{\mathrm{poly}}(X)$: We also have a projection $\epsilon : \mathrm{D}^\bullet_{\mathrm{poly}}(X) \to \mathcal{O}_X$. On $U = \operatorname{Spec} R$ with local coordinates $x_1, \ldots, x_m$, this takes

$$C_0(x_1, \ldots, x_m) + \sum_{I_1,\ldots,I_r} C_{I_1,\ldots,I_r}(x_1, \ldots, x_m)\partial_{I_1} \otimes \cdots \otimes \partial_{I_r} \text{ to } C_0(x_1, \ldots, x_m).$$

We now have the following fact

Proposition 2. *The multiplication m, the comultiplication Δ, the unit η and the counit ϵ together make $D^\bullet_{\mathrm{poly}}(X)$ a Hopf-algebra in* $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$.

Proof. Clearly, ϵ is an algebra homomorphism. Also, $\Delta(\mathrm{id}) = \mathrm{id} \otimes_{\mathcal{O}_X} \mathrm{id}$ tells us that η is a coalgebra homomorphism. The fact that Δ is an algebra homomorphism is exactly analogous to the fact that the comultiplication of the tensor algebra of a vector space over a field of characteristic 0 is an algebra homomorphism. Similarly, the fact that m is a co-algebra homomorphism is proven in exactly the same way by which one proves that the product in the graded tensor algebra of a vector space over a field of characteristic 0 is a co-algebra homomorphism.

The only things that remain to be checked are that the differential follows the Leibniz rule and respects co-multiplication — the latter fact following from the fact that the Hochschild *boundary* is a graded derivation with respect to the shuffle product (see Loday [4, Proposition 4.2.2]). Let $U = \operatorname{Spec} \mathrm{R}$ be an open affine subscheme of X. Let $a_1, \ldots, a_{k+l+1} \in R$. Let $D_1 \in \mathrm{D}^k_{\mathrm{poly}}(U)$ and $D_2 \in \mathrm{D}^l_{\mathrm{poly}}(U)$. The following calculation verifies that the differential d on $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$ obeys the Leibniz rule with respect to the multiplication m.

$$
\begin{aligned}
&d(D_1 \otimes D_2)(a_1 \otimes \cdots \otimes a_{k+l+1})\\
&\quad = a_1(D_1 \otimes D_2)(a_2 \otimes \cdots \otimes a_{k+l+1}) - (D_1 \otimes D_2)(a_1a_2 \otimes \cdots \otimes a_{k+l+1})\\
&\qquad + \cdots + (-1)^k (D_1 \otimes D_2)(a_1 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_{k+l+1})\\
&\qquad + \cdots + (-1)^{k+l+1} (D_1 \otimes D_2)(a_1 \otimes \cdots \otimes a_{k+l}) a_{k+l+1}\\
&\quad = a_1 D_1(a_2 \otimes \cdots \otimes a_{k+1}) \cdot D_2(a_{k+2} \otimes \cdots \otimes a_{k+l+1})\\
&\qquad - D_1(a_1 \cdot a_2 \otimes \cdots \otimes a_{k+1}) D_2(a_{k+2} \otimes \cdots \otimes a_{k+l+1})\\
&\qquad + \cdots + (-1)^k D_1(a_1 \otimes \cdots \otimes a_k a_{k+1}) D_2(a_{k+2} \otimes \cdots \otimes a_{k+l+1})\\
&\qquad + (-1)^{k+1} D_1(a_1 \otimes \cdots \otimes a_k) a_{k+1} D_2(a_{k+2} \otimes \cdots \otimes a_{k+l+1})\\
&\qquad + (-1)^{k} D_1(a_1 \otimes \cdots \otimes a_k) a_{k+1} D_2(a_{k+2} \otimes \cdots \otimes a_{k+l+1})\\
&\qquad + (-1)^{k+1} D_1(a_1 \otimes \cdots \otimes a_k) D_2(a_{k+1} a_{k+2} \otimes \cdots \otimes a_{k+l+1})\\
&\qquad + \cdots + (-1)^{k+l+1} D_1(a_1 \otimes \cdots \otimes a_k) D_2(a_{k+1} \otimes \cdots \otimes a_{k+l}) a_{k+l+1}\\
&\quad = dD_1(a_1 \otimes \cdots \otimes a_{k+1}) D_2(a_{k+2} \otimes \cdots \otimes a_{k+l+1})\\
&\qquad + (-1)^k D_1(a_1 \otimes \cdots \otimes a_k) dD_2(a_{k+1} \otimes \cdots \otimes a_{k+l+1}).
\end{aligned}
$$

□

Corollary 2. *The maps in $D^+(X)$ induced by m, Δ η and ϵ make $D^\bullet_{\mathrm{poly}}(X)$ a Hopf-algebra in $D^+(X)$.*

Remark. In fact, the Hopf algebra structure on $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$ is that of the graded tensor algebra. As $\mathcal{O}_X$-modules $\mathrm{D}^n_{\mathrm{poly}}(X)$ is isomorphic to $\mathrm{D}^1_{\mathrm{poly}}(X)^{\otimes_{\mathcal{O}_X} n}$ (Proposition 1). Thus, as far as the Hopf algebra structure is concerned, $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$ is isomorphic to the tensor algebra $T(\mathrm{D}^1_{\mathrm{poly}}(X))$ generated over $\mathcal{O}_X$ by $\mathrm{D}^1_{\mathrm{poly}}(X)$ in degree 1. They are isomorphic as Hopf algebras in $\mathrm{Gr}(\mathrm{mod} - \mathcal{O}_X)$. But there is a nontrivial differential (Hochschild cochain differential) on $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$.

4. A Decomposition of $\mathrm{D}^{\bullet}_{\mathrm{poly}}(X)$

4.1. *The decomposition*

Recall that $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ denotes the free Lie algebra generated over $\mathcal{O}_X$ by $\mathrm{D}^1_{\mathrm{poly}}(X)$ concentrated in degree 1. As graded $\mathcal{O}_X$ modules, it is a submodule of $\mathrm{D}^{\bullet}_{\mathrm{poly}}(X)$. Moreover,

Proposition 3. *The differential on $D^{\bullet}_{\mathrm{poly}}(X)$ preserves $L(D^1_{\mathrm{poly}}(X))$.*

Proof. By Proposition 2, the differential d on $\mathrm{D}^{\bullet}_{\mathrm{poly}}(X)$ obeys Leibniz rule with respect to the multiplication m on $\mathrm{D}^{\bullet}_{\mathrm{poly}}(X)$. It follows that it obeys Leibniz rule with respect to the Lie bracket $[,]$ induced by m on $\mathrm{D}^{\bullet}_{\mathrm{poly}}(X)$. The restriction of $[,]$ to $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ is precisely the Lie bracket on $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ and will also be denoted by $[,]$. Since $\mathrm{D}^1_{\mathrm{poly}}(X)$ (in degree 1) generates $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ as a Lie algebra over $\mathcal{O}_X$, and d obeys the Leibniz rule with respect to $[,]$, it is enough to check that $d(\mathrm{D}^1_{\mathrm{poly}}(X))$ is contained in $L(\mathrm{D}^1_{\mathrm{poly}}(X))$.

Since the differential d and the bracket $[,]$ are $\mathcal{O}_X$-linear, we only need to check that if $U = \operatorname{Spec} R$ with local coordinates $x_1, \ldots, x_m$ and if I is a multi-index, and if ∂_I denotes the corresponding partial derivative, then

$$d(\partial_I(a_1 \otimes a_2)) \in L(\mathrm{D}^1_{\mathrm{poly}}(U)).$$

Recall that if $I = (p_1, \ldots, p_m)$ and if $J = (q_1, \ldots, q_m)$ then $J \prec I$ if $q_i \leq p_i$ $\forall i$ and $J \neq I$. Let $I - J := (p_1 - q_1, \ldots, p_m - q_m)$. Now,

$$\begin{aligned} d(\partial_I(a_1 \otimes a_2)) &= a_1 \partial_I a_2 - \partial_I(a_1 \cdot a_2) + a_2 \cdot \partial_I(a_1) \\ &= -\frac{1}{2} \sum_{J \prec I} C_{IJ}(\partial_J(a_1) \cdot \partial_{I-J}(a_2) + \partial_{I-J}(a_1)\partial_J(a_2) \\ &= -\frac{1}{2} \sum_{J \prec I} C_{IJ}[\partial_J, \partial_{I-J}](a_1 \otimes a_2). \end{aligned}$$

Here C_{IJ} are some rational constants. This completes the desired verification. □

Symmetrization map I: We have a symmetrization map

$$I : \oplus_k \mathrm{Sym}^k(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \to \mathrm{D}^{\bullet}_{\mathrm{poly}}(X).$$

Let $U = \operatorname{Spec} R$ be an open affine subscheme of X, and let $z_1, \ldots, z_k$ be homogenous elements of $L(\mathrm{D}^1_{\mathrm{poly}}(U))$ of degrees $d_1, \ldots, d_k$ respectively. If $s(\sigma)$ is the sign such that $z_1, \ldots, z_k = s(\sigma) z_{\sigma(1)}, \ldots, z_{\sigma(k)} \in \mathrm{Sym}^k(L(\mathrm{D}^1_{\mathrm{poly}}(U)))$ then

$$I(z_1, \ldots, z_k) = \frac{1}{k!} \sum_{\sigma \in S_k} s(\sigma) z_{\sigma(1)} \otimes \cdots \otimes z_{\sigma(k)} \in \mathrm{D}^{\bullet}_{\mathrm{poly}}(X).$$

Note that $s(\sigma)$ depends on $d_1, \ldots, d_k$ and σ.

We now have the following key lemma

Lemma 1. *The symmetrization map I is an isomorphism in* $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$.

Proof. It follows directly from Propositions 2 and 3 and the definition of I that I is a map of complexes of $\mathcal{O}_X$ modules. It therefore suffices to show that I is a map of graded $\mathcal{O}_X$-modules. This can be verified locally.

Note that $\mathrm{D}^1_{\mathrm{poly}}(X)$ is locally free. Let $U = \mathrm{Spec}\, R$ be an affine open subscheme of X such that $\mathrm{D}^1_{\mathrm{poly}}(U)$ is trivial on U. Then, $\mathrm{D}^1_{\mathrm{poly}}(U) = V \otimes_K \mathcal{O}_U$. If $L(V)$ is the free Lie algebra generated over K by V in degree 1 and if $T(V)$ is the tensor algebra of V, then $L(\mathrm{D}^1_{\mathrm{poly}}(U)) = L(V) \otimes_K \mathcal{O}_U$ and $\mathrm{D}^\bullet_{\mathrm{poly}}(U) = T(V) \otimes_K \mathcal{O}_U$ as graded $\mathcal{O}_U$-modules.

Let $I_V : \oplus_k \mathrm{Sym}^k(L(V)) \to T(V)$ be the symmetrization map. Let $L_1, \dots, L_k$ be homogenous elements of $L(V)$ of degrees $d_1, \dots, d_k$ respectively. Let $s(\sigma)$ be the sign such that $L_1, \dots, L_k = s(\sigma) L_{\sigma(1)}, \dots, L_{\sigma(k)}$ in $\mathrm{Sym}^k(L(V))$. Then,

$$I_V(L_1, \dots, L_k) = \frac{1}{k!} \sum_{\sigma \in S_k} s(\sigma) L_{\sigma(1)} \otimes \cdots \otimes L_{\sigma(k)}.$$

Note that $\oplus \mathrm{Sym}^k(L(\mathrm{D}^1_{\mathrm{poly}}(U))) = \oplus_k \mathrm{Sym}^k(L(V)) \otimes_K \mathcal{O}_U$ and that $I = I_V \otimes \mathrm{id}_{\mathcal{O}_U}$. It thus suffices to show that I_V is an isomorphism of graded K-vector spaces. This is Proposition 17 of the Appendix to this paper. $\square$

4.2. *The Hochcshild–Kostant–Rosenberg map*

Throughout this subsection, let $U = \mathrm{Spec}\,\mathrm{R}$ be an affine open subscheme of X with local coordinates $x_1, \dots, x_m$. Recall (Yekutieli [1, Theorem 4.8]) that the Hochschild–Kostant–Rosenberg map $\mathrm{I}_{HKR} : \oplus_i \wedge^i T_X[-i] \to \mathrm{D}^\bullet_{\mathrm{poly}}(X)$ is a map of complexes which is a quasi-isomorphism. Also recall (Yekutieli [1, page 14]) that if $a_1, \dots, a_k \in R$, then

$$\mathrm{I}_{HKR}\left(\frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_k}}\right)(a_1 \otimes \cdots \otimes a_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \frac{\partial a_1}{\partial x_{i_{\sigma(1)}}} \cdots \frac{\partial a_k}{\partial x_{i_{\sigma(k)}}}.$$

Note that we have a map of complexes $\beta : T_X[-1] \to L(\mathrm{D}^1_{\mathrm{poly}}(X))$ such that on U, $\beta(\frac{\partial}{\partial x_i})(a) = \frac{\partial a}{\partial x_i}$ $\forall a \in R$. Note that β induces a map of complexes

$$\mathrm{Sym}^\bullet \beta : \oplus_i \wedge^i T_X[-i] \to \oplus_k \mathrm{Sym}^k(L(\mathrm{D}^1_{\mathrm{poly}}(X))).$$

Proposition 4. $\mathrm{Sym}^\bullet \beta$ *is a quasi-isomorphism.*

Proof. Observe that if $a_1, \dots, a_k \in R$, then

$$I \circ \mathrm{Sym}^\bullet \beta \left(\frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_k}}\right)(a_1 \otimes \cdots \otimes a_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \frac{\partial a_1}{\partial x_{i_{\sigma(1)}}} \cdots \frac{\partial a_k}{\partial x_{i_{\sigma(k)}}}.$$

Thus $I \circ \mathrm{Sym}^\bullet \beta = \mathrm{I}_{HKR}$.

Now I_{HKR} is a quasi-isomorphism (by Yekutieli [1, Theorem 4.8]) and I is an isomorphism of complexes of $\mathcal{O}_X$ modules by Lemma 1 and therefore a quasi-isomorphism. Thus, $\mathrm{Sym}^\bullet\beta$ is a quasi-isomorphism. □

Proposition 5. $\mathrm{Sym}^k\beta : \wedge^k T_X[-k] \to \mathrm{Sym}^k(L(D^1_{\mathrm{poly}}(X)))$ *is a quasi-isomorphism.*

Proof. Given a complex $\mathcal{M}$ in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$, let $\mathcal{H}^*(\mathcal{M})$ denote its cohomology, which is a graded $\mathcal{O}_X$ module. Then $\mathcal{H}^*(\wedge^k T_X[-k]) = \wedge^k T_X[-k]$ as $\wedge^k T_X[-k]$ has zero differential. Note that $\mathrm{Sym}^\bullet\beta$ induces a map $\mathrm{Sym}^\bullet\beta_* : \oplus_k \wedge^k T_X[-k] \to \mathcal{H}(\oplus_k \mathrm{Sym}^k(L(\mathrm{D}^1_{\mathrm{poly}}(X))))$ which is an isomorphism of graded $\mathcal{O}_X$-modules by Proposition 4. But $\mathrm{Sym}^\bullet\beta = \oplus_k \mathrm{Sym}^k\beta$ by definition. Thus, $\mathrm{Sym}^\bullet\beta_* = \oplus_k \mathrm{Sym}^k\beta_*$ where

$$\mathrm{Sym}^k\beta_* : \wedge^k T_X[-k] \to \mathcal{H}(\mathrm{Sym}^k(L(\mathrm{D}^1_{\mathrm{poly}}(X))))$$

is the map induced on cohomology by $\mathrm{Sym}^k\beta$. Since $\mathrm{Sym}^\bullet\beta_*$ is an isomorphism of graded $\mathcal{O}_X$-modules, it follows that for all k, $\mathrm{Sym}^k\beta_*$ is an isomorphism of graded $\mathcal{O}_X$-modules. □

In particular $\beta : T_X[-1] \to L(\mathrm{D}^1_{\mathrm{poly}}(X))$ is a quasi-isomorphism. We state this as a separate corollary in order to highlight it. Thus,

Corollary 3. $\beta : T_X[-1] \to L(D^1_{\mathrm{poly}}(X))$ *is a quasi-isomorphism.*

5. Theorem 1

5.1. *Precise statement and proof of Theorem 1*

Throughout this section let $U = \mathrm{Spec}\,\mathrm{R}$ be an arbitrary affine subscheme of X. Let $z_1, \ldots, z_k, y$ be homogenous elements of $L(\mathrm{D}^1_{\mathrm{poly}}(U))$ of degrees $d_1, \ldots, d_k, d$ respectively. Let

$$\omega : \oplus_k \mathrm{Sym}^k(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \otimes L(\mathrm{D}^1_{\mathrm{poly}}(X)) \to \oplus_k \mathrm{Sym}^k(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \otimes L(\mathrm{D}^1_{\mathrm{poly}}(X))$$

be the morphism in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$ such that on U,

$$\omega(z_1, \ldots, z_k \otimes y) = \sum_{i=1}^{i=k} (-1)^{d_i(d_{i+1}+\cdots+d_k)} z_1, \ldots, \widehat{z_i}, \ldots, z_k \otimes [z_i, y].$$

Note that if μ denotes the multiplication on $\mathrm{Sym}^\bullet(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$, then $\mu \circ \omega = \mathrm{ad}$ where ad denotes the right adjoint action of $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ on $\mathrm{Sym}^\bullet(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$. Let I be as in Sec. 4.

More generally, if M is a locally free $\mathcal{O}_X$ module, let $T(M)$ denote the (graded) tensor algebra of M. Let $L(M)$ denote the free Lie algebra generated over $\mathcal{O}_X$ by M in degree 1. On an open affine subscheme $U = \mathrm{Spec}\,R$, let $q_1, \ldots, q_k, r$ be homogenous elements of $L(M|_U)$ of degrees $d_1, \dots, d_k, d$ respectively. Let

$$\omega_M : \mathrm{Sym}^\bullet(L(M)) \otimes L(M) \to \mathrm{Sym}^\bullet(L(M)) \otimes L(M)$$

be the morphism of $\mathcal{O}_X$ modules such that

$$\omega_M(q_1, \dots, q_k \otimes r) = \sum_{i=1}^{i=k} (-1)^{d_i(d_{i+1}+\cdots+d_k)} q_1, \dots, \widehat{q_i}, \dots, q_k \otimes [q_i, r].$$

Let $I_M : \mathrm{Sym}^\bullet(L(M)) \to T(M)$ denote the symmetrization map such that

$$I_M(q_1, \dots, q_k) = \sum_{\sigma \in S_k} s(\sigma) q_{\sigma(1)} \otimes \cdots \otimes q_{\sigma(k)},$$

where $s(\sigma)$ is the sign such that $q_1, \dots, q_k = s(\sigma) q_{\sigma(1)}, \dots, q_{\sigma(k)}$ in $\mathrm{Sym}^k(L(M))$.

If V is a vector space over K, let $T(V)$ be the (graded) tensor algebra generated by V over K. Let $L(V)$ denote the free Lie algebra generated over K by V in degree 1. Let $v_1, \dots, v_k, w$ be homogenous elements of $L(V)$ of degrees $d_1, \dots, d_k, d$ respectively. Let

$$\omega_V : \mathrm{Sym}^\bullet(L(V)) \otimes L(V) \to \mathrm{Sym}^\bullet(L(V)) \otimes L(V)$$

be the map such that

$$\omega_V(v_1, \dots, v_k \otimes w) = \sum_{i=1}^{i=k} (-1)^{d_i(d_{i+1}+\cdots+d_k)} v_1, \dots, \widehat{v_i}, \dots, v_k \otimes [v_i, w].$$

Let $I_V : \mathrm{Sym}^\bullet(L(V)) \to T(V)$ denote the symmetrization map such that

$$I_V(v_1, \dots, v_k) = \sum_{\sigma \in S_k} s(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)},$$

where $s(\sigma)$ is the sign such that $v_1, \dots, v_k = s(\sigma) v_{\sigma(1)}, \dots, v_{\sigma(k)}$ in $\mathrm{Sym}^k(L(V))$.

Theorem 1. *The following diagram commutes in* $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$.

$$\begin{tikzcd}
D^\bullet_{\mathrm{poly}}(X) \otimes L(D^1_{\mathrm{poly}}(X)) \arrow[r, "m"] & D^\bullet_{\mathrm{poly}}(X) \\
\mathrm{Sym}^\bullet(L(D^1_{\mathrm{poly}}(X))) \otimes L(D^1_{\mathrm{poly}}(X)) \arrow[u, "I\otimes\mathrm{id}"] \arrow[r, "\mu\circ\frac{\omega}{1-e^{-\omega}}"] & \mathrm{Sym}^\bullet(L(D^1_{\mathrm{poly}}(X))) \arrow[u, "I"']
\end{tikzcd}$$

Step 1 of Proof. (Checking that all morphisms involved commute with the relevant differentials.)

Before we proceed, we note that the product m on $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$ and I commute with the relevant differentials by Proposition 2 and Lemma 1 respectively.

Further, we need to see that $\mu \circ \frac{\omega}{1-e^{-\omega}}$ commutes with the relevant differentials. Since μ commutes with the relevant differentials, we only need to check that $\frac{\omega}{1-e^{-\omega}}$ commutes with the relevant differentials. The latter expression is a power series in ω. It is therefore enough to verify that ω commutes with the relevant differentials. This only needs to be checked locally. The following calculations are done to complete the check

$$\omega(z_1z_2,\dots,z_k\otimes y)=\sum_i(-1)^{d_i(d_{i+1}+\dots+d_k)}z_1,\dots,\widehat{z_i},\dots,z_k\otimes[z_i,y],$$

$$\begin{aligned}d(\omega(z_1z_2,\dots,z_k\otimes y))=\sum_i(-1)^{d_i(d_{i+1}+\dots+d_k)}&\left(\sum_{j\neq i}(-1)^{d_1+\dots\widehat{d_i}\dots+d_{j-1}}\right.\\ &\times z_1,\dots,dz_j,\dots,\widehat{z_i},\dots,z_n\otimes[z_i,y]\\ &\left.+(-1)^{d_1+\dots\widehat{d_i}\dots+d_k}z_1,\dots,\widehat{z_i},\dots,z_k\otimes d([z_i,y])\right),\end{aligned}$$

$$\begin{aligned}d(z_1z_2,\dots,z_k\otimes y)=&\left(\sum_j(-1)^{d_1+\dots+d_{j-1}}z_1,\dots,dz_j,\dots,z_k\otimes y\right)\\ &+(-1)^{\sum d_i}z_1,\dots,z_k\otimes dy,\end{aligned}$$

$$\begin{aligned}\omega(d(z_1z_2,\dots,z_k\otimes y))=\sum_j\Bigg\{&\left(\sum_{i<j}(-1)^{d_1+\dots+d_{j-1}}(-1)^{d_i(d_{i+1}+\dots+d_k+1)}\right.\\ &\times z_1,\dots,\widehat{z_i},\dots,dz_j,\dots,z_k\otimes[z_i,y]\\ &+\sum_{i>j}(-1)^{d_1+\dots+d_{j-1}}(-1)^{d_i(d_{i+1}+\dots+d_k)}\\ &\left.\times z_1,\dots,dz_j,\dots,\widehat{z_i},\dots,z_k\otimes[z_i,y]\right)\\ &+(-1)^{d_1+\dots+d_{j-1}}(-1)^{(d_j+1)(d_{j+1},\dots,d_k)}\\ &\times z_1,\dots,\widehat{z_j},\dots,z_k\otimes[dz_j,y]\\ &+(-1)^{\sum d_i}(-1)^{d_j(d_{j+1}+\dots+d_k)}z_1,\dots,\widehat{z_j},\dots,z_k\otimes[z_j,dy]\Bigg\}.\end{aligned}$$

We now compare the coefficients of $[z_i,y]$, $[dz_i,y]$ and $[z_i,dy]$ in $d(\omega(z_1,\dots,z_k\otimes y))$ and $\omega(d(z_1,\dots,z_k\otimes y))$ and check that they are equal.

Step 2 of Proof. (Reduction to an analogous result for the graded free Lie algebra generated by a vector space over a field of characteristic 0.)

Having checked that all morphisms in the diagram given in the proposition are morphisms in $\mathrm{Ch}^+(\mathcal{O}_X-\mathrm{mod})$, it suffices to verify that the above diagram commutes in the category of graded $\mathcal{O}_X$-modules. In other words, we can "forget the differentials involved". Recall that $\mathrm{D}^1_{\mathrm{poly}}(X)$ is a locally free $\mathcal{O}_X$ module. It therefore suffices to prove that if M is a locally free $\mathcal{O}_X$ module, the following diagram commutes in the category of graded $\mathcal{O}_X$ modules. The map m_M in the diagram below is the multiplication in $T(M)$.

$$\begin{array}{ccc} T(M) \otimes L(M) & \xrightarrow{m_M} & T(M) \\ \uparrow{\scriptstyle I_M \otimes \mathrm{id}} & & {\scriptstyle I_M}\uparrow \\ \mathrm{Sym}^\bullet(L(M)) \otimes L(M) & \xrightarrow{\mu \circ \frac{\omega_M}{1-e^{-\omega_M}}} & \mathrm{Sym}^\bullet(L(M)) \end{array}$$

All morphisms in the diagram in this proposition are $\mathcal{O}_X$ module homomorphisms. It therefore suffices to check the claim that the above diagram commutes locally. We may therefore, without loss of generality, assume that M is a free $\mathcal{O}_X$ module i.e. $M = \mathcal{O}_X \otimes_K V$ for some K vector space V. Then, $T(M) = \mathcal{O}_X \otimes_K T(V)$, $L(M) = \mathcal{O}_X \otimes_K L(V)$ and $\mathrm{Sym}^\bullet(L(M)) = \mathcal{O}_X \otimes_K \mathrm{Sym}^\bullet(L(V))$.

Since the morphisms in the commutative diagram before the previous paragraph are all $\mathcal{O}_X$ linear, it suffices to check that the following diagram commutes in the category of graded K vector spaces. The map m_V in the diagram below is the multiplication in $T(V)$.

$$\begin{array}{ccc} T(V) \otimes L(V) & \xrightarrow{m_V} & T(V) \\ \uparrow{\scriptstyle I_V \otimes \mathrm{id}} & & {\scriptstyle I_V}\uparrow \\ \mathrm{Sym}^\bullet(L(V)) \otimes L(V) & \xrightarrow{\mu \circ \frac{\omega_V}{1-e^{-\omega_V}}} & \mathrm{Sym}^\bullet(L(V)) \end{array}$$

This is Theorem 6 of the Appendix. □

5.2. *Some remarks on Theorem 1*

Let A be an associative algebra in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$. In other words, there is a multiplication morphism $\mu_A : A \otimes A \to A$ in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$ which is associative i.e. $\mu_A \circ (\mu_A \otimes \mathrm{id}) = \mu_A \circ (\mathrm{id} \otimes \mu_A)$ as morphisms in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$ from $A \otimes A \otimes A$ to A.

A Lie algebra L in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$ is an object in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$ equipped with a morphism $[,]_L : L \otimes L \to L$ in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$ such that

(i) $[,]_L = -[,]_L \circ \tau$ where $\tau : L \otimes L \to L \otimes L$ is the swap map.
(ii) $[,]_L \circ (\mathrm{id} \otimes [,]_L) = [,]_L \circ ([,]_L \otimes \mathrm{id}) + [,]_L \circ (\mathrm{id} \otimes [,]_L) \circ (\tau \otimes \mathrm{id})$.

Note that any algebra A in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$ has a Lie algebra structure with

$$[,]_A = \mu_A \circ (\mathrm{id} - \tau).$$

Note that by Proposition 2, $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$ is an algebra in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$ and $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ is a Lie algebra in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$.

Given a Lie algebra L in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$, its universal enveloping algebra (if it exists) is an algebra $U(L)$ in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$ together with a morphism $i : L \to U(L)$ of Lie algebras such that given any morphism $f : L \to A$ of Lie algebras from L to an algebra A in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$, there exists a unique morphism $\bar{f} : U(L) \to A$ of algebras in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$ such that $f = \bar{f} \circ i$. We now prove the following lemma.

Lemma 2. *$D^{\bullet}_{\mathrm{poly}}(X)$ is the universal enveloping algebra of $L(D^1_{\mathrm{poly}}(X))$ in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$.*

Proof. Let A be an algebra in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$. Let $f : L(\mathrm{D}^1_{\mathrm{poly}}(X)) \to A$ be a morphism of Lie algebras in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$. In other words, $f \circ [,]_L = \mu_A \circ (\mathrm{id} - \tau) \circ (f \otimes f)$ as morphisms in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$. Let A^n denote the degree n term of the complex A. Let $f^n : L(\mathrm{D}^1_{\mathrm{poly}}(X)) \cap \mathrm{D}^n_{\mathrm{poly}}(X) \to A^n$ be the degree n component of the morphism f. In particular, $f^1 : \mathrm{D}^1_{\mathrm{poly}}(X) \to A^1$.

Note that $f^{1^{\otimes n}} : \mathrm{D}^n_{\mathrm{poly}}(X) = \mathrm{D}^1_{\mathrm{poly}}(X)^{\otimes n} \to (A^1)^{\otimes n}$ is a map of $\mathcal{O}_X$-modules. Note that the n-fold multiplication $\mu_{n,A} := \mu_A \circ (\mu_A \otimes \mathrm{id}) \circ \cdots \circ (\mu_A \otimes \mathrm{id}^{\otimes n-1}) : A^{\otimes n} \to A$ maps $(A^1)^{\otimes n}$ to A^n. Set $\bar{f}^n : \mathrm{D}^n_{\mathrm{poly}}(X) \to A^n$ to be the composite $\mu_{n,A} \circ f^{1^{\otimes n}}$.

Let $\bar{f} : \mathrm{D}^{\bullet}_{\mathrm{poly}}(X) \to A$ be the map of graded $\mathcal{O}_X$-modules whose degree n component is $\bar{f}^n$. We need to check that $\bar{f}$ is indeed a map of complexes of $\mathcal{O}_X$-modules. This can be checked locally. Suppose $U = \mathrm{Spec}\, R$ is an affine open subscheme of X, and if $D_1, \dots, D_n$ are differential operators on U. If d and d_A denote the differentials on $\mathrm{D}^{\bullet}_{\mathrm{poly}}(U)$ and $A|_U$ respectively, then

$$\begin{aligned} d_A(f^1(D_1), \dots, f^1(D_n)) &= \sum_{i=1}^{i=n} (-1)^{i-1} f^1(D_1), \dots, d_A(f^1(D_i)), \dots, f^1(D_n) \\ &= \sum_{i=1}^{i=n} (-1)^{i-1} f^1(D_1), \dots, f^2(d(D_i)), \dots, f^1(D_n). \end{aligned}$$

The last equality holds because f is a map of complexes.

Let $L(\mathrm{D}^1_{\mathrm{poly}}(U))^k$ denote the degree k term of the complex $L(\mathrm{D}^1_{\mathrm{poly}}(U))$. Note that $d(D_i) \in L(\mathrm{D}^1_{\mathrm{poly}}(U))^2$ by Proposition 3. Also, if $[,]$ denotes the bracket on $L(\mathrm{D}^1_{\mathrm{poly}}(U))$, then $f^2 \circ [,] = \mu_A \circ (\mathrm{id} - \tau) \circ f^1 \otimes f^1$ since f is a Lie algebra homomorphism in $\mathrm{D}^+(X)$. Also, $(\mathrm{id} - \tau) \circ (f^1 \otimes f^1) = (f^1 \otimes f^1) \circ (\mathrm{id} - \tau)$. Moreover, $[,] : \mathrm{D}^1_{\mathrm{poly}}(U)^{\otimes 2} \to L(\mathrm{D}^1_{\mathrm{poly}}(U))^2$ is surjective. It follows that $f^2(d(D_i)) = (f^1 \otimes f^1) d(D_i)$. ($d(D_i)$ on the right-hand side is thought of as an element of $\mathrm{D}^2_{\mathrm{poly}}(U)$ after identifying $L(\mathrm{D}^1_{\mathrm{poly}}(U))^2$ with its image in $\mathrm{D}^2_{\mathrm{poly}}(U)$.)

It follows that

$$\begin{aligned} d_A(f^1(D_1), \dots, f^1(D_n)) &= \sum_{i=1}^{i=n} (-1)^{i-1} f^1(D_1), \dots, f^2(d(D_i)), \dots, f^1(D_n) \\ &= \sum_{i=1}^{i=n} (-1)^{i-1} f^1(D_1), \dots, (f^1 \otimes f^1) d(D_i), \dots, f^1(D_n) \\ &= \sum_{i=1}^{i=n} f^{1^{\otimes n+1}}(D_1 \otimes \cdots \otimes d(D_i) \otimes \cdots \otimes D_n) \\ &= \bar{f}^{n+1} d(D_1 \otimes \cdots \otimes D_n). \end{aligned}$$

This shows that $\bar{f}$ is indeed a map of complexes. $\bar{f}$ is a map of algebras in $\mathrm{Ch}^+(\mathcal{O}_X-$mod) is immediate from its construction.

If $i : L(\mathrm{D}^1_{\mathrm{poly}}(X)) \to \mathrm{D}^\bullet_{\mathrm{poly}}(X)$ is the restriction of I to $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ we need to check that $\bar{f}\circ i = f$. Since the maps involved are maps of complexes of $\mathcal{O}_X$-modules, it is enough to check that $\bar{f}\circ i = f$ as maps of graded $\mathcal{O}_X$-modules. This can again be checked locally.

Let τ_k denote the k-cycle $(n-k+1,\, n-k+2, \dots, n)$ of S_n. Consider the element $\sigma_n := (1-\tau_n), \dots, (1-\tau_2)$ of group ring KS_n of S_n. Recall that S_n acts on $A^{1^{\otimes n}}$ on the right by $a_1 \otimes \cdots \otimes a_n \rightsquigarrow \mathrm{sgn}(\sigma) a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}$. By K-linearity, this extends to an action of KS_n on $A^{1^{\otimes n}}$.

Suppose $U = \operatorname{Spec} R$ is an affine open subscheme of X, and if $D_1, \dots, D_n$ are differential operators on U. Let $[,]_n : \mathrm{D}^1_{\mathrm{poly}}(U) \to L(\mathrm{D}^1_{\mathrm{poly}}(U))$ denote the "n-fold bracket" i.e. the map taking $D_1 \otimes \cdots \otimes D_n$ to $[D_1, [D_2[\dots[D_{n-1}, D_n]]]]$. The following diagram commutes in the category of graded R-modules since f is a Lie algebra homomorphism in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$.

$$\begin{array}{ccc}
\mathrm{D}^1_{\mathrm{poly}}(U)^{\otimes n} & \xrightarrow{f^{1\otimes n}} & A^{1^{\otimes n}} \\
{\scriptstyle [,]_n}\big\downarrow & & \big\downarrow{\scriptstyle \mu_{n,A}\circ\sigma_n} \\
L(\mathrm{D}^1_{\mathrm{poly}}(U))^n & \xrightarrow{f^n} & A^n
\end{array}$$

Note that S_n acts on the right on $\mathrm{D}^1_{\mathrm{poly}}(U)^{\otimes n}$ as well. Also, $f^{1^{\otimes n}}\circ\sigma_n = \sigma_n \circ f^{1^{\otimes n}}$. Also $i \circ [,]_n = \sigma_n : \mathrm{D}^1_{\mathrm{poly}}(U)^{\otimes n} \to \mathrm{D}^1_{\mathrm{poly}}(U)^{\otimes n} = \mathrm{D}^n_{\mathrm{poly}}(U)$. It follows that

$$f^n = f^{1^{\otimes n}} \circ i : L(\mathrm{D}^1_{\mathrm{poly}}(U))^n \to A^n.$$

This proves that $\bar{f}\circ i = f$.

Finally, we need to prove that $\bar{f}$ is the unique map with the required properties. Suppose that $g : \mathrm{D}^\bullet_{\mathrm{poly}}(X) \to A$ is a morphism of algebras in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$ such that $g \circ i = f$. Then, the restriction of g to $\mathrm{D}^1_{\mathrm{poly}}(X)$ is precisely f^1. It then follows from the fact that g is an algebra morphism in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$ that the restriction of g to $\mathrm{D}^n_{\mathrm{poly}}(X) = \mathrm{D}^1_{\mathrm{poly}}(X)^{\otimes n}$ is precisely $\bar{f}^n$. This proves that $g = \bar{f}$. □

5.2.1. *Meaning of Theorem 1*

Let $\mathfrak{g}$ denote a finite-dimensional Lie algebra over a field K of characteristic 0. Let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$. Let $\mathrm{I}_{PBW} : \mathrm{Sym}^\bullet\mathfrak{g} \to U(\mathfrak{g})$ be the symmetrization map. If $g_1, \dots, g_k \in \mathfrak{g}$ then,

$$\mathrm{I}_{PBW}(g_1, \dots, g_k) = \sum_{\sigma\in S_k} \frac{1}{k!} g_{\sigma(1)} *, \dots, * g_{\sigma(k)}$$

where $*$ denotes the multiplication in $U(\mathfrak{g})$. Let $\exp : \mathfrak{g} \to U(\mathfrak{g})$ denote the exponential map.

Consider the following calculation of $d(\exp)^{-1}$. Note first that $\exp(v) = \mathrm{I}_{PBW} \circ e^v \ \forall\, v \in \mathfrak{g}$, where $e^v = 1 + v + \frac{v^2}{2!} + \cdots \in \mathrm{Sym}^\bullet \mathfrak{g}$. Also,

$$d(\exp)^{-1}_{\exp(X)}(Y) = \frac{d}{dt}\bigg|_{t=0} \exp^{-1}(\exp(X) * \exp(tY)).$$

Thus $\exp^{-1}(\exp(X) * \exp(ty)) = X + d(\exp)^{-1}_{\exp(X))}(Y).t + h.o.t.$

It follows that $\mathrm{I}^{-1}_{PBW}(\mathrm{I}_{PBW}(e^X) * \mathrm{I}_{PBW}(e^{tY})) = e^{X + d(\exp)^{-1}_{\exp(X)}(Y).t + h.o.t}$.

Taking the derivative with respect to t at $t = 0$ on both sides of the previous equation, we get $\mathrm{I}^{-1}_{PBW}(\mathrm{I}_{PBW}(e^X) * Y) = e^X.d(\exp)^{-1}_{\exp(X))}(Y)$ where the multiplication on the right is the product in $\mathrm{Sym}^\bullet \mathfrak{g}$.

Another way of looking at this phenomenon is to say that the calculation of $d(\exp)^{-1}$ is equivalent to specifying φ in the following commutative diagram.

$$\begin{array}{ccc} U\mathfrak{g} \otimes \mathfrak{g} & \xrightarrow{\mu} & U\mathfrak{g} \\ {\scriptstyle \mathrm{I}_{PBW} \otimes \mathrm{id}} \uparrow & & \uparrow {\scriptstyle \mathrm{I}_{PBW}} \\ \mathrm{Sym}^\bullet \mathfrak{g} \otimes \mathfrak{g} & \xrightarrow{\varphi} & \mathrm{Sym}^\bullet \mathfrak{g} \end{array}$$

Let

$$\omega_{\mathfrak{g}} : \mathrm{Sym}^\bullet \mathfrak{g} \otimes \mathfrak{g} \to \mathrm{Sym}^\bullet \mathfrak{g} \otimes \mathfrak{g}$$

be the map such that for $g_1, \ldots, g_k, y \in \mathfrak{g}$,

$$\omega_{\mathfrak{g}}(g_1, \ldots, g_k \otimes y) = \sum_i g_1, \ldots, \widehat{i}, \ldots, g_k \otimes [g_i, y].$$

Note that if μ is the natural product in $\mathrm{Sym}^\bullet(\mathfrak{g})$, then $\mu \circ \omega_g = \mathrm{ad}$ where ad denotes the right adjoint action of $\mathfrak{g}$ on $\mathrm{Sym}^\bullet \mathfrak{g}$. In the classical situation, we know that $d(\exp)^{-1}_{\exp(X))}(Y) = \frac{\mathrm{ad}(X)}{1 - e^{-\mathrm{ad}(X)}} Y$. It follows then that $\varphi = \mu \circ \frac{\omega_{\mathfrak{g}}}{1 - e^{-\omega_{\mathfrak{g}}}}$.

This in short describes how I_{PBW} fails to commute with multiplication in the classical situation. By Lemma 2, $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$ is the universal enveloping algebra of $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$. I is the symmetrization map, and is the direct analog in our situation of I_{PBW}. Theorem 1, therefore, specifies how I fails to commute with multiplication and says that the error term measuring this failure has the same "$d(\exp^{-1})$ like" form as the corresponding error term in the classical situation. Since measuring how I_{PBW} fails to commute with multiplication calculates $d(\exp)^{-1}$ in the classical situation, we can call Theorem 1 the calculation of $d(\exp)^{-1}$ for the Lie algebra $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ of $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$.

6. The Atiyah Class of T_X

Recall that $\mathrm{D}^+(X)$ denotes the derived category of bounded below complexes of $\mathcal{O}_X$-modules with coherent cohomology. Let E be a vector bundle on X, and let $J_1(E)$ be the bundle of first jets of E. Recall that the Atiyah class of E is the element in $\mathrm{Hom}_{\mathrm{D}^+(X)}(E, E \otimes_{\mathcal{O}_X} \Omega[1])$ arising out of the exact sequence $0 \to E \otimes_{\mathcal{O}_X} \Omega \to J_1(E) \to E \to 0$. We denote the Atiyah class of E by $\alpha(E)$.

We can extend the definition of the Atiyah class of a vector bundle to elements of $\mathrm{D}^+(X)$ given by classes of perfect complexes of vector bundles on X. If $E^\bullet$ is such a complex, we have an exact sequence of complexes of vector bundles

$$0 \to E^\bullet \otimes_{\mathcal{O}_X} \Omega \xrightarrow{f} J_1(E^\bullet) \xrightarrow{g} E^\bullet \to 0.$$

We thus have a quasi-isomorphism of complexes

$$Q : \mathrm{tot}(0 \to E^\bullet \otimes_{\mathcal{O}_X} \Omega \to J_1(E^\bullet)) \to E^\bullet$$

such that the map $J_1(E^n) \oplus E^{n-1} \otimes_{\mathcal{O}_X} \Omega \to E^n$ is the composite

$$J_1(E^n) \oplus E^{n-1} \to J_1(E^n) \xrightarrow{g^n} E^n$$

where the first arrow is projection to the first factor. On the other hand, we have a map of complexes

$$R : J_1(E^\bullet) \oplus E^\bullet \otimes_{\mathcal{O}_X} \Omega[1] \to E^\bullet \otimes_{\mathcal{O}_X} \Omega[1]$$

given by projection to the second factor. Consider the element

$$R \circ Q^{-1} \in \mathrm{Hom}_{\mathrm{D}^+(X)}(E^\bullet, E^\bullet \otimes_{\mathcal{O}_X} \Omega[1]).$$

One checks that replacing $E^\bullet$ by a complex quasi-isomorphic to $E^\bullet$ does not give us a different element in $\mathrm{Hom}_{\mathrm{D}^+(X)}(E^\bullet, E^\bullet \otimes_{\mathcal{O}_X} \Omega[1])$. In case $E^\bullet$ is a complex comprising a vector bundle E concentrated at degree 0, we check that this gives us the Atiyah class of E. We can this call this element of $\mathrm{Hom}_{\mathrm{D}^+(X)}(E^\bullet, E^\bullet \otimes_{\mathcal{O}_X} \Omega[1])$ the Atiyah class of $E^\bullet$, and denote it by $\alpha(E^\bullet)$.

We now prove the following proposition that is stated without proof in Markarian [3]. It has also been proven in [14].

Proposition 6. *Let $E^\bullet$ be a complex of vector bundles with differential d on X such that every term of $E^\bullet$ has a global connection ∇. Then the Atiyah class $\alpha(E^\bullet)$ is given by $\{(-1)^n(-\nabla d + d\nabla)\} \in Hom_{D^+(X)}(E^\bullet, E^\bullet \otimes_{\mathcal{O}_X} \Omega[1])$.*

Proof. We recall that a connection ∇ on a vector bundle E is a K-linear map $\nabla : E \to E \otimes_{\mathcal{O}_X} \Omega$ so that for a section e of E over an open set U, $\nabla(f.e) = f(\nabla e) + e \otimes df$.

Also recall that $J_1(E) = p_{2*}(p_1^* E \otimes \mathcal{O}_{X \times_K X}/\mathcal{I}^2)$ where $\mathcal{I}$ is the kernel of the multiplication map $\mathcal{O}_X \otimes_K \mathcal{O}_X \to \mathcal{O}_X$. On an open subset $U = \mathrm{Spec}\, R$ of X where the sections of E on U are given by an R-module M, $J_1(E) = M \otimes_R (R \otimes R)/I^2$ with R-module structure given by multiplication with the second factor of $R \otimes_K R$. For a, b in R, let $a \hat{\otimes} b$ be the image of $a \otimes b$ in $R \otimes_K R/I^2$. Then, we have a map $p : E \to J_1(E)$ such that $m \rightsquigarrow m \otimes 1 \hat{\otimes} 1$. We observe that for a morphism of vector bundles $f : E \to E'$, $p \circ f = J_1(f) \circ p$.

Consider a complex $E^\bullet$ of vector bundles on X as in this proposition. Note that, since E^n has a global connection ∇_n, the exact sequence $0 \to E^n \otimes_{\mathcal{O}_X} \Omega \to J_1(E^n) \to E^n \to 0$ splits. The splitting map is given by $p - \nabla_n$ where $p : E^n \to J_1(E^n)$ is as in the previous paragraph. This splitting gives rise to a map

$\varphi_n : J_1(E^n) \to E^n \otimes_{\mathcal{O}_X} \Omega$ of $\mathcal{O}_X$-modules. We observe that $\{(-1)^n \varphi_n\}$ gives us a homotopy between the second projection $J_1(E^\bullet) \oplus E^\bullet \otimes_{\mathcal{O}_X} \Omega[1] \to E^\bullet \otimes_{\mathcal{O}_X} \Omega[1]$ and the map

$$\psi : J_1(E^\bullet) \oplus E^\bullet \otimes_{\mathcal{O}_X} \Omega[1] \to E^\bullet \otimes_{\mathcal{O}_X} \Omega[1]$$

given by

$$\psi(x, y) = (-1)^{n+1} \varphi_{n+1} dx + (-1)^n d\varphi_n x.$$

We next note that the map $p : E^\bullet \to J_1(E^\bullet)$ is a map of complexes of sheaves of K vector spaces inverting the quasi-isomorphism $\mathrm{tot}(0 \to E^\bullet \otimes_{\mathcal{O}_X} \Omega \to J_1(E^\bullet)) \to E^\bullet$. Therefore, in the category of complexes of sheaves of K vector spaces, $\alpha(E^\bullet)$ is homotopy equivalent to $\psi \circ p = \{(-1)^n(-\nabla d + d\nabla)\}$. We note that the latter is a morphism of complexes of $\mathcal{O}_X$-modules. Thus $\alpha(E^\bullet) = \{(-1)^n(-\nabla d + d\nabla)\}$ as a morphism in $\mathrm{D}^+(X)$. $\square$

We recall Kapranov [6, Proposition 1.2.2] which amount to saying that the Atiyah class of T_X which is a morphism in $\mathrm{D}^+(X)$ from $T_X[-1] \otimes T_X[-1] \to T_X[-1]$ equips $T_X[-1]$ with the structure of a Lie algebra in $\mathrm{D}^+(X)$.

Let β be as in Corollary 3, Sec. 4. By Corollary 3 of Sec. 4, $\beta : T_X[-1] \to L(\mathrm{D}^1_{\mathrm{poly}}(X))$ is a quasi-isomorphism provided that the right-hand side is equipped with the Hochschild co-boundary as differential. Thus, in $\mathrm{D}^+(X)$, $T_X[-1]$ is identified with $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ equipped with Hochschild co-boundary. We use this to realize $\alpha(T_X)$ explicitly as a map of complexes from $L(\mathrm{D}^1_{\mathrm{poly}}(X)) \otimes L(\mathrm{D}^1_{\mathrm{poly}}(X))$ to $L(\mathrm{D}^1_{\mathrm{poly}}(X))$.

Theorem 2. *The Atiyah class $\alpha(T_X)$ corresponds to the natural Lie bracket in $L(D^1_{\mathrm{poly}}(X))$ under the quasi-isomorphism β. In other words, the following diagram commutes in $D^+(X)$.*

$$\begin{tikzcd}
T_X[-1] \otimes_{\mathcal{O}_X} T_X[-1] \arrow[r, "\beta\otimes\beta"] \arrow[d, "\alpha(T_X)"'] & L(D^1_{\mathrm{poly}}(X)) \otimes_{\mathcal{O}_X} L(D^1_{\mathrm{poly}}(X)) \arrow[d, "{[,]}"] \\
T_X[-1] \arrow[r, "\beta"] & L(D^1_{\mathrm{poly}}(X))
\end{tikzcd}$$

Note that all maps in the diagram in Theorem 2 except for $\alpha(TX)$ arise out of maps in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$. The rows in the diagram of Theorem 2 are quasi-isomorphisms.

Proof. First of all, we note that $\mathrm{D}^n_{\mathrm{poly}}(X)$ has a natural connection for any n. To see this, let $U = \mathrm{Spec}\,\mathrm{A}$ be an affine open subscheme of X. If $f \in \mathrm{D}^n_{\mathrm{poly}}(U)$, then we can define ∇f by $\nabla_Y f(a_1, \ldots, a_n) = \partial_Y f(a_1, \ldots, a_n)\ \forall\ a_1, \ldots, a_n \in A$. It suffices to check this theorem locally. Further, one notes that $\alpha(T_X) = \alpha(T_X[-1]) = \alpha(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$. The second equality is by Corollary 3 of Sec. 4. Note that if $f \in \mathrm{D}^m_{\mathrm{poly}}(U)$ and $g \in \mathrm{D}^n_{\mathrm{poly}}(U)$, then $\nabla_Y[f, g] = [\nabla_Y f, g] + [f, \nabla_Y g]$. This implies that the connection ∇ on $\mathrm{D}^n_{\mathrm{poly}}(X)$ restricts to a connection on

$L(\mathrm{D}^1_{\mathrm{poly}}(X)) \cap \mathrm{D}^n_{\mathrm{poly}}(X)$. We now make the following calculation.

$$\nabla_Y(d\partial_I(a_1, a_2)) = \nabla_Y(a_1\partial_I a_2 - \partial_I(a_1a_2) + (\partial_I a_1)a_2)$$

$$d(\nabla_Y(\partial_I(a_1, a_2))) = a_1\partial_Y\partial_I a_2 - \partial_Y\partial_I(a_1a_2) + (\partial_Y\partial_I a_1)a_2$$

$$\partial_Y(a_1\partial_I a_2) = (\partial_Y a_1)(\partial_I a_2) + a_1\partial_Y\partial_I a_2$$

$$\partial_Y((\partial_I a_1)a_2) = (\partial_Y\partial_I a_1)a_2 + (\partial_I a_1)(\partial_Y a_2).$$

This tells us that $(d\nabla - \nabla d)(\partial_I \otimes \partial_Y) = [\partial_I, \partial_Y]$.

Now let $D_1 \in \mathrm{D}^m_{\mathrm{poly}}(U)$ and $D_2 \in \mathrm{D}^n_{\mathrm{poly}}(U)$. Suppose that $(d\nabla_Y - \nabla_Y d)D_1 = (-1)^m[D_1, \partial_Y]$ and $(d\nabla_Y - \nabla_Y d)D_2 = (-1)^n[D_2, \partial_Y]$. Then, by Proposition 2 and the fact that $[D_1, D_2] = D_1 \otimes D_2 - (-1)^{mn}D_2 \otimes D_1$,

$$d[D_1, D_2] = [dD_1, D_2] + (-1)^m[D_1, dD_2].$$

Further,

$$\nabla_Y[D_1, D_2] = [\nabla_Y D_1, D_2] + [D_1, \nabla_Y D_2].$$

Thus,

$$\begin{aligned}(d\nabla_Y - \nabla_Y d)[D_1, D_2] &= [(d\nabla_Y - \nabla_Y d)D_1, D_2] + (-1)^m[D_1, (d\nabla_Y - \nabla_Y d)D_2]\\ &= (-1)^m[[D_1, \partial_Y], D_2] + (-1)^{m+n}[D_1, [D_2, \partial_Y]]\\ &= (-1)^{m+n}[[D_1, D_2], \partial_Y]\end{aligned}$$

Thus, $(-1)^{m+n}(d\nabla_Y - \nabla_Y d)[D_1, D_2] = [[D_1, D_2], \partial_Y]$. Using induction on the degree of $D \in L(\mathrm{D}^1_{\mathrm{poly}}(U))$, together with the fact that $\mathrm{D}^1_{\mathrm{poly}}(U)$ generates $L(\mathrm{D}^1_{\mathrm{poly}}(U))$ as a Lie algebra over A, we see that $(-1)^{|D|}(d\nabla_Y - \nabla_Y d)D = [D, \partial_Y]$. This is exactly the desired theorem. □

Corollary 1 now follows immediately from Theorems 1 and 2.

7. The Universal Enveloping Algebra of $T_X[-1]$ in $\mathrm{D}^+(X)$

We would like to understand how Corollary 1 helps in relating the big Chern classes to the Chern character. This requires Theorem 3, which is proven in this section. A less tangible consequence of Theorem 3 is the ability to give Corollary 1 an interpretation along the lines of that given in Sec. 5.2.1 to Theorem 1.

An associative algebra A in $\mathrm{D}^+(X)$ is an object of $\mathrm{D}^+(X)$ such that there is a multiplication morphism $\mu_A : A \otimes A \to A$ in $\mathrm{D}^+(X)$ which is associative i.e.

$$\mu_A \circ (\mu_A \otimes \mathrm{id}) = \mu_A \circ (\mathrm{id} \otimes \mu_A) \text{ as morphisms in } \mathrm{D}^+(X) \text{ from } A \otimes A \otimes A \text{ to } A.$$

A Lie algebra L in $\mathrm{D}^+(X)$ is an object in $\mathrm{D}^+(X)$ equipped with a morphism$[,]_L : L \otimes L \to L$ in $\mathrm{D}^+(X)$ such that

(i) $[,]_L = -[,]_L \circ \tau$ where $\tau : L \otimes L \to L \otimes L$ is the swap map.
(ii) $[,]_L \circ (\mathrm{id} \otimes [,]_L) = [,]_L \circ ([,]_L \otimes \mathrm{id}) + [,]_L \circ (\mathrm{id} \otimes [,]_L) \circ (\tau \otimes \mathrm{id})$.

Note that any algebra A in $\mathrm{D}^+(X)$ has a Lie algebra structure with $[,]_A = \mu_A \circ (\mathrm{id} - \tau)$.

By Proposition 2, $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$ is an algebra in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$ and $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ is a Lie algebra in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$. Since $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$ and $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ are complexes with bounded below coherent co-homology by Proposition 5 and Corollary 3, they represent objects in $\mathrm{D}^+(X)$ denoted again by $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$ and $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ respectively. The algebra structure of $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$ in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$ induces an algebra structure in $\mathrm{D}^+(X)$. The Lie algebra structure of $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$ induces a Lie algebra structure in $\mathrm{D}^+(X)$. Corollary 3 of Sec. 4 says that $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ is isomorphic to $T_X[-1]$ in $\mathrm{D}^+(X)$. Theorem 2 says that the Lie algebra structure on $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ described in this paragraph coincides with the Lie algebra structure on $T_X[-1]$ induced by $\alpha(TX)$ after identifying $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ with $T_X[-1]$ via the quasi-isomorphism β of Corollary 3 of Sec. 4.

Given a Lie algebra L in $\mathrm{D}^+(X)$, its universal enveloping algebra (if it exists) is an algebra $U(L)$ in $\mathrm{D}^+(X)$ together with a morphism $i : L \to U(L)$ of Lie algebras such that given any morphism $f : L \to A$ of Lie algebras from L to an algebra A in $\mathrm{D}^+(X)$, there exists a unique morphism $\bar{f} : U(L) \to A$ of algebras in $\mathrm{D}^+(X)$ such that $f = \hat{f} \circ i$. We now state and prove Theorem 3

Theorem 3. *$D^\bullet_{\mathrm{poly}}(X)$ is the universal enveloping algebra of $T_X[-1]$ in $D^+(X)$. In other words, let A be an associative algebra in $D^+(X)$. If $f : T_X[-1] \to A$ is a morphism in $D^+(X)$ making the following diagram commute in $D^+(X)$,*

$$\begin{array}{ccc} T_X[-1] \otimes_{\mathcal{O}_X} T_X[-1] & \xrightarrow{\alpha(T_X)} & T_X[-1] \\ {\scriptstyle f\otimes f}\downarrow & & \downarrow{\scriptstyle f} \\ A \otimes_{\mathcal{O}_X} A & \xrightarrow{[,]_A} & A \end{array}$$

then there exists a unique morphism $\bar{f} : D^\bullet_{\mathrm{poly}}(X) \to A$ of algebras in $D^+(X)$ so that the composite $\bar{f} \circ I_{HKR} = f$.

Remark This theorem should be compared to Lemma 2. The reason why proving this is harder is that an algebra in $\mathrm{D}^+(X)$ may be realized as a complex of $\mathcal{O}_X$-modules, but may not be realizable as an algebra in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$. For instance, given an algebra A in $\mathrm{D}^+(X)$ and a complex $F^\bullet$ quasi-isomorphic to A, any map $F^\bullet \otimes F^\bullet \to F^\bullet$ representing μ_A may be associative only up to homotopy.

7.1. *Proof of Theorem 3*

Let $\beta : T_X[-1] \to L(\mathrm{D}^1_{\mathrm{poly}}(X))$ be as defined in Sec. 4.2. Note that β is a quasi-isomorphism by Proposition 5, and therefore induces an isomorphism in $\mathrm{D}^+(X)$. Moreover, by Theorem 2, β is a morphism of Lie algebras in $\mathrm{D}^+(X)$. It follows that we can replace $T_X[-1]$ by $L(\mathrm{D}^1_{\mathrm{poly}}(X))$. Suppose that A is an associative algebra in $\mathrm{D}^+(X)$ and that $f : L(\mathrm{D}^1_{\mathrm{poly}}(X)) \to A$ is a morphism of Lie algebras in $\mathrm{D}^+(X)$.

Step 0 (Conventions, notations and some observations).

(1) An inclusion in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$ will mean a map of complexes that is injective term by term.
(2) By convention, $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ is equipped with the Hochschild co-boundary.
(3) If M is any object in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$, $\mathcal{L}(M)$ denotes the free Lie algebra generated in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$ over $\mathcal{O}_X$ by M. The only differential on $\mathcal{L}(M)$ arises out of the differential on M. $\mathcal{T}(M)$ will denote the tensor algebra generated by M over $\mathcal{O}_X$. Again, the only differential on $\mathcal{T}(M)$ arises out of the differential on M.
(4) $\mathrm{Sym}^{\bullet}(M)$ will denote the symmetric algebra generated by M over $\mathcal{O}_X$ in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$. We have a map of complexes $\mathrm{J}(M) : \mathrm{Sym}^{\bullet}(M) \to \mathcal{T}(M)$. Given an open subscheme $U = \operatorname{Spec} R$ of X, if $m_1, \ldots, m_k$ are sections of $M|_U$ of degrees $d_1, \ldots, d_k$ respectively, then

$$\mathrm{J}(M)(m_1, \ldots, m_k) = \frac{1}{k!} \sum_{\sigma \in S_k} s(\sigma) m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(k)}$$

where $s(\sigma)$ is the sign (depending on $d_1, \ldots, d_k$ and σ) such that

$$m_1, \ldots, m_k = s(\sigma) m_{\sigma(1)}, \ldots, m_{\sigma(k)} \in \mathrm{Sym}^k(M).$$

Also, we have a map of complexes $\mathrm{B}(M) : \mathrm{Sym}^{\bullet}(\mathcal{L}(M)) \to \mathcal{T}(M)$. If $U = \operatorname{Spec} R$ is an open affine subscheme of X, and if $z_1, \ldots, z_k$ are sections of $\mathcal{L}(M)|_U$ of degrees $d_1, \ldots, d_k$ respectively, then

$$\mathrm{B}(M)(z_1, \ldots, z_k) = \frac{1}{k!} \sum_{\sigma \in S_k} s(\sigma) z_{\sigma(1)} \otimes \cdots \otimes z_{\sigma(k)}.$$

Here, the z_i's on the right-hand side are thought of as sections of $\mathcal{T}(M)|_U$.
$s(\sigma)$ is the sign (depending on $d_1, \ldots, d_k$ and σ) such that

$$z_1, \ldots, z_k = s(\sigma) z_{\sigma(1)}, \ldots, z_{\sigma(k)} \in \mathrm{Sym}^k(\mathcal{L}(M)).$$

(5) Recall from Lemma 1 that the symmetrization map $I : \oplus_k \mathrm{Sym}^k \times (L(\mathrm{D}^1_{\mathrm{poly}}(X))) \to \mathrm{D}^{\bullet}_{\mathrm{poly}}(X)$ is an isomorphism in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$. Denote its inverse by G.
(6) Denote the restriction of I to $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ by I_1. This is an inclusion in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$.
(7) Denote that natural inclusion from $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ to $\mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$ by I_2.
(8) The object in $\mathrm{D}^+(X)$ represented by an object M in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$ will be denoted by M itself provided it exists.
(9) The Hochschild–Kostant–Rosenberg theorem implies that all objects in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$ mentioned in this proof represent objects in $\mathrm{D}^+(X)$.

Step 1 (Construction of $\hat{f} : \mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \to A$ in $\mathrm{D}^+(X)$).

We have $f^{\otimes k} : L(\mathrm{D}^1_{\mathrm{poly}}(X))^{\otimes k} \to A^{\otimes k}$. Since A is an algebra in $\mathrm{D}^+(X)$, we have the k fold multiplication $\mu_{k,A} : A^{\otimes k} \to A$ in $\mathrm{D}^+(X)$. We thus get a morphism

$\mu_{k,A} \circ f^{\otimes k} : L(\mathrm{D}^1_{\mathrm{poly}}(X))^{\otimes k} \to A$ in $\mathrm{D}^+(X)$. Putting these together we get a morphism $\hat{f} : \mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \to A$ in $\mathrm{D}^+(X)$. By construction, $\hat{f}$ is an algebra homomorphism in $\mathrm{D}^+(X)$ and the composite $\hat{f} \circ I_2$ equals f.

Step 2 (Construction of $\mathcal{I} : \mathrm{D}^\bullet_{\mathrm{poly}}(X) \to \mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$.

We have another kind of symmetrization map

$$\mathrm{J}((L(\mathrm{D}^1_{\mathrm{poly}}(X)))) : \mathrm{Sym}^\bullet(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \to \mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X))).$$

Consider the composite

$$\mathrm{J}(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \circ G : \mathrm{D}^\bullet_{\mathrm{poly}}(X) \to \mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X))).$$

Denote it by $\mathcal{I}$. Note that $\mathcal{I} \circ I_1 = I_2$.

Step 3 (Construction of $\bar{f} : \mathrm{D}^\bullet_{\mathrm{poly}}(X) \to A$ in $\mathrm{D}^+(X)$).

We define $\bar{f} := \hat{f} \circ \mathcal{I}$. By construction the composite $\bar{f} \circ I_1 = \hat{f} \circ \mathcal{I} \circ I_1 = \hat{f} \circ I_2 = f$.

To complete the proof of Theorem 1, we only need to check that $\bar{f}$ is an algebra homomorphism in $\mathrm{D}^+(X)$.

Step 4 (Construction of an algebra homomorphism $\lambda : \mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \to \mathrm{D}^\bullet_{\mathrm{poly}}(X)$ in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$).

Let $m_k : \mathrm{D}^\bullet_{\mathrm{poly}}(X)^{\otimes k} \to \mathrm{D}^\bullet_{\mathrm{poly}}(X)$ denote the k-fold product of $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$.

Consider the composite map

$$\lambda_k : L(\mathrm{D}^1_{\mathrm{poly}}(X))^{\otimes k} \xrightarrow{I_1{}^{\otimes k}} \mathrm{D}^\bullet_{\mathrm{poly}}(X)^{\otimes k} \xrightarrow{m_k} \mathrm{D}^\bullet_{\mathrm{poly}}(X).$$

This map is a map in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$ as I_1 and m_k are maps in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$. Putting these together, we obtain a map $\lambda : \mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \to \mathrm{D}^\bullet_{\mathrm{poly}}(X)$. By construction, λ is a morphism of algebras in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$. Also, the composite of λ_k with the restriction of $\mathrm{J}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$ to $\mathrm{Sym}^k(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$ is just the restriction of the symmetrization map I to $\mathrm{Sym}^k(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$ Thus, $\lambda \circ \mathrm{J}(L(\mathrm{D}^1_{\mathrm{poly}}(X))) = I$. It follows that

$$\lambda \circ \mathcal{I} = \lambda \circ \mathrm{J}(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \circ G = I \circ G = \mathrm{id}.$$

Step 5 (Uniqueness of $\bar{f}$).

Suppose that $\bar{f}_1 : \mathrm{D}^\bullet_{\mathrm{poly}}(X) \to A$ and $\bar{f}_2 : \mathrm{D}^\bullet_{\mathrm{poly}}(X) \to A$ are two morphisms of algebras in $\mathrm{D}^+(X)$ such that $\bar{f}_1 \circ I_1 = \bar{f}_2 \circ I_1 = f$. Then, by the construction of the map $\lambda : \mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \to \mathrm{D}^\bullet_{\mathrm{poly}}(X)$ in Step 4,

$$\bar{f}_1 \circ \lambda = \bar{f}_2 \circ \lambda = \hat{f} : \mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \to A.$$

It follows from this and the fact (demonstrated in Step 4) $\lambda \circ \mathcal{I} = \mathrm{id}$ that

$$\hat{f} \circ \mathcal{I} = \bar{f}_1 = \bar{f}_1 \circ \lambda \circ \mathcal{I} = \bar{f}_2 \circ \lambda \circ \mathcal{I} = \bar{f}_2.$$

This proves that $\bar{f}_1$ and $\bar{f}_2$ are identical morphisms in $\mathrm{D}^+(X)$.

Step 6

We now return to proving that $\bar{f}$ is a morphism of algebras in $\mathrm{D}^+(X)$. Proving this will completely prove Theorem 3. We claim that the following diagram commutes in $\mathrm{Ch}^+(\mathcal{O}_X-\mathrm{mod})$.

$$\begin{tikzcd}
\mathrm{D}^\bullet_{\mathrm{poly}}(X)\otimes\mathrm{D}^\bullet_{\mathrm{poly}}(X) \arrow[r, "\mathcal{I}\otimes\mathcal{I}"] \arrow[d, "m"'] & \mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))\otimes\mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \arrow[d, "\mu_{\mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))}"] \\
\mathrm{D}^\bullet_{\mathrm{poly}}(X) & \mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \arrow[l, "\lambda"]
\end{tikzcd}$$

This is immediate from the fact $\lambda\circ\mathcal{I}=\mathrm{id}$ and the fact λ is an algebra homomorphism in $\mathrm{Ch}^+(\mathcal{O}_X-\mathrm{mod})$. Both these facts were demonstrated in Step 4.

It follows that the following diagram commutes in $\mathrm{D}^+(X)$.

$$\begin{tikzcd}
\mathrm{D}^\bullet_{\mathrm{poly}}(X)\otimes\mathrm{D}^\bullet_{\mathrm{poly}}(X) \arrow[r, "\mathcal{I}\otimes\mathcal{I}"] \arrow[d, "m"'] & \mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))\otimes\mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \arrow[r, "\hat{f}\otimes\hat{f}"] \arrow[d, "\mu_{\mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))}"'] & A\otimes A \arrow[d, "\mu_A"] \\
\mathrm{D}^\bullet_{\mathrm{poly}}(X) & \mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \arrow[l, "\lambda"] \arrow[r, "\hat{f}"] & A
\end{tikzcd}$$

The arrows in the square on the left arise out of morphisms in $\mathrm{Ch}^+(\mathcal{O}_X-\mathrm{mod})$ itself.

Now observe that in $\mathrm{D}^+(X)$,

$$\begin{aligned}
\mu_A\circ(\bar{f}\otimes\bar{f}) &= \mu_A\circ[(\hat{f}\circ\mathcal{I})\otimes(\hat{f}\circ\mathcal{I})]\\
&= \hat{f}\circ\mu_{\mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))}\circ(\mathcal{I}\otimes\mathcal{I})\\
&= \hat{f}\circ(\mathrm{id}-\mathcal{I}\circ\lambda)\circ\mu_{\mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))}\circ(\mathcal{I}\otimes\mathcal{I})\\
&\quad+\hat{f}\circ\mathcal{I}\circ\lambda\circ\mu_{\mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))}\circ(\mathcal{I}\otimes\mathcal{I}).
\end{aligned}$$

But $\lambda\circ\mu_{\mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))}\circ(\mathcal{I}\otimes\mathcal{I})=m$. Thus,

$$\begin{aligned}
\mu\circ\bar{f}\otimes\bar{f} &= \hat{f}\circ(\mathrm{id}-\mathcal{I}\circ\lambda)\circ\mu_{\mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))}\circ(\mathcal{I}\otimes\mathcal{I})+\hat{f}\circ\mathcal{I}\circ m\\
&= \hat{f}\circ(\mathrm{id}-\mathcal{I}\circ\lambda)\circ\mu_{\mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))}\circ(\mathcal{I}\otimes\mathcal{I})+\bar{f}\circ m.
\end{aligned}$$

Therefore, to show that $\bar{f}$ is a homomorphism of algebras in $\mathrm{D}^+(X)$, it suffices to show the following proposition.

Proposition 7. *The composite*

$$\mathcal{T}(L(D^1_{\mathrm{poly}}(X)))\xrightarrow{\mathrm{id}-\mathcal{I}\circ\lambda}\mathcal{T}(L(D^1_{\mathrm{poly}}(X)))\xrightarrow{\hat{f}}A$$

is 0 *in* $D^+(X)$.

We note that the first arrow $\mathrm{id}-\mathcal{I}\circ\lambda$ is an arrow arising out of a morphism in $\mathrm{Ch}^+(\mathcal{O}_X-\mathrm{mod})$. We break the proof of Proposition 7 into the following easier propositions.

Note that any morphism in $\mathrm{Ch}^+(\mathcal{O}_X-\mathrm{mod})$ can be thought of as morphism of graded $\mathcal{O}_X$ modules. Denote the map $\mathrm{B}(L(\mathrm{D}^1_{\mathrm{poly}}(X))) : \mathrm{Sym}^\bullet(\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))) \to \mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$ by B. Note that B is a morphism in $\mathrm{Ch}^+(\mathcal{O}_X-\mathrm{mod})$. Denote the restriction of B to $\mathrm{Sym}^k(\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(X))))$ by B^k. In particular, B^1 is a map in $\mathrm{Ch}^+(\mathcal{O}_X-\mathrm{mod})$ from $\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$ to $\mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$.

Proposition 8. *B^1 is a morphism of Lie algebras in $\mathrm{Ch}^+(\mathcal{O}_X-\mathrm{mod})$.*

Proof. Let $\mathrm{B}(M)^k$ denote the restriction of $\mathrm{B}(M)$ to $\mathrm{Sym}^k(\mathcal{L}(M))$. Then, we claim that $\mathrm{B}(M)^1$ is a morphism of Lie algebras in $\mathrm{Ch}^+(\mathcal{O}_X-\mathrm{mod})$. Let $U=\mathrm{Spec}\,\mathrm{R}$ be an open affine subscheme of X and let z_1,z_2 be sections of $\mathcal{L}(M)|_U$ of degrees d_1,d_2 respectively. Then $\mathrm{B}(M)^1[z_1,z_2]=z_1\otimes z_2-(-1)^{d_1d_2}z_2\otimes z_1$ and $\mathrm{B}(M)^1(z_i)=z_i$. This proves the desired proposition. $\square$

Recall that we have a natural inclusion of complexes $L(\mathrm{D}^1_{\mathrm{poly}}(X)) \to \mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$. This just treats a section of $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ as a section of $\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$.

Proposition 9. (i) *As a morphism of graded $\mathcal{O}_X$ modules, $\lambda\circ B^1$ maps $\mathcal{L}(L(D^1_{\mathrm{poly}}(X)))$ to $I_1(L(D^1_{\mathrm{poly}}(X)))$. Let π denote $G\circ\lambda\circ B^1$.*

(ii) *$\pi:\mathcal{L}(L(D^1_{\mathrm{poly}}(X)))\to L(D^1_{\mathrm{poly}}(X))$ is a map of Lie algebras in* $\mathrm{Ch}^+(\mathcal{O}_X-\mathrm{mod})$.

(iii) *The composite $L(D^1_{\mathrm{poly}}(X))\to\mathcal{L}(L(D^1_{\mathrm{poly}}(X)))\xrightarrow{\pi}L(D^1_{\mathrm{poly}}(X))$ is the identity.*

(iv) $\lambda\circ B=I\circ\mathrm{Sym}^\bullet(\pi)$.

Proof. Note that $\mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$ and $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$ are Lie algebras in $\mathrm{Ch}^+(\mathcal{O}_X-\mathrm{mod})$. Their Lie algebra structures are induced by their algebra structures in $\mathrm{Ch}^+(\mathcal{O}_X-\mathrm{mod})$. Since $\lambda:\mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))\to\mathrm{D}^\bullet_{\mathrm{poly}}(X)$ is a morphism of algebras in $\mathrm{Ch}^+(\mathcal{O}_X-\mathrm{mod})$, λ is also a morphism of Lie algebras in $\mathrm{Ch}^+(\mathcal{O}_X-\mathrm{mod})$. Also, by its very construction, λ maps $I_2(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$ identically to $I_1(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$. It follows that λ maps $B^1(\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(X))))$ to the Lie subalgebra of $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$ generated over $\mathcal{O}_X$ by $I_1(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$ which is $I_1(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$ itself. This proves (i).

Observe that λ is a Lie algebra homomorphism in $\mathrm{Ch}^+(\mathcal{O}_X-\mathrm{mod})$. Further, since $I_1:L(\mathrm{D}^1_{\mathrm{poly}}(X))\to I(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$ is a morphism of Lie algebras in $\mathrm{Ch}^+(\mathcal{O}_X-\mathrm{mod})$, $G:I(L(\mathrm{D}^1_{\mathrm{poly}}(X)))\to L(\mathrm{D}^1_{\mathrm{poly}}(X))$ is also a morphism of Lie algebras in $\mathrm{Ch}^+(\mathcal{O}_X-\mathrm{mod})$. Since B^1 is a morphism of Lie algebras in $\mathrm{Ch}^+(\mathcal{O}_X-\mathrm{mod})$ by Proposition 8, (ii) follows.

(iii) is immediate from the construction of λ.

Let $U=\mathrm{Spec}\,R$ be an affine open subscheme of X. Let $x_1,\dots,x_k$ be homogenous sections of $\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(U)))$ of degrees $d_1,\dots,d_k$ respectively. For a permutation $\sigma\in S_k$ let $s(\sigma)$ be the sign such that $x_1,\dots,x_k=s(\sigma)x_{\sigma(1)},\dots,x_{\sigma(k)}$ in

$\mathrm{Sym}^k(\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(U))))$. Then,

$$B(x_1,\dots,x_k) = \frac{1}{k!}\sum_{\sigma\in S_k} s(\sigma)x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(k)}$$

$$\lambda\circ B(x_1,\dots,x_k) = \frac{1}{k!}\sum_{\sigma\in S_k} s(\sigma)\lambda(x_{\sigma(1)})\otimes\cdots\otimes \lambda(x_{\sigma(k)}).$$

But, by the definition of π, if $x\in\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(U)))$, then $\lambda(x) = I_1(\pi(x))$. It follows that $\lambda\circ B(x_1,\dots,x_k)$ is precisely $I(\pi(x_1),\dots,\pi(x_k))$ in $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$. This proves (iv). $\square$

Recall that we have the map

$$\mathrm{J}(\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))) : \mathrm{Sym}^\bullet(\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))) \to \mathcal{T}(\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))).$$

Let $\mathrm{J}(\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(X))))^k$ denote its restriction to $\mathrm{Sym}^k(\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(X))))$. Let $\mathrm{J}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))^k$ denote the restriction of $\mathrm{J}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$ to $\mathrm{Sym}^k(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$.

Let $\pi^\bullet$ denote the morphism $\oplus_k\pi^{\otimes k} : \mathcal{T}(\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))) \to \mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$.

Proposition 10. *The following diagram commutes in* $\mathrm{Ch}^+(\mathcal{O}_X-\mathrm{mod})$.

$$\begin{array}{ccc}
\mathrm{Sym}^k(\mathcal{L}(L(D^1_{\mathrm{poly}}(X)))) & \xrightarrow{J(\mathcal{L}(L(D^1_{\mathrm{poly}}(X))))^k} & \mathcal{T}(\mathcal{L}(L(D^1_{\mathrm{poly}}(X)))) \\
\Big\downarrow{\scriptstyle \mathrm{Sym}^k(\pi)} & & \Big\downarrow{\scriptstyle \pi^\bullet} \\
\mathrm{Sym}^k(L(D^1_{\mathrm{poly}}(X))) & \xrightarrow{J(L(D^1_{\mathrm{poly}}(X)))^k} & \mathcal{T}(L(D^1_{\mathrm{poly}}(X)))
\end{array}$$

Proof. This is immediate from the definitions of π, $\mathrm{J}(\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(X))))^k$ and $\mathrm{J}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))^k$. $\square$

Note that the symmetrization map

$$\mathrm{B}(L(\mathrm{D}^1_{\mathrm{poly}}(X))) : \mathrm{Sym}^\bullet(\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))) \to \mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$$

is an isomorphism in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$. This is proven in a fashion similar to Lemma 1. Denote the inverse of $\mathrm{B}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$ by W. Let $\mathcal{Z}$ denote $\mathrm{J}(\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(X))))\circ W$. We have the following corollary of Propositions 9 and 10.

Corollary 4. *The following diagram commutes in the category of differential graded* $\mathcal{O}_X$ *modules.*

$$\begin{array}{ccc}
\mathcal{T}(L(D^1_{\mathrm{poly}}(X))) & \xrightarrow{\mathcal{Z}} & \mathcal{T}(\mathcal{L}(L(D^1_{\mathrm{poly}}(X)))) \\
\Big\downarrow{\scriptstyle \lambda} & & \Big\downarrow{\scriptstyle \pi^\bullet} \\
D^\bullet_{\mathrm{poly}}(X) & \xrightarrow{\mathcal{I}} & \mathcal{T}(L(D^1_{\mathrm{poly}}(X)))
\end{array}$$

Proof. Observe that Proposition 9(iv) tells us that $\lambda \circ B = I \circ \mathrm{Sym}^\bullet(\pi)$. Thus, $G \circ \lambda \circ B = \mathrm{Sym}^\bullet(\pi)$. Thus,

$$\mathrm{J}(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \circ G \circ \lambda \circ B \circ W = \mathrm{J}(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \circ \mathrm{Sym}^\bullet(\pi) \circ W.$$

But $\mathcal{I} = \mathrm{J}(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \circ G$ and $B \circ W = \mathrm{id}$. Thus, $\mathcal{I} \circ \lambda = \mathrm{J}(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \circ \mathrm{Sym}^\bullet(\pi) \circ W$. By Proposition 10, $\mathrm{J}(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \circ \mathrm{Sym}^\bullet(\pi) = \pi^\bullet \circ \mathrm{J}(\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(X))))$. Thus,

$$\mathcal{I} \circ \lambda = \pi^\bullet \circ \mathrm{J}(\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))) \circ W = \pi^\bullet \circ \mathcal{Z}. \qquad \square$$

We have the multiplication map $\mu_{\mathcal{T}}$ from $\mathcal{T}(\mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X))))$ to $\mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$. This arises from the tensor product in $\mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$. Let $B : \mathrm{Sym}^\bullet(\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))) \to \mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$ be as in Proposition 9. The map B^1 yields us a map $\mathcal{T}(B^1) : \mathcal{T}(\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))) \to \mathcal{T}(\mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X))))$ in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$. This is a map of algebras in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$ by construction. Denote the composite $\mu_{\mathcal{T}} \circ \mathcal{T}(B^1) : \mathcal{T}(\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))) \to \mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$ by $\mathcal{G}$. Since $\mu_{\mathcal{T}}$ and $\mathcal{T}(B^1)$ are morphisms of algebras in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$, so is $\mathcal{G}$.

Proposition 11. *With $\mathcal{G}$ as defined above*, $\mathcal{G} \circ \mathcal{Z} = \mathrm{id}$.

Proof. First note that $\mathcal{G} \circ \mathrm{J}(\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))) = \mathrm{B}(\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(X))))$. To see this, let $U = \mathrm{Spec}\, R$ be an open affine subscheme of X, and let $z_1, \ldots, z_k$ be sections of $\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))|_U$ of degrees $d_1, \ldots, d_k$ respectively. Then, by the construction of G, $G(z_1 \otimes \cdots \otimes z_k) = z_1 \otimes \cdots \otimes z_k$ where the tensor product on the right is that in $\mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$ and where the z_i's on the right are treated as sections of $\mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))|_U$.

Now, $\mathcal{G} \circ \mathcal{Z} = \mathcal{G} \circ \mathrm{J}(\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))) \circ W = \mathrm{B}(\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))) \circ W = \mathrm{id}$. $\square$

The following corollary is obtained from the above proposition and Corollary 4.

Corollary 5. $\mathrm{id} - \mathcal{I} \circ \lambda = (\mathcal{G} - \pi^\bullet) \circ \mathcal{Z}$.

Note that all the commutative diagrams in Propositions 8–11 and their corollaries are diagrams in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$. They induce corresponding commutative diagrams in $\mathrm{D}^+(X)$.

From Corollary 5, it is clear that to prove Proposition 7, it suffices to prove the following proposition.

Proposition 12. *$\hat{f} \circ \mathcal{G} = \hat{f} \circ \pi^\bullet$ in $D^+(X)$.*

Proof. We have a natural inclusion of complexes $\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))^{\otimes k} \to \mathcal{T}(\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(X))))$. Denote the composite of $\mathcal{G}$ with this inclusion by $\mathcal{G}_k$. This is a morphism in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$.

To prove this proposition, it suffices to show that $\hat{f} \circ \mathcal{G}_k = \hat{f} \circ \pi^{\otimes k}$ in $\mathrm{D}^+(X)$. Since $\mathcal{G}$ is a homomorphism of algebras in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$, $\mathcal{G}_k = \mathcal{G}_1^{\otimes k}$. Moreover, $\hat{f}$ is a morphism of algebras in $\mathrm{D}^+(X)$. It is therefore sufficient to show that $\hat{f} \circ \mathcal{G}_1 = \hat{f} \circ \pi$.

This is done in Propositions 13 and 14 that follow. $\square$

We recall that if V is a vector space over a field of characteristic 0, S_n acts on the right on $V^{\otimes n}$. If σ is a permutation on S_n, $\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$. This extends to an action of S_n on $\mathcal{A}^{\otimes n}$ for any complex $\mathcal{A}$ of $\mathcal{O}_X$ modules. The action descends to an action on $\mathcal{B}^{\otimes n}$ for any element $\mathcal{B}$ of $\mathrm{D}^+(X)$.

Observation 1. Note that if A is an associative algebra in $\mathrm{D}^+(X)$, with multiplication μ_A, the Lie Bracket $[,] : A \otimes A \to A$ is defined as $\mu_A \circ (\mathrm{id} - (12))$ where (12) is the swap applied to $A^{\otimes 2}$. Let $\tau_k \in S_n$ be the k-cycle $(n-k+1\ n-k+2\ , \ldots, n)$. Let e_n be the element $(\mathrm{id} - \tau_n) \circ \cdots \circ (\mathrm{id} - \tau_2) \in KS_n$. e_n is a quasi-idempotent, in the group ring KS_n of S_n. If $\mu_{n,A} : A^{\otimes n} \to A$ denotes the n-fold multiplication on A, then $\mu_{n,A} \circ e_n : A^{\otimes n} \to A$ is the n-fold Lie bracket on A. Note that this is only a morphism in $\mathrm{D}^+(X)$.

Denote by L_n the morphism in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$ from $L(\mathrm{D}^1_{\mathrm{poly}}(X))^{\otimes n}$ to $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ such that for sections $z_1, \ldots, z_n$ of $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ over an affine open subscheme U of X, $L_n(z_1 \otimes \cdots \otimes z_n) = [z_1, [z_2, [\ldots, [z_{n-1}, z_n]]]]$.

The following proposition is a direct consequence of the fact that f is a morphism of Lie algebras in $\mathrm{D}^+(X)$.

Proposition 13. *The following diagram commutes in $D^+(X)$.*

$$\begin{array}{ccc} L(D^1_{\mathrm{poly}}(X))^{\otimes n} & \xrightarrow{f^{\otimes n}} & A^{\otimes n} \\ \downarrow{\scriptstyle L_n} & & \downarrow{\scriptstyle \mu_{n,A} \circ e_n} \\ L(D^1_{\mathrm{poly}}(X)) & \xrightarrow{f} & A \end{array}$$

Proof. For $n = 2$, this is exactly the statement that f is a morphism of Lie algebras in $\mathrm{D}^+(X)$. For other n, it is proven by induction on n using the facts that $L_2 \circ (\mathrm{id} \otimes L_{n-1}) = L_n$ and $\mu_{n,A} \circ e_n = (\mu_A \circ e_2) \circ (\mathrm{id} \otimes [\mu_{n-1,A} \circ e_{n-1}])$. □

Proposition 14. *The following diagram also commutes in* $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$, *and hence in $D^+(X)$.*

$$\begin{array}{ccc} \mathcal{L}(L(D^1_{\mathrm{poly}}(X))) & \xrightarrow{B^1} & \mathcal{T}(L(D^1_{\mathrm{poly}}(X))) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \oplus_n \frac{1}{n} L_n} \\ L(D^1_{\mathrm{poly}}(X)) & \xrightarrow{\mathrm{id}} & L(D^1_{\mathrm{poly}}(X)) \end{array}$$

Proof. Let L denote $\oplus_n \frac{1}{n} L_n$. Denote the multiplication on $\mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$ by $m_{\mathcal{T}}$. The n-fold multiplication will be denoted by $m_{n,\mathcal{T}} : \mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))^{\otimes n} \to \mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$. All these are maps in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$.

Let e_n be as in Observation 1 prior to Proposition 13. It is immediate from Reutenauer [11, Theorem 8.16] that $\frac{1}{n} e_n$ is an idempotent. Moreover, it is a projection from $\mathcal{T}(W)$ to $\mathcal{L}(W)$ for any $W \in \mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$. Let $z_1, \ldots, z_n$

be sections of $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ over an open affine subscheme U of X. Then, $B^1([z_1,[z_2[\dots[z_{n-1},z_n]]]]) = e_n(z_1 \otimes \cdots \otimes z_n)$ by the definition of e_n. On the other hand, $\pi(B^1([z_1,[z_2[\dots[z_{n-1},z_n]]]])) = [z_1,[z_2[\dots[z_{n-1},z_n]]]]$ where the bracket on the right-hand side is that of $L(\mathrm{D}^1_{\mathrm{poly}}(X))$. To verify this proposition, it suffices to check that $L_n \circ \frac{1}{n}e_n = L_n$. Let $C : B^1(\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))) \to \mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$ be the left inverse of B^1. Now, $L_n(z_1 \otimes \cdots \otimes z_n) = \pi \circ C \circ e_n(z_1 \otimes \cdots \otimes z_n)$. Thus, $L_n \circ \frac{1}{n}e_n = \pi \circ C \circ e_n \circ \frac{1}{n}e_n = \pi \circ C \circ e_n = L_n$. This proves the desired proposition. $\square$

Proof (Final steps to proving Proposition 7).

Note that $f \circ \pi = \hat{f} \circ \pi$. Now combining Propositions 13 and 14, we see that $\hat{f} \circ \pi$ equals the following composition of morphisms in $\mathrm{D}^+(X)$.

$$\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \xrightarrow{B^1} \mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \xrightarrow{\oplus_n f^{\otimes n}} T(A) \xrightarrow{\oplus_n(\mu_{n,A}\circ e_n)} A.$$

Now we can see that if $h : \mathcal{B} \to \mathcal{C}$ is any morphism in $\mathrm{D}^+(X)$ and $\sigma \in S_n$ then $h^{\otimes n} \circ \sigma = \sigma \circ h^{\otimes n}$. This is verified by checking the corresponding fact at the level of complexes of $\mathcal{O}_X$ modules.

Let $\mu_{\bullet,A} = \oplus_n \mu_{n,A} : T(A) \to A$. It follows that $f \circ \pi$ is given by the following composition of morphisms in $\mathrm{D}^+(X)$.

$$\mathcal{L}(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \xrightarrow{B^1} \mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \xrightarrow{\oplus_n \frac{1}{n}e_n} \mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$$
$$\xrightarrow{\oplus_n f^{\otimes n}} T(A) \xrightarrow[\mu_{\bullet,A}]{} A.$$

But $\oplus_n \frac{1}{n}e_n$ is a projector from $T(W)$ to $L(W)$ for any complex of $\mathcal{O}_X$ modules W. This is immediate from Reutenauer [11, Theorem 8.16]. It follows that the composite of the first two maps in the previous composition is just $\mathcal{G}_1$. The map $\mu_{\bullet,A} \circ \oplus_n f^{\otimes n}$ is precisely $\hat{f}$. This proves that $\hat{f} \circ \mathcal{G}_1 = \hat{f} \circ \pi$, thereby finally proving Proposition 7 and therefore, Theorem 1. $\square$

Remark. Taking $\mathcal{I} \circ \lambda$ essentially amounts to taking brackets among elements of $L(\mathrm{D}^1_{\mathrm{poly}}(X))$. The fact that $\bar{f}$ when restricted to $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ is a Lie algebra homomorphism implies that $\bar{f} \circ \mathcal{I} \circ \lambda = \bar{f}$ in $\mathrm{D}^+(X)$ which is exactly what we want. This is the "hand waving" argument for Proposition 7 that is made rigorous by the proofs of Propositions 8–14.

8. The Chern Character as a Character of a Representation

If E is a vector bundle on X, $\mathcal{E}\mathrm{nd}(E)$ concentrated in degree 0 is an algebra in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$. The algebra structure on $\mathcal{E}\mathrm{nd}(E)$ is given by the composition map $\circ : \mathcal{E}\mathrm{nd}(E) \otimes \mathcal{E}\mathrm{nd}(E) \to \mathcal{E}\mathrm{nd}(E)$. We recall from Kapranov [6] that the Atiyah class of E endows it with the structure of a module over the Lie algebra $T_X[-1]$ in $\mathrm{D}^+(X)$. In other words, if we identify $\mathrm{Hom}_{\mathrm{D}^+(X)}(E \otimes T_X[-1], E)$

with $\mathrm{Hom}_{\mathrm{D}^+(X)}(T_X[-1], \mathcal{E}\mathrm{nd}(E))$, then $\alpha(E) : T_X[-1] \to \mathcal{E}\mathrm{nd}(E)$ is a morphism of Lie algebras in $\mathrm{D}^+(X)$.

The multiplication on $\mathcal{E}\mathrm{nd}(E)$ induces a k-fold multiplication $\circ_k : \mathcal{E}\mathrm{nd}(E)^{\otimes k} \to \mathcal{E}\mathrm{nd}(E)$. For an element $\alpha \in \mathrm{Hom}_{\mathrm{D}^+(X)}(T_X[-1], \mathcal{E}\mathrm{nd}(E))$, let $\alpha^{\circ k}$ denote the composite

$$T_X^{\otimes k}[-k] \xrightarrow{\alpha^{\otimes k}} \mathcal{E}\mathrm{nd}(E)^{\otimes k} \xrightarrow{\circ_k} \mathcal{E}\mathrm{nd}(E)$$

which is an element of $\mathrm{Hom}_{\mathrm{D}^+(X)}(T_X^{\otimes k}[-k], \mathcal{E}\mathrm{nd}(E))$. Denote $\alpha(E)^{\circ k}$ by $\tilde{\mathrm{t}_k}(E)$.

Let $p : \wedge^k T_X[-k] \to T_X^{\otimes k}[-k]$ be the morphism of complexes such that if $v_1, \dots, v_k$ are sections of T_X over an open affine subscheme U of X, then

$$p(v_1 \wedge \cdots \wedge v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}.$$

Then, $\alpha(E)^{\circ k} \circ p \in \mathrm{Hom}_{\mathrm{D}^+(X)}(\wedge^k T_X[-k], \mathcal{E}\mathrm{nd}(E))$. Denote $\alpha(E)^{\circ k} \circ p$ by $\tilde{\mathrm{ch}_k}(E)$.

Note that we have a map of $\mathcal{O}_X$-modules $\mathrm{tr} : \mathcal{E}\mathrm{nd}(E) \to \mathcal{O}_X$. This is a map in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$ if $\mathcal{E}\mathrm{nd}(E)$ and $\mathcal{O}_X$ are thought of as complexes concentrated in degree 0. Then, $\mathrm{tr} \circ \tilde{\mathrm{t}_k}(E)$ is an element in $\mathrm{Hom}_{\mathrm{D}^+(X)}(T_X^{\otimes k}[-k], \mathcal{O}_X)$. Denote it by $\mathrm{t}_k(E)$. Similarly, $\mathrm{tr} \circ \tilde{\mathrm{ch}_k}(E)$ is an element of $\mathrm{Hom}_{\mathrm{D}^+(X)}(\wedge^k T_X[-k], \mathcal{O}_X)$. Denote it by $\mathrm{ch}_k(E)$. The isomorphism of $\mathrm{Hom}_{\mathrm{D}^+(X)}(\wedge^k T_X[-k], \mathcal{O}_X)$ with $\mathrm{H}^k(X, \wedge^k \Omega)$ maps $\mathrm{ch}_k(E)$ to the degree k component of the Chern character of E.

Let $\mathrm{I}^1_{HKR} : T_X[-1] \to \mathrm{D}^\bullet_{\mathrm{poly}}(X)$ denote the composite of I_{HKR} with the inclusion of $T_X[-1]$ in $\oplus_k \wedge^k T_X[-k]$ as a direct summand. Since $\alpha(E) : T_X[-1] \to \mathcal{E}\mathrm{nd}(E)$ is a morphism of Lie algebras in $\mathrm{D}^+(X)$, Theorem 3 implies that there exists a morphism $\theta_E : \mathrm{D}^\bullet_{\mathrm{poly}}(X) \to \mathcal{E}\mathrm{nd}(E)$ of algebras in $\mathrm{D}^+(X)$ such that $\theta_E \circ \mathrm{I}^1_{HKR} = \alpha_E$. Let $\varphi_E := \mathrm{tr} \circ \theta_E : \mathrm{D}^\bullet_{\mathrm{poly}}(X) \to \mathcal{O}_X$.

Let $\mathrm{J} : \oplus_k T_X^{\otimes k}[-k] \to \mathrm{D}^\bullet_{\mathrm{poly}}(X)$ be the morphism of complexes such that if $v_1, \dots, v_k$ are sections of T_X over an open affine subscheme U of X, then

$$\mathrm{J}(v_1 \otimes \cdots \otimes v_k) = v_1 \otimes \cdots \otimes v_k \in \mathrm{D}^n_{\mathrm{poly}}(U).$$

We now have the following theorem.

Theorem 4. (i) $\tilde{t_k(E)} = \theta_E \circ J$,

(ii) $\tilde{\mathrm{ch}_k(E)} = \theta_E \circ I_{HKR}$,

(iii) $\oplus_n t_n(E) = \varphi_E \circ J$,

(iv) $\mathrm{ch}(E) = \varphi_E \circ I_{HKR}$.

Proof. Let J^n denote the composite of J with the inclusion of $T_X^{\otimes n}[-n]$ in $\oplus_k T_X^{\otimes k}[-k]$ as a direct summand. Then, by the definition of J, $\mathrm{J}^n = {\mathrm{I}^1_{HKR}}^{\otimes n}$. Further, θ_E is a morphism of algebras in $\mathrm{D}^+(X)$ and $\theta_E \circ \mathrm{I}^1_{HKR} = \alpha(E)$. It follows that the following diagram commutes in $\mathrm{D}^+(X)$.

$$\begin{array}{ccc} T_X[-1]^{\otimes n} & \xrightarrow{\mathrm{J}^n} & \mathrm{D}^\bullet_{\mathrm{poly}}(X) \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \theta_E} \\ T_X[-1])^{\otimes n} & \xrightarrow{\alpha(E)^{\circ n}} & \mathcal{E}\mathrm{nd}(E) \end{array}$$

Also, the following diagram commutes in $\mathrm{D}^+(X)$ by the definitions of I_{HKR} and J.

$$\begin{array}{ccc} \wedge^n T_X[-n] & \xrightarrow{p} & T_X[-1]^{\otimes n} \\ {\scriptstyle \mathrm{I}_{HKR}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{J}_n} \\ \mathrm{D}^\bullet_{\mathrm{poly}}(X) & \xleftarrow{\mathrm{id}} & \mathrm{D}^\bullet_{\mathrm{poly}}(X)) \end{array}$$

It follows that $\theta_E \circ \mathrm{J} = \oplus_k \alpha(E)^{\circ k} \circ \mathrm{id}$.

Thus, $\varphi_E \circ \mathrm{J} = \oplus_k \mathrm{tr} \circ \alpha(E)^{\circ k}) = \oplus_k \mathrm{t}_k(E)$.

This proves (i) and (iii). For (ii) and (iv), we use the fact $\mathrm{J} \circ p = \mathrm{I}_{HKR}$ to see that $\theta_E \circ \mathrm{J} \circ p = \oplus_k \alpha(E)^{\circ k} \circ \mathrm{id} \circ p$. Composing both sides of this with the trace map from $\mathrm{End}(E)$ to $\mathcal{O}_X$, we see that $\varphi_E \circ \mathrm{I}_{HKR} = \mathrm{tr} \circ \oplus_k \alpha(E)^{\circ k} \circ p$. This proves (ii) and (iv). □

In the classical situation, if $\mathfrak{g}$ is an ordinary Lie algebra over a field of characteristic 0 and E is a finite-dimensional representation of $\mathfrak{g}$, we have a Lie algebra homomorphism $\theta_E : \mathfrak{g} \to \mathrm{End}(E)$. This induces a map $U\mathfrak{g} \to \mathrm{End}(E)$ of algebras where $U\mathfrak{g}$ is the universal enveloping algebra of E. One has the trace $\mathrm{End} \to K$. One can therefore compose these to get a map $\varphi_E : U\mathfrak{g} \to K$. This is the character of the representation E of $\mathfrak{g}$. The analogy with the Chern character is now clear. By Kapranov [6] any vector bundle E is a representation of the Lie algebra $T_X[-1]$ in $\mathrm{D}^+(X)$. By Theorem 4(iv), the Chern character of the vector bundle is the character in the Representation theoretic sense of the representation E of $T_X[-1]$.

Theorem 4 also enables us to prove some properties of the big Chern classes shown by Ramadoss [8] in a more general framework. In this section, we shall reprove the fact that the big Chern classes commute with the Adams operations. The new proof would make the parallel of this fact with the Representation theoretic identity $\chi_{\psi^p V}(g) = \chi_V(g^p)$ transparent.

For this, we need a digression on Adams operations in commutative Hopf algebras. Let $\mathcal{H}$ be a commutative Hopf algebra, with multiplication μ and comultiplication Δ. Let μ_k and Δ^k denote the k fold multiplication and k fold comultiplication respectively. Then, the maps $\psi^k : \mathcal{H} \to \mathcal{H}$ are ring homomorphisms. Moreover, $\psi^p \circ \psi^q = \psi^{pq}$. These maps can therefore be thought of as Adams operations. If $\mathcal{H}$ is noncommutative but is cocommutative and primitively generated, then these maps are not ring homomorphisms though they satisfy $\psi^p \circ \psi^q = \psi^{pq}$.

Coming back to the classical picture, recall that the character of a representation E of a Lie algebra $\mathfrak{g}$ is a K linear map $\varphi_E : U(\mathfrak{g}) \to K$. Let us restrict φ_E to elements of $U\mathfrak{g}$ that are of the form $\exp(tv)$, $t \in \mathbb{R}$, $v \in \mathfrak{g}$, v fixed. This yields a character (in the usual "character of a representation of a group" sense) of the representation E of the one parameter group $\exp(tv) \subset U\mathfrak{g}$. In this case we denote the character φ_E restricted to the one parameter group by χ_E to keep notation more standard.

From the fact that $\chi_{\psi^p E}(g) = \chi_E(g^p)$ it follows that $\chi_{\psi^p E}\exp(tv) = \chi_E\exp(ptv)$. If $*$ denotes the multiplication in $U\mathfrak{g}$ and $v^k := v * \cdots * v\,(k \text{ times})$, then $\chi_{\psi^p E}\sum_k \frac{1}{k!}t^k v^k = \sum_k \frac{1}{k!}t^k p^k v^k$. Note that $U\mathfrak{g}$ is a co-commutative Hopf algebra with $\Delta^p(v) = v \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes v \in U\mathfrak{g}^{\otimes p}$ for $v \in \mathrm{I}_{PBW}(\mathfrak{g})$. Since Δ^p is an algebra homomorphism, it follows that

$$\Delta^p(v^k) = \sum_{\{(k_1,\ldots,k_p) | k_i \geq 0\ \forall\ i \text{and} \sum k_i = k\}} v^{k_1} \otimes \cdots \otimes v^{k_p} \in U\mathfrak{g}^{\otimes p}.$$

Thus, $\mu_p \circ \Delta^p(v^k) = p^k v^k$. Since $\{\mathrm{I}_{PBW}(v^k) | v \in \mathfrak{g}, k \geq 0\}$ spans $U\mathfrak{g}$, the following diagram commutes.

$$\begin{array}{ccc} U\mathfrak{g} & \xrightarrow{\psi^p} & U\mathfrak{g} \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \varphi_E} \\ U\mathfrak{g} & \xrightarrow{\varphi_{\psi^p E}} & K \end{array}$$

The commuting of this diagram is equivalent to the fact that $\chi_{\psi^p E}(g) = \chi_E(g^p)$. We will now prove the same in $\mathrm{D}^+(X)$ for the Lie algebra $T_X[-1]$. Recall, from Proposition 2 that $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$ is a Hopf algebra in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$. It is therefore a Hopf algebra in $\mathrm{D}^+(X)$ with all operations induced by the corresponding operations in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$. It is co-commutative but non-commutative.

Proposition 15. *The following diagram commutes in $D^+(X)$.*

$$\begin{array}{ccc} D^\bullet_{\mathrm{poly}}(X) & \xrightarrow{\psi^p} & D^\bullet_{\mathrm{poly}}(X) \\ {\scriptstyle \varphi_{\psi^p E}}\downarrow & & \downarrow{\scriptstyle \varphi_E} \\ \mathcal{O}_X & \xrightarrow{\mathrm{id}} & \mathcal{O}_X \end{array}$$

This result explains why the big Chern classes commute with Adams operations [8] without recourse to hands on computation as was done in [8].

Proof. Let I^k_{HKR} denote the composite of I_{HKR} with the inclusion of $\wedge^k T_X[-k]$ into $\oplus_i \wedge^i T_X[-i]$ as a direct summand. Let U be an affine open subscheme of X and let $v_1, \ldots, v_k$ be sections of $T_X|_U$. If $(k_1, \ldots, k_p)$ is a p-tuple of nonnegative integers such that $\sum_i k_i = k$, define a $(k_1, \ldots, k_p)$-multi-shuffle to be a permutation

σ of $\{1,\ldots,k\}$ such that $\sigma(1) < \cdots < \sigma(k_1), \sigma(k_1+1) < \cdots < \sigma(k_1+k_2), \ldots, \sigma(k_1+\cdots+k_{p-1}+1) < \cdots < \sigma(k)$.

Then

$$\Delta^p(v_1 \otimes \cdots \otimes v_k) = \sum_{\{(k_1,\ldots,k_p) \mid \sum_i k_i = k\}} \sum_{\sigma \text{ a } (k_1,\ldots,k_p)\text{-shuffle}} \operatorname{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k_1)} \bigotimes \cdots \bigotimes v_{\sigma(k_1+\cdots+k_{p-1}+1)} \otimes \cdots \otimes v_{\sigma(k)}.$$

We follow the convention that if $k_i = 0$, then $v_{\sigma(k_1+\cdots+k_{i-1}+1)} \otimes \cdots \otimes v_{\sigma(k_1+\cdots+k_i)} = 1$.

It follows that if m_p denotes the p-fold multiplication on $\mathrm{D}^\bullet_{\text{poly}}(X)$, then

$$m_p \circ \Delta^p(v_1 \otimes \cdots \otimes v_k) = \sum_{\{(k_1,\ldots,k_p) \mid \sum_i k_i = k\}} \sum_{\sigma \text{ a } (k_1,\ldots,k_p)\text{-shuffle}} \operatorname{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}.$$

Thus,

$$\begin{aligned} m_p \circ \Delta^p &\left(\sum_{\tau \in S_k} \operatorname{sgn}(\tau) v_{\tau(1)} \otimes \cdots \otimes v_{\tau(k)} \right) \\ &= \sum_{\tau \in S_k} \sum_{\{(k_1,\ldots,k_p) \mid \sum_i k_i = k\}} \sum_{\sigma \text{ a } (k_1,\ldots,k_p)\text{-shuffle}} \operatorname{sgn}(\tau)\operatorname{sgn}(\sigma) v_{\sigma(\tau(1))} \otimes \cdots \otimes v_{\sigma(\tau(k))}. \end{aligned}$$

Since there are p^k $(k_1,\ldots,k_p)$-shuffles such that $\sum_i k_i = k$,

$$\begin{aligned} &\sum_{\tau \in S_k} \sum_{\{(k_1,\ldots,k_p) \mid \sum_i k_i = k\}} \sum_{\sigma \text{ a } (k_1,\ldots,k_p)\text{-shuffle}} \operatorname{sgn}(\tau)\operatorname{sgn}(\sigma) v_{\sigma(\tau(1))} \otimes \cdots \otimes v_{\sigma(\tau(k))} \\ &\quad = p^k \sum_{\tau \in S_k} \operatorname{sgn}(\tau) v_{\tau(1)} \otimes \cdots \otimes v_{\tau(k)}. \end{aligned}$$

From the fact that $\mathrm{I}_{HKR}(v_1 \wedge \cdots \wedge v_k) = \frac{1}{k!} \sum_{\tau \in S_k} \operatorname{sgn}(\tau) v_{\tau(1)} \otimes \cdots \otimes v_{\tau(k)}$ and the fact that $m_p \circ \Delta^p = \psi^p$, it follows that $\psi^p \circ \mathrm{I}^k_{HKR} = p^k \mathrm{I}^k_{HKR}$.

Now, $\varphi_E \circ \mathrm{I}^k_{HKR} = \mathrm{ch}_k(E)$ by Theorem 4(iv). Thus

$$\varphi_E \circ \psi^p \circ \mathrm{I}^k_{HKR} = p^k \mathrm{ch}_k(E) = \mathrm{ch}_k(\psi^p E) = \varphi_{\psi^p E} \circ \mathrm{I}^k_{HKR}.$$

This together with the facts $\oplus_k \mathrm{I}^k_{HKR} = \mathrm{I}_{HKR}$ and I_{HKR} is a quasi-isomorphism and therefore, an isomorphism in $\mathrm{D}^+(X)$ prove the desired proposition. $\square$

The following corollary is now immediate.

Corollary 6. *The following diagram commutes.*

$$\begin{array}{ccc} \oplus_k T_X^{\otimes k}[-k] & \xrightarrow{\psi^p} & T_X^{\otimes k}[-k] \\ \Big\downarrow \oplus_k t_k(\psi^p E) & & \oplus_k t_k(E) \Big\downarrow \\ \mathcal{O}_X & \xrightarrow{\mathrm{id}} & \mathcal{O}_X \end{array}$$

This is the statement that the big Chern classes commute with Adams operations.

9. A Formula for the Big Chern Classes

This section proves a formula for the big Chern classes in terms of the components of the Chern character for vector bundles over an arbitrary smooth scheme over a field of characteristic 0. The existence of such a formula was proven in my thesis [8] for smooth projective varieties using the existence of an ample line bundle together with combinatorial arguments. The method used here is very different from that of [8]. It is also more general, and works for vector bundles over smooth complex manifolds as well.

9.1. $\mu \circ \frac{\omega}{1-e^{-\omega}}$ *as an element in the PROP* $\mathrm{END}_{T[-1]}$

9.1.1. *A proposition*

We note that there is a PROP $\mathrm{END}_{T[-1]}$ where $\mathrm{END}_{T[-1]}(n,m) := \mathrm{Hom}_{\mathrm{D}^+(X)}(T_X^{\otimes n}[-n], T_X^{\otimes m}[-m])$. Let $\varphi \in \mathrm{END}_{T[-1]}(n,m)$ and $\zeta \in \mathrm{END}_{T[-1]}(n,p)$. Clearly, we have a composition $\odot : \mathrm{END}_{T[-1]}(m,p) \otimes \mathrm{END}_{T[-1]}(n,m)$ which takes $\zeta \otimes \varphi$ to $\zeta \circ \varphi$. A permutation σ of S_n gives rise to two elements of $\mathrm{END}_{T[-1]}(n,n)$: $l(\sigma)$ is induced at the level of complexes by the map $v_1 \otimes \cdots \otimes v_n \rightsquigarrow \mathrm{sgn}(\sigma) v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$ and $r(\sigma)$ is induced by $v_1 \otimes \cdots \otimes v_n \rightsquigarrow \mathrm{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$ for sections $v_1, \ldots, v_n$ of T_X over an open affine subscheme U of X. If $\sigma \in S_n$ and $\tau \in S_m$ then $\sigma(\varphi) := \varphi \circ l(\sigma)$ and $(\varphi)\tau := r(\tau) \circ \varphi$.

We also have a juxtaposition map

$$\times : \mathrm{END}_{T[-1]}(n,m) \otimes \mathrm{END}_{T[-1]}(n',m') \to \mathrm{END}_{T[-1]}(n+n', m+m').$$

If $\eta \in \mathrm{END}_{T[-1]}(n+n', m+m')$, then $\varphi * \eta := \varphi \otimes \eta$.

Let ΠLIE denote the PROP generated by the Lie operad.

We recall from [6, Theorem 3.5.1] that there is a morphism of PROPs

$$\Upsilon : \Pi\mathrm{LIE} \to \mathrm{END}_{T[-1]}$$

so that

$$[x1, x2] \in \Pi\mathrm{LIE}(2,1) \to \alpha_{T_X} \in \mathrm{END}_{T[-1]}(2,1) := \mathrm{Hom}_{\mathrm{D}^+(X)}(T_X[-1]^{\otimes 2}, T_X[-1]).$$

We note that Theorem 2 tells us that the bracket $[,]$ on $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ is really $\alpha_{T_X} \in \mathrm{END}_{T[-1]}(2,1)$ after identifying $L(\mathrm{D}^1_{\mathrm{poly}}(X))$ with $T_X[-1]$ in $\mathrm{D}^+(X)$ via the map β described in Corollary 3, Sec. 4. Let ω be as in Theorem 1. We now want to look at ω as an element of $\oplus_{1\leq m\leq n}\mathrm{END}_{T[-1]}(n,m)$.

Let $\{z_i\}, y, \{d_i\}, d$ be as in the proof of Theorem 1. Let $\hat{\omega} : \mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \otimes L(\mathrm{D}^1_{\mathrm{poly}}(X)) \to \mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \otimes L(\mathrm{D}^1_{\mathrm{poly}}(X))$ be the map

$$z_1 \otimes \cdots \otimes z_k \otimes y \rightsquigarrow \sum_i (-1)^{d_i(d_{i+1}+\cdots+d_k)} z_1 \otimes \cdots \widehat{z_i} \cdots \otimes z_k \otimes [z_i, y].$$

Let $\hat{\mu}(z_1 \otimes \cdots \otimes z_k \otimes y) = \frac{1}{k}\sum_i (-1)^{d(d_{i+1}+\cdots+d_k)} z_1 \otimes \cdots \otimes z_i \otimes y \otimes z_{i+1} \otimes \cdots \otimes z_k$.

Recall the definition of the map $J(L(\mathrm{D}^1_{\mathrm{poly}}(X))) : \mathrm{Sym}^\bullet(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \to \mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \otimes L(\mathrm{D}^1_{\mathrm{poly}}(X))$ from Sec. 7.1(4). Unlike Sec. 7, we will denote $J(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$ by B to avoid confusing it with $\mathrm{J} : \oplus_n T_X^{\otimes n}[-n] \to \mathrm{D}^\bullet_{\mathrm{poly}}(X)$.

Proposition 16. *The following diagrams commute in* $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$, *and hence in $D^+(X)$.*

$$\begin{array}{ccc}
\mathcal{T}(L(D^1_{\mathrm{poly}}(X))) \otimes L(D^1_{\mathrm{poly}}(X)) & \xrightarrow{\hat{\omega}} & \mathcal{T}(L(D^1_{\mathrm{poly}}(X))) \otimes L(D^1_{\mathrm{poly}}(X)) \\
\uparrow{\scriptstyle B\otimes \mathrm{id}} & & \uparrow{\scriptstyle B\otimes \mathrm{id}} \\
\mathrm{Sym}^\bullet(L(D^1_{\mathrm{poly}}(X))) \otimes L(D^1_{\mathrm{poly}}(X)) & \xrightarrow{\omega} & \mathrm{Sym}^\bullet(L(D^1_{\mathrm{poly}}(X))) \otimes L(D^1_{\mathrm{poly}}(X)).
\end{array}$$

$$\begin{array}{ccc}
\mathcal{T}(L(D^1_{\mathrm{poly}}(X))) \otimes L(D^1_{\mathrm{poly}}(X)) & \xrightarrow{\hat{\mu}} & \mathcal{T}(L(D^1_{\mathrm{poly}}(X))) \\
\uparrow{\scriptstyle B\otimes \mathrm{id}} & & {\scriptstyle B}\uparrow \\
\mathrm{Sym}^\bullet(L(D^1_{\mathrm{poly}}(X))) \otimes L(D^1_{\mathrm{poly}}(X)) & \xrightarrow{\mu} & \mathrm{Sym}^\bullet(L(D^1_{\mathrm{poly}}(X))).
\end{array}$$

Proof. Since $z_1, \ldots, z_k = z_1, \ldots, \hat{z_j}, \ldots, z_k z_j$ up to a sign, the coefficient of $[z_k, y]$ in $\hat{\omega} \circ (B \otimes \mathrm{id})(z_1, \ldots, z_k)$ will be equal to the coefficient of $[z_j, y]$ in $\hat{\omega} \circ (B \otimes \mathrm{id})$ up to the same sign. The same observation holds with $(B \otimes \mathrm{id}) \circ \omega(z_1, \ldots, z_k)$ instead of $\hat{\omega} \circ (B \otimes \mathrm{id})$. We thus, need to compare the coefficient of $[z_k, y]$ in $\hat{\omega} \circ (B \otimes \mathrm{id})(z_1, \ldots, z_k)$ and $(B \otimes \mathrm{id}) \circ \omega(z_1, \ldots, z_k)$.

In the second map, the coefficient of $[z_k, y]$ is simply $B(z_1, \ldots, z_{k-1})$. For a permutation $\sigma \in S_k$, let $s(\sigma)$ be the sign such that $z_1, \ldots, z_k = s(\sigma) z_{\sigma(1)}, \ldots, z_{\sigma(k)}$. Then

$$B(z_1, \ldots, z_k) = \frac{1}{k!} \sum_{\sigma \in S_k} s(\sigma) z_{\sigma(1)} \otimes \cdots \otimes z_{\sigma(k)}.$$

Let $\tau \in S_{k-1}$ let τ_i be the permutation in S_k such that $\tau_i(j) = \tau(j)$ for $j \leq i-1$, $\tau_i(i) = k$ and $\tau_i(j) = \tau(j-1)$ for $j > i$. Then $s(\tau_i) = (-1)^{d_k(d_{\tau(i+1)}+\cdots+d_{\tau(k-1)})} s_\tau$.

Thus, the coefficient of $[z_k, y]$ in $\hat{\omega} \circ (B \otimes \mathrm{id})$ is

$$\begin{aligned}
&\frac{1}{k!} \sum_{\tau \in S_{k-1}} \sum_i (-1)^{d_k(d_{\tau(i+1)}+\cdots+d_{\tau(k-1)})} s(\tau_i) z_{\tau(1)} \otimes \cdots \otimes z_{\tau(k-1)} \\
&\quad = \frac{1}{k!} \sum_{\tau \in S_{k-1}} \sum_i s(\tau) z_{\tau(1)} \otimes \cdots \otimes z_{\tau(k-1)} \\
&\quad = \frac{1}{(k-1)!} \sum_{\tau \in S_{k-1}} s(\tau) z_{\tau(1)} \otimes \cdots \otimes z_{\tau(k-1)}.
\end{aligned}$$

This is just $B(z_1, \ldots, z_{k-1})$.

This shows that the first square commutes. The commuting of the second square is checked by a similar easier calculation. □

9.1.2. $\hat{\omega}$ *as an element in* $END_{T[-1]}$

Let $\sigma_{k,n}$ be the permutation of $\{1, \ldots, n\}$ such that $\sigma_{k,n}(n) = k$ and $\sigma_{k,n}(j) = j$ if $j < k$ and $\sigma_{k,n}(j) = j + 1$ otherwise. The map $\sigma \rightsquigarrow l(\sigma)$ gives us a map from KS_n to $\Pi(n, n)$ for any PROP Π.

The map $\hat{\omega} : \mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \otimes L(\mathrm{D}^1_{\mathrm{poly}}(X)) \to \mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \otimes L(\mathrm{D}^1_{\mathrm{poly}}(X))$ is clearly the action of the following element of $\oplus_{n,m} \mathrm{END}_{T[-1]}(n, m)$

$$\hat{\omega} : \left(\sum_{r \geq 1} \mathrm{id} \times^{r-1} \times \Upsilon([x_1, x_2]) \right) \circ \sum_n \sum_{k=1}^{k=n} l(\sigma^{-1}_{k,n}) \in \oplus_{1 \leq m < n} \mathrm{END}_{T[-1]}(n, m).$$

Note that $\hat{\omega} = \Upsilon(\zeta)$ where $\zeta \in \oplus_{1 \leq m < n} \Pi\mathrm{LIE}(n, m)$ is given by

$$\left(\sum_{r \geq 1} \mathrm{id} \times^{r-1} \times [x_1, x_2] \right) \circ \sum_n \sum_{k=1}^{k=n} l(\sigma^{-1}_{k,n}).$$

By convention, if $p \neq n$ then

$$\circ : \mathrm{END}_{T[-1]}(p, q) \otimes \mathrm{END}_{T[-1]}(m, n) \to \oplus \mathrm{END}_{T[-1]}(m, n) = 0.$$

With this convention, we can think of $\zeta^{\circ k} \in \oplus_{1 \leq m < n} \Pi\mathrm{LIE}(n, m)$ and $\hat{\omega}^k \in \oplus_{1 \leq m < n} \mathrm{END}_{T[-1]}(n, m)$. Denote $\zeta^{\circ k}$ by ζ^k. Clearly, $\hat{\omega}^k = \Upsilon(\zeta^k)$. This enables us to look at any power series in $\hat{\omega}$ as an element of $\oplus_{1 \leq m < n} \mathrm{END}_{T[-1]}(n, m)$. Similarly, any power series in ζ can be seen as an element in $\oplus_{1 \leq m < n} \Pi\mathrm{LIE}(n, m)$.

We also note that $\hat{\mu} = \oplus_n \frac{1}{n} \sum_k l(\sigma_{k,n}) \in \oplus_m \mathrm{END}_{T[-1]}(m, m)$.

Recalling once again that $L(\mathrm{D}^1_{\mathrm{poly}}(X)) = T_X[-1]$ as Lie algebras in $\mathrm{D}^+(X)$, we see that the map $\hat{\mu} \circ \frac{\hat{\omega}}{1 - e^{-\hat{\omega}}} : \mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X))) \otimes L(\mathrm{D}^1_{\mathrm{poly}}(X)) \to \mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$ is just the action of the element

$$\Theta := \hat{\mu} \circ \frac{\hat{\omega}}{1 - e^{-\hat{\omega}}} \in \oplus_{1 \leq m < n} \mathrm{END}_{T[-1]}(n, m).$$

Note that $\Theta = \Upsilon(\hat{\Theta})$ where

$$\hat{\Theta} := \hat{\mu} \circ \frac{\zeta}{1 - e^{-\zeta}} \in \oplus_{1 \leq m < n} \Pi\mathrm{LIE}(n, m).$$

Let $p : \oplus_i \wedge^i T_X[-i] \to T(T_X[-1])$ be as in Sec. 8. Since $L(\mathrm{D}^1_{\mathrm{poly}}(X)) = T_X[-1]$ in $\mathrm{D}^+(X)$, we get the following corollary of Proposition 16.

Corollary 7. *The following diagram commutes in $D^+(X)$.*

$$\begin{tikzcd}
T(T_X[-1]) \otimes T_X[-1] \arrow[r, "\Theta"] & T(T_X[-1]) \\
\oplus_i \wedge^i T_X[-i] \otimes T_X[-1] \arrow[u, "p\otimes \mathrm{id}"] \arrow[r, "\mu\circ\frac{\bar{\omega}}{1-e^{-\bar{\omega}}}"] & \oplus_i \wedge^i T_X[-i]. \arrow[u, "p"']
\end{tikzcd}$$

9.2. *A return to Theorem 1*

Corollary 1 can be rephrased to say that the following diagram commutes in $\mathrm{D}^+(X)$.

$$\begin{tikzcd}
\mathrm{D}^\bullet_{\mathrm{poly}}(X) \otimes \mathrm{D}^\bullet_{\mathrm{poly}}(X) \arrow[r, "m"] & \mathrm{D}^\bullet_{\mathrm{poly}}(X) \\
\oplus_i \wedge^i T_X[-i] \otimes T_X[-1] \arrow[u, "\mathrm{I}_{HKR}\otimes \mathrm{I}_{HKR}"] \arrow[r, "\mu\circ\frac{\bar{\omega}}{1-e^{-\bar{\omega}}}"] & \oplus_i \wedge^i T_X[-i]. \arrow[u, "\mathrm{I}_{HKR}"']
\end{tikzcd}$$

Let m_k denote the k-fold multiplication on $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$. It follows from the above commutative diagram that the following diagram commutes for all k.

$$\begin{tikzcd}
\mathrm{D}^\bullet_{\mathrm{poly}}(X) \otimes \mathrm{D}^\bullet_{\mathrm{poly}}(X) \arrow[r, "m"] & \mathrm{D}^\bullet_{\mathrm{poly}}(X) \\
\oplus_i \wedge^i T_X[-i] \otimes T_X^{\otimes k}[-k] \arrow[u, "\mathrm{I}_{HKR}\otimes (m_k \circ \mathrm{I}^{\otimes k}_{HKR})"] \arrow[r, "(\mu\circ\frac{\bar{\omega}}{1-e^{-\bar{\omega}}}\otimes \mathrm{id}^{k-1})\circ\cdots\circ(\mu\circ\frac{\bar{\omega}}{1-e^{-\bar{\omega}}})"] & \oplus_i \wedge^i T_X[-i]. \arrow[u, "\mathrm{I}_{HKR}"']
\end{tikzcd}$$

The following diagram commutes in $\mathrm{D}^+(X)$ since it does so in $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$.

$$\begin{tikzcd}
\mathrm{D}^\bullet_{\mathrm{poly}}(X) \arrow[r, "1\otimes(\mu\circ \mathrm{I}^{\otimes k}_{HKR})"] & \mathrm{D}^\bullet_{\mathrm{poly}}(X) \otimes \mathrm{D}^\bullet_{\mathrm{poly}}(X) \\
T_X^{\otimes k}[-k] \arrow[u, "m_k\circ \mathrm{I}^{\otimes k}_{HKR}"] \arrow[r, "1\otimes \mathrm{id}"] & \oplus_i \wedge^i T_X[-i] \otimes T_X^{\otimes k}[-k]. \arrow[u, "\mathrm{I}_{HKR}\otimes (m_k\circ \mathrm{I}^{\otimes k}_{HKR})"']
\end{tikzcd}$$

Combining this diagram with the previous one we get the following corollary.

Corollary 8. *The following diagram commutes in $D^+(X)$ for all k.*

$$\begin{tikzcd}
D^\bullet_{\mathrm{poly}}(X) \arrow[r, "\mathrm{id}"] & D^\bullet_{\mathrm{poly}}(X) \\
T_X^{\otimes k}[-k] \arrow[u, "(m_k\circ I^{\otimes k}_{HKR})"] \arrow[r, "(\mu\circ\frac{\bar{\omega}}{1-e^{-\bar{\omega}}}\otimes \mathrm{id}^{k-1})\circ\cdots\circ(\mu\circ\frac{\bar{\omega}}{1-e^{-\bar{\omega}}})\circ(1\otimes \mathrm{id})"] & \oplus_i \wedge^i T_X[-i]. \arrow[u, "I_{HKR}"']
\end{tikzcd}$$

By Corollary 7, the following diagram commutes in $\mathrm{D}^+(X)$.

$$\begin{array}{ccc} T(T_X[-1]) \otimes T_X^{\otimes k}[-k] & \xrightarrow{(\Theta\times \mathrm{id}^{\times k-1})\circ\cdots\circ\Theta} & T(T_X[-1]) \\ \uparrow{\scriptstyle p\otimes \mathrm{id}} & & \uparrow{\scriptstyle p} \\ \oplus_i \wedge^i T_X[-i] \otimes T_X^{\otimes k}[-k] & \xrightarrow{(\mu\circ\frac{\bar{\omega}}{1-e^{-\bar{\omega}}}\otimes \mathrm{id}^{k-1})\circ\cdots\circ(\mu\circ\frac{\bar{\omega}}{1-e^{-\bar{\omega}}})\circ(1\otimes \mathrm{id})} & \oplus_i \wedge^i T_X[-i]. \end{array}$$

Further, the following diagram commutes in $\mathrm{D}^+(X)$.

$$\begin{array}{ccc} T_X^{\otimes k}[-k] & \xrightarrow{1\otimes \mathrm{id}} & T(T_X[-1]) \otimes T_X^{\otimes k}[-k] \\ \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle p\otimes \mathrm{id}} \\ T_X^{\otimes k}[-k] & \xrightarrow{1\otimes \mathrm{id}} & \oplus_i \wedge^i T_X[-i] \otimes T_X^{\otimes k}[-k]. \end{array}$$

Note that the upper morphism in the above diagram just expresses $T_X^{\otimes k}[-k]$ as a summand of $T(T_X[-1]) \otimes T_X^{\otimes k}[-k]$. We can therefore conclude from the above diagram and the one before that

Corollary 9. *The following diagram commutes in $D^+(X)$.*

$$\begin{array}{ccc} T_X^{\otimes k}[-k] & \xrightarrow{(\Theta\times \mathrm{id}^{\times k-1})\circ\cdots\circ\Theta} & T(T_X[-1]) \\ {\scriptstyle \mathrm{id}}\uparrow & & {\scriptstyle \gamma}\uparrow \\ T_X^{\otimes k}[-k] & \xrightarrow{(\mu\circ\frac{\bar{\omega}}{1-e^{-\bar{\omega}}}\otimes \mathrm{id}^{k-1})\circ\cdots\circ(\mu\circ\frac{\bar{\omega}}{1-e^{-\bar{\omega}}})\circ(1\otimes \mathrm{id})} & \oplus_i \wedge^i T_X[-i]. \end{array}$$

Let

$$\Psi_k := (\Theta \times \mathrm{id}^{\times k-1}) \circ \cdots \circ \Theta \in \oplus_{m\leq n}\mathrm{END}_{T[-1]}(n, m).$$

Note that $\Psi_k = \Upsilon(\hat{\Psi_k})$ where

$$\hat{\Psi_k} := (\hat{\Theta} \times \mathrm{id}^{\times k-1}) \circ \cdots \circ \hat{\Theta} \in \oplus_{m\leq n}\Pi\mathrm{LIE}(n, m).$$

Let Ψ_{kl} denote the component of Ψ in $\mathrm{END}_{T[-1]}(k, l)$. Let $\pi : T(T_X[-1]) \to \oplus_i \wedge^i T_X[-i]$ be the standard projection.

Theorem 5. *Let X be a smooth scheme over a field of characteristic 0. Let E be a vector bundle on X. Then,*

(i) $\tilde{t_k}(E) = \tilde{\mathrm{ch}_k}(E) \circ \pi + \sum_{l<k} \tilde{\mathrm{ch}_l}(E) \circ \pi \circ \Psi_{kl}$,
(ii) $t_k(E) = \mathrm{ch}_k(E) \circ \pi + \sum_{l<k} \mathrm{ch}_l(E) \circ \pi \circ \Psi_{kl}$.

Proof. Note that $\pi \circ p = \mathrm{id}$. By this observation and Corollary 9,

$$p \circ \Psi = \left(\mu \circ \frac{\bar{\omega}}{1 - e^{-\bar{\omega}}} \otimes \mathrm{id}^{k-1}\right) \circ \cdots \circ \left(\mu \circ \frac{\bar{\omega}}{1 - e^{-\bar{\omega}}}\right) \circ (1 \otimes \mathrm{id}) : T_X^{\otimes k}[-k] \to \oplus_i \wedge^i T_X[-i]$$

in $\mathrm{D}^+(X)$.

Let $\mathrm{J} : \oplus_k T_X^{\otimes k}[-k] \to \mathrm{D}^{\bullet}_{\mathrm{poly}}(X)$ and J^k be as in Theorem 4. We note that $m_k \circ \mathrm{I}_{HKR}^{\otimes k} = \mathrm{J}^k$. We also recall that for any vector bundle E, we have a morphism $\theta_E : \mathrm{D}^{\bullet}_{\mathrm{poly}}(X) \to \mathcal{E}\mathrm{nd}(E)$ in $\mathrm{D}^+(X)$ so that $\theta_E \circ \mathrm{J}^k = \tilde{\mathrm{t}_k}(E)$ and $\theta_E \circ \mathrm{I}_{HKR} = \tilde{\mathrm{ch}}(E)$ by Theorem 4. By these observations and Corollary 8,

$$\tilde{\mathrm{t}_k}(E) = \tilde{\mathrm{ch}}(E) \circ \pi \circ \Psi_k.$$

Note that as $\Psi_k \in \oplus_{m \le n} \mathrm{END}_{T[-1]}(n,m)$, the contribution of $\mathrm{ch}_l(E) \circ \pi \circ \Psi_{kl}$ to $\mathrm{t}_k(E)$ vanishes when $l > k$. It only remains to show that $\pi \circ \Psi_{kk} = \pi$.

Note that $\Theta = \hat{\mu} \circ \sum_k c_k \hat{\omega}^k$ for some constants c_k. Also note that $\hat{\omega} \in \oplus_{m<n} \mathrm{END}_{T[-1]}(n,m)$. From the observation that composing an element of $\oplus_{m \le n} \mathrm{END}_{T[-1]}(n,m)$ with one of $\oplus_{m<n} \mathrm{END}_{T[-1]}(n,m)$ gives an element of $\oplus_{m<n} \mathrm{END}_{T[-1]}(n,m)$, it follows that the only component of Θ in $\oplus_m \mathrm{END}_{T[-1]}(m,m)$ is $\hat{\mu}$. It follows that the component of Ψ_k in $\oplus_m \mathrm{END}_{T[-1]}(m,m)$ is $(\hat{\mu} \times \mathrm{id}^{\times k-1}) \circ \cdots \circ \hat{\mu}$. This map applied to $T_X^{\otimes k}[-k]$ is just the symmetrization map from $T_X^{\otimes k}[-k]$ to itself. It follows that the contribution of $\tilde{\mathrm{ch}}_k(E) \circ \pi \circ \Psi_{kk}$ to $\tilde{\mathrm{t}_k}(E)$ is precisely $\tilde{\mathrm{ch}}_k(E) \circ \pi = \tilde{\mathrm{ch}}_k(E)$. This proves (i).

(ii) follows immediately from (i) and from the facts that if $\mathrm{tr} : \mathcal{E}\mathrm{nd}(E) \to \mathcal{O}_X$ is the trace map, $\mathrm{t}_k(E) = \mathrm{tr} \circ \tilde{\mathrm{t}_k}(E)$ and $\mathrm{ch}_k(E) = \mathrm{tr} \circ \tilde{\mathrm{ch}}_k(E)$. □

Remark. This theorem gives a formula for the Big Chern classes in terms of the components of the Chern character for arbitrary smooth schemes. The same proof will go through for complex manifolds as well. This generalizes a similar, more vaguely stated formula t_k in terms of ch_l for $l \le k$ that I obtained in my theses for vector bundles over smooth *projective* varieties by some combinatorial methods [8]. The method there makes it difficult to see the explicit formula for Ψ_{kl}. It also requires the existence of an ample line bundle on the variety for which we are deducing this formula. Even in the smooth projective case, it is difficult to see Theorem 5(i) using the methods of [8].

9.3. *Proper subfunctors of the Hodge functors $H^q(X, \Omega^p)$, p,q ≥ 2*

The formula for t_k in terms of ch_l, $l \le k$ also easily gives us a method for finding an increasing chain of proper contravariant subfunctors of the Hodge functors $\mathrm{H}^q(X, \Omega^p)$ for smooth schemes over a field of characteristic 0. We note that

$$\Psi_{kl} \in \mathrm{Hom}_{\mathrm{D}^+(X)}(T_X^{\otimes k}[-k], T_X^{\otimes l}[-l]) = \mathrm{Hom}_{\mathrm{D}^+(X)}(\Omega^{\otimes l}[l], \Omega^{\otimes k}[k]).$$

Further, $\pi : T_X^{\otimes l}[-l] \to \wedge^l T_X[-l]$ is identified with $\gamma \cdot k! : \wedge^l \Omega_X[l] \to \Omega^{\otimes l}[l]$ where γ is the symmetrization map. Henceforth, in this subsection, we think of $\mathrm{ch}^l(E)$ and $\mathrm{t}_l(E)$ as elements in $\mathrm{Hom}_{\mathrm{D}^+(X)}(\mathcal{O}_X, \wedge^l \Omega_X[l])$ and $\mathrm{Hom}_{\mathrm{D}^+(X)}(\mathcal{O}_X, T_X^{\otimes l}[l])$ respectively.

With this convention, the first formula in Theorem 4 can be rewritten as

$$\mathrm{t}_k(E) = \gamma \circ k! \mathrm{ch}_k(E) + \sum_{l<k} \Psi_{kl} \circ \gamma \circ l! \mathrm{ch}_l(E).$$

The second formula in Theorem 4 may be rewritten in an identical fashion as well, though that does not concern us now.

In this picture, $\Psi_{kl} \circ \gamma$ yields a map from $\mathrm{H}^l(X, \Omega^l)$ to $\mathrm{H}^k(X, \Omega^{\otimes k})$. Denote this map by D_{kl}. Applying Theorem 4 to $\psi^p E$, we get

$$\mathrm{t}_k(\psi^p E) = p^k k!\mathrm{ch}_k(E) + \sum_{l<k} p^l D_{kl}(\mathrm{ch}_l(E)) \ \forall \ p \geq 1.$$

On the other hand,

$$\mathrm{t}_k(\psi^p E) = \sum_{l \leq k} p^l \mathrm{t}_k(\mathrm{ch}^{-1}(\mathrm{ch}_l(E))).$$

It follows that $D_{kl}(\mathrm{ch}_l(E)) = \mathrm{t}_k(\mathrm{ch}^{-1}(\mathrm{ch}_l(E)))$. It was shown in [8] that if $X = G(r, n)$ a Grassmannian of r-dimensional quotient spaces of an n-dimensional vector space over a field of characteristic 0, and if n is large enough, and if $E = Q$, the canonical quotient bundle of X, then $\mathrm{t}_k(\mathrm{ch}^{-1}(\mathrm{ch}_l(E))) \neq 0$ if $l \geq 2$. Therefore, the operator D_{kl} does not kill $\mathrm{H}^{l,l}$ in general. On the other hand, the Atiyah class $\alpha(T_X) = 0$ is X is an Abelian variety (a torus, for example). In such a case $D_{kl} = 0$ if $k \neq l$.

Therefore, if $X = G(r, n) \times T$ where T is a torus, then $\mathrm{ch}^{-1}(\mathrm{ch}_l(p_1^* Q))$ is not in the kernel of D_{kl}. On the other hand, $D_{kl}(p_2^* Y) = 0 \ \forall \ Y \in \mathrm{H}^{l,l}(T)$. We thus see that $\mathrm{H}^{l,l}_k := \ker D_{kl} : \mathrm{H}^{l,l} \to \mathrm{H}^k(X, \Omega^{\otimes k})$ is a proper subfunctor of $\mathrm{H}^{l,l}$ (as a theory) for all $l \geq 2$.

Given our current convention, in which we think of Ψ_{kl} as an element of $\mathrm{Hom}_{\mathrm{D}^+(X)}(\Omega^{\otimes l}[l], \Omega^{\otimes k}[k])$. If $p > q$, $(\Psi_{kq} \otimes \mathrm{id}^{\otimes p-q}) \circ \gamma$ yields a map from $\mathrm{H}^{p,q}$ to $\mathrm{H}^k(X, \Omega^{\otimes k})$. Denote this morphism by D_{kq}. If $p < q$, D_{kp} will denote the map yielded by the element $\Psi_{kp} \circ \gamma \in \mathrm{Hom}_{\mathrm{D}^+(X)}(\wedge^p \Omega_X[p], \Omega^{\otimes k}[k])$ from $\mathrm{H}^{p,q}$ to $\mathrm{H}^{k-p+q}(X, \Omega^{\otimes k})$.

We see that $\mathrm{H}^{p,q}_k$, given by $\ker(D_{kq}) : \mathrm{H}^{p,q} \to \mathrm{H}^k(X, \Omega^{\otimes k+p-q})$ if $p > q$ and $\ker(D_{kp}) : \mathrm{H}^{p,q} \to \mathrm{H}^{k+q-p}(X, \Omega^{\otimes k})$ otherwise, is a proper subfunctor (as a theory) of $\mathrm{H}^{p,q}$. To see this, again consider the case when $X = G(r, n) \times T$ as before, T a suitable torus. If $p > q$, and $\alpha_q = \mathrm{ch}^{-1}\mathrm{ch}_q$, then

$$D_{kq}(\alpha_q(p_1^* Q) \cup p_2^* Y) = (D_{kq}(\mathrm{ch}_q(p_1^* Q))) \cup p_2^* Y = \mathrm{t}_k(\alpha_q Q) \cup p_2^* Y \neq 0$$

where Y is a nonzero element of $\mathrm{H}^{p-q,0}(T)$. On the other hand, if $Z \in \mathrm{H}^{p,q}(T)$ then $D_{kq}((p_2^* Z)) = 0$. This shows that $\mathrm{H}^{p,q}_k$ is a proper subfunctor of $\mathrm{H}^{p,q}$ if $p > q$. If $p < q$ note that if $Y \in \mathrm{H}^{0,q-p}(T)$ is nonzero, then $D_{kp}((\alpha_p(p_1^* Q) \cup p_2^* Y)) = (D_{kp}(\alpha_p(p_1^* Q))) \cup p_2^* Y = \mathrm{t}_k(\alpha_p(p_1^* Q)) \cup p_2^* Y \neq 0$ and that if $Z \in \mathrm{H}^{p,q}(T)$, then $D_{kp}(p_2^* Z) = 0$. This proves that $\mathrm{H}^{p,q}_k$ is a proper subfunctor of $\mathrm{H}^{p,q}$ for all $k > q$ where $p, q \geq 2$.

Appendix

This appendix is meant to collect some facts about graded free Lie algebras used in Lemma 1 and Theorem 1. Proposition 17 is standard. Since I have not seen Theorem 6 in the literature, I have included it here as a theorem.

Let V be a vector space over a field K of characteristic 0. Let $T(V)$ denote the (graded) tensor algebra generated over K by V in degree 1. Let $L(V)$ be the free Lie algebra generated over K by V in degree 1.

Let $I_V : \mathrm{Sym}^\bullet(L(V)) \to T(V)$ and $\omega_V : \mathrm{Sym}^\bullet(L(V)) \otimes L(V) \to \mathrm{Sym}^\bullet(L(V)) \otimes L(V)$ be as in Sec. 5. Then,

Proposition 17. *I_V is an isomorphism of graded K-vector spaces.*

Proof. This is a form of the PBW theorem for $L(V)$ proven in Bahturin [17]. By Bakhturin [17, Theorem 2.10], $T(V)$ is the universal enveloping algebra of $L(V)$. Let $L(V)_+$ denote the subspace of $L(V)$ spanned by elements of even degree. Let $L(V)_-$ denote the subspace of $L(V)$ spanned by elements of odd degree. Let $z_1, z_2, \dots,$ be a homogenous ordered basis of $L(V)$. We recall from the PBW theorem ([17, Theorem 2.2]) that the elements $z_{i_i} \otimes \cdots \otimes z_{i_n}$ such that $i_j \leq i_{j+1}$ for all j and $z_{i_j} \neq z_{i_{j+1}}$ if $z_{i_j} \in L(V)_-$ form a basis of $T(V)$. Note that the elements $z_{i_1}, \dots, z_{i_n}$ such that $i_j \leq i_{j+1}$ for all j and $z_{i_j} \neq z_{i_{j+1}}$ if $z_{i_j} \in L(V)_-$ form a basis of $\mathrm{Sym}^\bullet(L(V))$.

For a multi-set $S = \{i_1, \dots, i_n\}$ such that $i_j \leq i_{j+1}$ for all j and $i_j \neq i_{j+1}$ if $z_{i_j} \in L(V)_-$, let z_S denote the element $z_{i_i} \otimes \cdots \otimes z_{i_n}$ of $T(V)$. Let $\pi(z_S)$ denote the element $z_{i_1}, \dots, z_{i_n} \in \mathrm{Sym}^\bullet(L(V))$. The cardinality $|S|$ of this multi-set is n. Then, let $G_V : T(V) \to \mathrm{Sym}^\bullet(L(V))$ be the map such that

$$G_V\left(\sum_{|S|\leq n, a_S\neq 0 \text{ for some } S \text{ such that } |S|=n} a_S z_S\right) = \sum_{|S|=n} a_s \pi(z_S).$$

G_V is a vector space isomorphism by the PBW theorem. Clearly, $G_V \circ I_V(\pi(z_S)) = \pi(z_S)$ by [17, proof of Theorem 2.2]. Since the $\pi(z_S)$ form a basis of $\mathrm{Sym}^\bullet(L(V))$, $G_V \circ I_V = \mathrm{id}$. This proves that I_V is a K-vector space isomorphism. $\square$

Theorem 6. *The following diagram commutes in the category of graded K vector spaces.*

$$\begin{array}{ccc} T(V)\otimes L(V) & \xrightarrow{m_V} & T(V) \\ \uparrow{\scriptstyle I_V\otimes\mathrm{id}} & & {\scriptstyle I_V}\uparrow \\ \mathrm{Sym}^\bullet(L(V))\otimes L(V) & \xrightarrow{\mu\circ\frac{\omega_V}{1-e^{-\omega_V}}} & \mathrm{Sym}^\bullet(L(V)) \end{array}$$

Proof. Let $T(L(V))$ denote the tensor algebra generated over K by $L(V)$. Let $\hat{\omega}_V : T(L(V)) \bigotimes L(V) \to T(L(V)) \otimes L(V)$ be the map

$$z_1 \otimes \cdots \otimes z_k \bigotimes y \rightsquigarrow \sum_i (-1)^{d_i(d_{i+1}+\cdots+d_k)} z_1 \otimes \cdots \widehat{i} \cdots \otimes z_k \bigotimes [z_i, y]$$

for homogenous elements $z_1, \dots, z_k$, of $L(V)$ of degrees $d_1, \dots, d_k, d$ respectively. Let $\hat{\mu} : T(L(V)) \bigotimes L(V) \to T(L(V))$ be the map such that

$$z_1 \otimes \cdots \otimes z_{k-1} \bigotimes z_k \rightsquigarrow \frac{1}{k} \sum_{i=1}^{i=k} (-1)^{d_k(d_i + \cdots + d_{k-1})} z_1 \otimes \cdots \otimes z_{i-1} \otimes z_k \otimes z_i \otimes \cdots \otimes z_{k-1}.$$

For a permutation σ of S_k, let $s(\sigma)$ be the sign such that $z_1, \dots, z_k = s(\sigma) z_{\sigma(1)}, \dots, z_{\sigma(k)}$. Let $J_V : \mathrm{Sym}^\bullet(L(V)) \to T(L(V))$ be the symmetrization map $z_1, \dots, z_k \rightsquigarrow \sum_{\sigma \in S_k} s(\sigma) z_{\sigma(1)} \otimes \cdots \otimes z_{\sigma(k)}$. Then the following diagram commutes.

Step 1

(An analog of Proposition 16)

$$\begin{tikzcd} T(L(V)) \otimes L(V) \arrow[r, "\hat{\mu} \circ \frac{\hat{\omega}_V}{1 - e^{\hat{\omega}_V}}"] & T(L(V)) \\ \mathrm{Sym}^\bullet(L(V)) \arrow[u, "J_V \otimes \mathrm{id}"] \arrow[r, "\mu \circ \frac{\omega_V}{1 - e^{-\omega_V}}"] & \mathrm{Sym}^\bullet(L(V)) \arrow[u, "J_V"'] \end{tikzcd}$$

This follows immediately from the fact that the following two diagrams commute.

$$\begin{tikzcd} T(L(V)) \otimes L(V) \arrow[r, "\hat{\omega}_V"] & T(L(V)) \otimes L(V) \\ \mathrm{Sym}^\bullet(L(V)) \otimes L(V) \arrow[u, "J_V \otimes \mathrm{id}"'] \arrow[r, "\omega_V"] & \mathrm{Sym}^\bullet(L(V)) \otimes L(V) \arrow[u, "J_V \otimes \mathrm{id}"] \end{tikzcd}$$

$$\begin{tikzcd} T(L(V)) \otimes L(V) \arrow[r, "\hat{\mu}"] & T(L(V)) \\ \mathrm{Sym}^\bullet(L(V)) \otimes L(V) \arrow[u, "J_V \otimes \mathrm{id}"'] \arrow[r, "\mu"] & \mathrm{Sym}^\bullet(L(V)) \arrow[u, "J_V"] \end{tikzcd}$$

The proof that the above two diagrams commute is word for word identical to that of Proposition 16 (Sec. 9) with $L(V)$ replacing $L(\mathrm{D}^1_{\mathrm{poly}}(X))$, $T(L(V))$ replacing $\mathcal{T}(L(\mathrm{D}^1_{\mathrm{poly}}(X)))$ and J_V replacing B.

Step 2

(Reduction to a combinatorial question)

The natural inclusion from $L(V)$ to $T(V)$ induces a map of graded algebras $\varphi : T(L(V)) \to T(V)$ such that $\varphi \circ J_V = I_V$. It follows from this that if m denotes the multiplication in $T(L(V))$, we only need to prove the following assertion

$$m \circ (J_V \otimes \mathrm{id}) = \hat{\mu} \circ \frac{\hat{\omega}_V}{1 - e^{\hat{\omega}_V}}.$$

Let $z_1, \dots, z_k, z_{k+1}$ be homogenous elements of $L(V)$ of degrees $d_1, \dots, d_k$ respectively. Note that $m \circ (J_V \otimes \mathrm{id})(z_1, \dots, z_k \otimes z_{k+1})$ and $\hat{\mu} \circ \frac{\hat{\omega}_V}{1 - e^{\hat{\omega}_V}}(z_1, \dots, z_k \otimes z_{k+1})$ are in the K-span of $\{z_{\sigma(1)} \otimes \cdots \otimes z_{\sigma(k+1)} | \sigma \in S_{k+1}\}$. Denote this subspace of $T(L(V))$ by $\mathcal{W}$.

Note that S_{k+1} has a right action on $\mathcal{W}$ such that for a permutation $\tau \in S_{k+1}$,

$$\tau(z_{\sigma(1)} \otimes \cdots \otimes z_{\sigma(k+1)}) = s(\tau, \sigma) z_{\sigma(\tau(1))} \otimes \cdots \otimes z_{\sigma(\tau(k+1))}$$

where $s(\tau, \sigma)$ is the sign such that $z_{\sigma(1)}, \ldots, z_{\sigma(k+1)} = s(\tau, \sigma) z_{\sigma(\tau(1))}, \ldots, z_{\sigma(\tau(k+1))}$ in $\mathrm{Sym}^{k+1}(L(V))$.

Let $\sigma(i, l, k+1)$ be the permutation in S_{k+1} such that $\sigma(i, l, k+1)(j) = j$ for $j \leq i-1$. $\sigma(i, l, k+1)(i-1+k) = n-l+k$ for $1 \leq k \leq l$ and $\sigma(i, l, k+1)(j) = j-l$ for $j \geq i+l$. Let $\nu(i, l, k+1)$ denote the inverse of $\sigma(i, l, k+1)$ in S_{k+1}. Let τ_l denote the l-cycle $(k-l+2\ k-l+1, \ldots, k+1)$.

Observation 1.

Identifying $T(L(V)) \otimes L(V)$ as a direct summand of $\oplus_k L(V)^{\otimes k+1}$ of $T(L(V))$, we have by a direct computation that

$$\begin{aligned} \hat{\mu} \circ \hat{\omega}_V^j = {} & \left(\sum_{i=1}^{i=k+1-j} \sigma(i, j+1, k+1) \right) \circ (\mathrm{id} - \tau_{j+1}) \circ \left(\sum_{i=1}^{i=k+2-j} \nu(i, j, k+1) \right) \\ & \circ \cdots \circ \left(\sum_{i=1}^{i=k} \sigma(i, 2, k+1) \right) \circ (\mathrm{id} - \tau_2) \circ \left(\sum_{i=1}^{i=k+1} \nu(i, 1, k+1) \right) \end{aligned}$$

on $\mathcal{W}$.

Note that $\hat{\mu} \circ \frac{\hat{\omega}_V}{1-e^{\hat{\omega}_V}} = \sum_j c_j \hat{\mu} \circ \hat{\omega}_V^j$ where $\frac{y}{1-e^{-y}} = \sum_j c_j y^j$. The above formula thus enables us to express $\hat{\mu} \circ \frac{\hat{\omega}_V}{1-e^{\hat{\omega}_V}}$ as the action of an explicit element in the group ring of S_{k+1} on $\mathcal{W}$.

Observation 2.

On the other hand,

$$m \circ (J_V \otimes \mathrm{id})(z_1, \ldots, z_k \otimes z_{k+1}) = \sum_{\varphi \in S_k} \frac{1}{k!} s(\varphi) z_{\varphi(1)} \otimes \cdots \otimes z_{\varphi(k)} \otimes z_{k+1}.$$

Let $\iota : S_k \to S_{k+1}$ be the homomorphism fixing $k+1$. It follows from both observations that we need to prove the following identity in KS_{k+1} for all k.

$$\begin{aligned} \sum_{\varphi \in S_k} \iota(\varphi) = {} & \left(\sum_{i=1}^{i=k+1-j} \sigma(i, j+1, k+1) \right) \circ (\mathrm{id} - \tau_{j+1}) \\ & \circ \left(\sum_{i=1}^{i=k+2-j} \nu(i, j, k+1) \right) \circ \cdots \circ \left(\sum_{i=1}^{i=k} \sigma(i, 2, k+1) \right) \\ & \circ (\mathrm{id} - \tau_2) \circ \left(\sum_{i=1}^{i=k+1} \nu(i, 1, k+1) \right) \circ \sum_{\varphi \in S_k} \iota(\varphi). \end{aligned}$$

Call this identity (***).

This finishes step 2.

Step 3

(Proving the combinatorial identity (***))

Let W (different from $\mathcal{W}$) be an infinite-dimensional vector space over K concentrated in degree 0. Let $L(W)$ and $T(W)$ be the free Lie algebra generated over K by W and the tensor algebra generated over K by W respectively. Let $I_W : \mathrm{Sym}^\bullet(L(W)) \to T(W)$ be the symmetrization map such that $I_W(z_1, \ldots, z_k) = \sum_{\sigma \in S_k} z_{\sigma(1)} \otimes \cdots \otimes z_{\sigma(k)}$ for all $z_1, \ldots, z_k \in L(W)$. Let μ denote the multiplication on $\mathrm{Sym}^\bullet(L(W))$ and let $\omega_W : \mathrm{Sym}^\bullet(L(W)) \otimes L(W) \to \mathrm{Sym}^\bullet(L(W)) \otimes L(W)$ be the map such that

$$\omega_W(z_1, \ldots, z_k \otimes y) = \sum_i z_1, \ldots, \widehat{i}, \ldots, z_k \otimes [z_i, y].$$

It follows from Reutenauer [11, Chap. 3] that the following diagram commutes in the category of K-vector spaces.

$$\begin{tikzcd}
T(W) \otimes L(W) \arrow[r, "m"] & T(W) \\
\mathrm{Sym}^\bullet(L(W)) \otimes L(W) \arrow[u, "I_W \otimes \mathrm{id}"] \arrow[r, "\mu \circ \frac{\omega_W}{1 - e^{-\omega_W}}"] & \mathrm{Sym}^\bullet(L(W)) \arrow[u, "I_W"']
\end{tikzcd}$$

Now, if $z_1, \ldots, z_{k+1}$ are linearly independent elements of $L(W)$ and if $\mathcal{W}$ denotes the K-span of $\{z_{\sigma(1)} \otimes \cdots \otimes z_{\sigma(k)} | \sigma \in S_{k+1}\}$, then S_{k+1} has a right action on $\mathcal{W}$ such that $\tau(z_{\sigma(1)} \otimes \cdots \otimes z_{\sigma(k)}) = z_{\sigma(\tau(1))} \otimes \cdots \otimes z_{\sigma(\tau(k))}$. Further, two elements α and β in KS_k are equal if and only if $\alpha(z_1 \otimes \cdots \otimes z_n) = \beta(z_1 \otimes \cdots \otimes z_n)$ in $\mathcal{W}$. Let $\iota : S_k \to S_{k+1}$ be as in the previous step. The identity (***) of Step 2 follows from this set up once we note that

$$m \circ (I_W \otimes \mathrm{id})(z_1, \ldots, z_k \otimes z_{k+1}) = \sum_{\varphi \in S_k} \iota(\varphi)(z_1 \otimes \cdots \otimes z_{k+1})$$

and

$$\begin{aligned}
&I_W \circ \left(\mu \circ \frac{\omega_W}{1 - e^{-\omega_W}}\right)(z_1, \ldots, z_k \otimes z_{k+1}) \\
&\quad = \left(\sum_{i=1}^{i=k+1-j} \sigma(i, j+1, k+1)\right) \circ (\mathrm{id} - \tau_{j+1}) \circ \left(\sum_{i=1}^{i=k+2-j} \nu(i, j, k+1)\right) \\
&\qquad \circ \cdots \circ \left(\sum_{i=1}^{i=k} \sigma(i, 2, k+1)\right) \circ (\mathrm{id} - \tau_2) \circ \left(\sum_{i=1}^{i=k+1} \nu(i, 1, k+1)\right) \\
&\qquad \circ \sum_{\varphi \in S_k} \iota(\varphi)(z_1 \otimes \cdots \otimes z_{k+1}).
\end{aligned}$$

The second of these two identities requires some work. Let $T(L(W))$ be the tensor algebra of $L(W)$. Let $\hat{\omega}_W : T(L(W)) \otimes L(W) \to T(L(W)) \otimes L(W)$ be the map $z_1 \otimes \cdots \otimes z_k \bigotimes y \rightsquigarrow \sum_i z_1 \otimes \cdots \widehat{i} \cdots \otimes z_k \otimes [z_i, y]$. Further, let $\hat{\mu} : T(L(W)) \otimes L(W) \to T(L(W))$ be the map such that $\hat{\mu}(z_1 \otimes \cdots \otimes z_{k-1} \bigotimes z_k) = \frac{1}{k} \sum_{i=1}^{i=k} z_1 \otimes \cdots \otimes z_{i-1} \otimes$

$z_k \otimes z_i \otimes \cdots \otimes z_{k-1}$. Further, let $J(W) : \mathrm{Sym}^\bullet(L(W)) \to T(L(W))$ be the map such that $z_1, \ldots, z_k \rightsquigarrow \sum_{\sigma \in S_k} z_{\sigma(1)} \otimes \cdots \otimes z_{\sigma(k)}$.

The following diagrams commute.

$$\begin{array}{ccc} T(L(W)) \otimes L(W) & \xrightarrow{\hat{\omega}_W} & T(L(W)) \otimes L(W) \\ \uparrow{\scriptstyle J_W \otimes \mathrm{id}} & & \uparrow{\scriptstyle J_W \otimes \mathrm{id}} \\ \mathrm{Sym}^\bullet(L(W)) \otimes L(W) & \xrightarrow{\omega_W} & \mathrm{Sym}^\bullet(L(W)) \otimes L(W) \end{array}$$

$$\begin{array}{ccc} T(L(W)) \otimes L(W) & \xrightarrow{\hat{\mu}} & T(L(W)) \\ \uparrow{\scriptstyle J_W \otimes \mathrm{id}} & & \uparrow{\scriptstyle J_W} \\ \mathrm{Sym}^\bullet(L(W)) \otimes L(W) & \xrightarrow{\mu} & \mathrm{Sym}^\bullet(L(W)). \end{array}$$

From these, we see that the following diagram commutes.

$$\begin{array}{ccc} T(L(W)) \otimes L(W) & \xrightarrow{\hat{\mu} \circ \frac{\hat{\omega}_W}{1 - e^{\hat{\omega}_W}}} & T(L(W)) \\ \uparrow{\scriptstyle J_W \otimes \mathrm{id}} & & \uparrow{\scriptstyle J_W} \\ \mathrm{Sym}^\bullet(L(W)) & \xrightarrow{\mu \circ \frac{\omega_W}{1 - e^{-\omega_W}}} & \mathrm{Sym}^\bullet(L(W)). \end{array}$$

Now, a direct computation shows us that

$$\begin{aligned} &\left(\hat{\mu} \circ \frac{\hat{\omega}_W}{1 - e^{\hat{\omega}_W}}\right) \circ (J_W \otimes \mathrm{id})(z_1 \otimes \cdots \otimes z_k \bigotimes z_{k+1}) \\ &\quad = \left(\sum_{i=1}^{i=k+1-j} \sigma(i, j+1, k+1)\right) \circ (\mathrm{id} - \tau_{j+1}) \circ \left(\sum_{i=1}^{i=k+2-j} \nu(i, j, k+1)\right) \\ &\qquad \circ \cdots \circ \left(\sum_{i=1}^{i=k} \sigma(i, 2, k+1)\right) \circ (\mathrm{id} - \tau_2) \circ \left(\sum_{i=1}^{i=k+1} \nu(i, 1, k+1)\right) \\ &\qquad \circ \sum_{\varphi \in S_k} \iota(\varphi)(z_1 \otimes \cdots \otimes z_{k+1}). \end{aligned}$$

Thus,

$$\begin{aligned} &I_W \circ \left(\mu \circ \frac{\omega_W}{1 - e^{-\omega_W}}\right)(z_1, \ldots, z_k \otimes z_{k+1}) \\ &\quad = \left(\sum_{i=1}^{i=k+1-j} \sigma(i, j+1, k+1)\right) \circ (\mathrm{id} - \tau_{j+1}) \circ \left(\sum_{i=1}^{i=k+2-j} \nu(i, j, k+1)\right) \\ &\qquad \circ \cdots \circ \left(\sum_{i=1}^{i=k} \sigma(i, 2, k+1)\right) \circ (\mathrm{id} - \tau_2) \circ \left(\sum_{i=1}^{i=k+1} \nu(i, 1, k+1)\right) \\ &\qquad \circ \sum_{\varphi \in S_k} \iota(\varphi)(z_1 \otimes \cdots \otimes z_{k+1}). \end{aligned}$$

Thereby proving Theorem 5. □

Acknowledgments

I thank Prof. Victor Ginzburg for introducing me to the works of Caldararu [10] and Markarian [3] and asking whether the material in these works had anything to do with the relation between the big Chern classes and the Chern character. In addition, Prof. Ginzburg went through the first draft of this work and gave me many helpful suggestions. I am grateful to Prof. Madhav Nori for many useful discussions. I also thank Prof. Kaan Akin, Dr. Victor Protsak and the referees of the first and current versions of this paper for their useful suggestions.

References

[1] A. Yekutieli, The continuous Hochschild cochain complex of a scheme, *Canad. J. Math.* **54**(6) (2002) 1319–1337.
[2] M. Kontsevich, Deformation quantization of Poisson manifolds I, preprint, math.QA/9709040.
[3] N. Markarian, Poincare–Birkhoff–Witt isomorphism, Hochschild homology and Riemann–Roch theorem, preprint (2001), MPI 2001-52, available at the Max Planck Institute's website.
[4] J. L. Loday, *Cyclic Homology*, 2nd edn. Grundlehren der Mathematischen Wissenchaften [Fundamental Principles of Mathematical Sciences], Vol. 301 (Springer-Verlag, Berlin, 1998).
[5] J. Milnor and J. C. Moore, On the structure of Hopf algebras, *Ann. Math. (2)* **81** (1965) 211–264.
[6] M. Kapranov, Rozansky–Witten invariants via Atiyah classes, *Compos. Math.* **115**(1) (1999) 71–113.
[7] J. Roberts, Rozansky-Witten theory, preprint, math.QA/0112209.
[8] A. Ramadoss, On the nonexistence of certain morphisms from Grassmannian to Grassmannian in characteristic 0, Ph.D thesis, preprint, math.AG/0501130.
[9] D. Thurston, Wheeling: A diagrammatic analogue of the Duflo isomorphism, preprint, math.QA/0006083.
[10] A. Caldararu. The Mukai pairing II, preprint, math.AG/0308080.
[11] C. Reutenauer, *Free Lie Algebras*, London Mathematical Society Monographs, Vol. 7 (Oxford University Press, 1993).
[12] A. Ramadoss, On the nonexistence of certain morphisms from Grassmannian to Grassmannian in characteristic 0, preprint.
[13] J. Roberts and S. Willerton, On the Rozansky-Witten weight systems, preprint, math.DG/0602653.
[14] N. Markarian, The Atiyah class, Hochschild cohomology and the Riemann–Roch theorem, preprint, math.AG/0610553.
[15] E. Getzler and J. D. S. Jones, Operads, homotopy algebras and iterated integrals of double loop spaces, preprint, hep-th/9403055.
[16] A. Ramadoss, The relative Riemann–Roch theorem from Hochschild homology, preprint, math.AG/0603127.
[17] Yu. A. Bahturin *et al.*, *Infinite Dimensional Lie Superalgebras*, de-Gruyter Expositions in Mathematics, Vol. 7 (De-Gruyter, 1992).