A generalized Hirzebruch Riemann-Roch theorem.

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Abstract

This short note proves a generalization of the Hirzebruch Riemann-Roch theorem equivalent to the Cardy condition described in [1]. This is done using an earlier result [4] that explicitly describes what the Mukai pairing in [1] descends to in Hodge cohomology via the Hochschild-Kostant-Rosenberg map twisted by the root Todd genus.

1 Statement of the generalized Riemann-Roch theorem.

Let X be a smooth proper scheme of dimension n over a field $\mathbb{K}$ of characteristic 0. Let $\mathcal{E}$ and $\mathcal{F}$ be elements of the bounded derived category $\mathrm{D}^b_{\mathrm{coh}}(X)$ of coherent $\mathcal{O}_X$-modules on X. Let v and w be elements of $\mathrm{End}_{\mathrm{D}^b_{\mathrm{coh}}(X)}(\mathcal{E})$ and $\mathrm{End}_{\mathrm{D}^b_{\mathrm{coh}}(X)}(\mathcal{F})$ respectively. Let $\mathrm{T}_{v,w}$ denote the endomorphism

$$a \mapsto w \circ a \circ v$$

of $\mathrm{RHom}_X(\mathcal{E}, \mathcal{F})$. Let $\mathrm{at}(\mathcal{E}) \in \mathrm{Hom}_{\mathrm{D}^b_{\mathrm{coh}}(X)}(\mathcal{E}, \mathcal{E} \otimes \Omega[1])$ denote the Atiyah class of $\mathcal{E}$. Let $\mathrm{ch}_v(\mathcal{E})$ denote the "twisted" Chern character

$$\mathrm{Tr}_{\mathcal{E}}(\exp(\mathrm{at}(\mathcal{E})) \circ v) \in \mathrm{Hom}_{\mathrm{D}^b_{\mathrm{coh}}(X)}(\mathcal{O}_X, \oplus_i \Omega^i_X[i]) \simeq \oplus_i \mathrm{H}^i(X, \Omega^i_X) \ .$$

Note that if $v = \mathrm{id}_{\mathcal{E}}$ then $\mathrm{ch}_v(\mathcal{E}) = \mathrm{ch}(\mathcal{E})$, the Chern character of $\mathcal{E}$. Let K be the involution on $\oplus_{p,q}\mathrm{H}^q(X, \Omega^p)$ that acts on the summand $\mathrm{H}^q(X, \Omega^p)$ by multiplication with $(-1)^q$. If x in an element of $\oplus_{p,q}\mathrm{H}^q(X, \Omega^p)$, we shall denote $K(x)$ by x^*. For any endomorphism T of $\mathrm{RHom}_X(\mathcal{E}, \mathcal{F})$, $\mathrm{str}(\mathrm{T})$ shall denote the alternated trace of T. If $f : X \to Y$ is a morphism of smooth proper schemes, f_*,f^* etc shall denote the corresponding derived functors unless explicitly mentioned otherwise. $\int_X$ shall denote the linear functional on $\oplus_{p,q}\mathrm{H}^q(X, \Omega^p)$ that coincides with the Serre duality trace on $\mathrm{H}^n(X, \Omega^n_X)$

and vanishes on all other direct summands. We have the following generalization of the Hirzebruch Riemann-Roch theorem.

Explicit Cardy condition.

$$str(T_{v,w}) = \int_X ch_v(\mathcal{E})^* ch_w(\mathcal{F}) td(X) \ .$$

Remark 1: Note that when $v = \mathrm{id}_{\mathcal{E}}$ and when $w = \mathrm{id}_{\mathcal{F}}$ the above statement amounts to the Hirzebruch Riemann-Roch theorem

$$\chi(\mathcal{E}, \mathcal{F}) = \int_X \mathrm{ch}(\mathcal{E})^* \mathrm{ch}(\mathcal{F}) \mathrm{td}(X) \ .$$

Remark 2: The proof of the above theorem will also enable us to see that the above theorem is in fact equivalent to the Cardy condition in [1] (see Theorem 7.9 of [1]). The more classical (compared to Theorem 7.9 of [1]) statement of the Cardy condition given here would be useful for those interested in an explicit version of the Cardy condition for computational purposes. Such computations are related to understanding string propagation between D-Branes with twisted boundary conditions in the situation where X is Calabi-Yau and the category of D-Branes is equivalent to $\mathrm{D}^b_{\mathrm{coh}}(X)$. An analogous computation in a different set-up seems to have been discussed at least implicitly in [5].

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2 The Mukai pairing and the Cardy condition.

Let $\Delta : X \to X \times X$ be the diagonal embedding. Let $\Delta_!$ denote the left adjoint of Δ^*. Recall that the i-th Hochschild homology $\mathrm{HH}_i(X)$ is the space $\mathrm{Hom}^i_{\mathrm{D}^b_{\mathrm{coh}}(X\times X)}(\Delta_!\mathcal{O}_X,\Delta_*\mathcal{O}_X)$. In his paper [1], Caldararu defined a (nondegenerate) Mukai pairing

$$\langle\ ,\ \rangle_M : \mathrm{HH}_i(X)\otimes HH_{-i}(X)\to\mathbb{K}\ \forall\ i\ .$$

In [1], we have a map

$$\iota^{\mathcal{H}} : \mathrm{Hom}_{\mathrm{D}^b_{\mathrm{coh}}(X)}(\mathcal{H},\mathcal{H})\to\mathrm{HH}_0(X)=\mathrm{Hom}_{\mathrm{D}^b_{\mathrm{coh}}(X\times X)}(\Delta_!\mathcal{O}_X,\Delta_*\mathcal{O}_X)$$

for any $\mathcal{H}\in\mathrm{D}^b_{\mathrm{coh}}(X)$. Also, $\Delta_!\mathcal{O}_X\simeq\Delta_*S_X^{-1}$ where S_X is the shifted line bundle tensoring with which yields the Serre duality functor on $\mathrm{D}^b_{\mathrm{coh}}(X)$. Let $\nu\in\mathrm{Hom}_{\mathrm{D}^b_{\mathrm{coh}}(X\times X)}(\Delta_*\mathcal{O}_X,\Delta_*S_X)$. For any smooth proper scheme Y and any $\mathcal{V}\in\mathrm{D}^b_{\mathrm{coh}}(Y)$, let $\mathrm{Tr}_Y:\mathrm{Hom}_{\mathrm{D}^b_{\mathrm{coh}}(Y)}(\mathcal{V},S_Y\otimes\mathcal{V})\to\mathbb{K}$ denote the Serre duality trace. If $u\in\mathrm{Hom}_{\mathrm{D}^b_{\mathrm{coh}}(X)}(\mathcal{H},\mathcal{H})$, then

Proposition 1. *(see the definition of $\iota^{\mathcal{H}}$ in Section 6.3 of [1])*

$$Tr_{X\times X}(\nu\circ\iota^{\mathcal{H}}(u)) = Tr_X(\pi_{2*}(\pi_1^*\mathcal{H}\otimes\nu)\circ u)\ .$$

The following statement of the Cardy condition appears as Theorem 7.9 in [1]. Let $\mathcal{E}$,$\mathcal{F}$,v,w be as in the previous subsection.

Theorem 1.

$$str(T_{v,w})=\langle\iota^{\mathcal{E}}(v),\iota^{\mathcal{F}}(w)\rangle_M\ .$$

Recall that the Adjunction $\Delta_!\dashv\Delta^*$ identifies $\mathrm{Hom}^i_{\mathrm{D}^b_{\mathrm{coh}}(X\times X)}(\Delta_!\mathcal{O}_X,\Delta_*\mathcal{O}_X)$ with $\mathrm{Hom}^i_{\mathrm{D}^b_{\mathrm{coh}}(X)}(\mathcal{O}_X,\Delta^*\Delta_*\mathcal{O}_X)$. Therefore, the Hochschild-Kostant-Rosenberg map

$$\mathrm{I}_{HKR}:\Delta^*\Delta_*\mathcal{O}_X\to\oplus_j\Omega^j[j]$$

induces maps

$$\mathrm{I}_{HKR}:HH_i(X)\to\oplus_j\mathrm{H}^{j-i}(X,\Omega^j_X)\ .$$

The following theorem appears implicitly in Markarian's work [3] and in the form stated below in an earlier paper of this author [4]. Let $x\in\mathrm{HH}_i(X)$ and let $y\in\mathrm{HH}_{-i}(X)$. Then,

Theorem 2.

$$\langle x,y\rangle_M=\int_X I_{HKR}(x)^*I_{HKR}(y)td(X)\ .$$

In order to prove the explicit Cardy condition, we therefore need to prove the following proposition.

Proposition 2.

$$I_{HKR}(\iota^{\mathcal{E}}(v)) = ch_v(\mathcal{E}) \ .$$

Note that setting $x = \iota^{\mathcal{E}}(v)$ and $y = \iota^{\mathcal{F}}(w)$ in Theorem 2 ,applying Theorem 1 and using the fact that $\mathrm{I}_{HKR}(x) = \mathrm{ch}_v(\mathcal{E})$ and that $\mathrm{I}_{HKR}(y) = \mathrm{ch}_w(\mathcal{F})$ by Proposition 2, we obtain the explicit Cardy condition. We now prove Proposition 2 below.

Proof. This proof is a straightforward modification of the proof of Theorem 4.5 in [2]. Remember that if $\mathcal{A}, \mathcal{H}, \mathcal{J} \in \mathrm{D}^b_{\mathrm{coh}}(X)$, there is a trace map (see Section 2.4 of [1])

$$\mathrm{Tr}_{\mathcal{A}} : \mathrm{Hom}_{\mathrm{D}^b_{\mathrm{coh}}(X)}(\mathcal{A} \otimes \mathcal{H}, \mathcal{A} \otimes \mathcal{J}) \to \mathrm{Hom}_{\mathrm{D}^b_{\mathrm{coh}}(X)}(\mathcal{H}, \mathcal{J}) \ .$$

For any $\theta \in \mathrm{Hom}_{\mathrm{D}^b_{\mathrm{coh}}(X)}(\mathcal{H}, \mathcal{J})$, we will abuse notation and denote the map $\mathrm{id}_{\mathcal{A}} \otimes \theta \in \mathrm{Hom}_{\mathrm{D}^b_{\mathrm{coh}}(X)}(\mathcal{A} \otimes \mathcal{H}, \mathcal{A} \otimes \mathcal{J})$ by $\mathcal{A} \otimes \theta$.

Recall that the Adjunction $\Delta_! \dashv \Delta^*$ identifies $\mathrm{Hom}_{\mathrm{D}^b_{\mathrm{coh}}(X \times X)}(\Delta_! \mathcal{O}_X, \Delta_* \mathcal{O}_X)$ with $\mathrm{Hom}_{\mathrm{D}^b_{\mathrm{coh}}(X)}(\mathcal{O}_X, \Delta^* \Delta_* \mathcal{O}_X)$. Let $\hat{\mathrm{ch}}_v(\mathcal{E})$ denote the image of $\iota^{\mathcal{E}}(v)$ under this identification. The desired proposition states that the image of $\hat{\mathrm{ch}}_v(\mathcal{E})$ under I_{HKR} is $\mathrm{ch}_v(\mathcal{E})$. Let ν be an arbitrary element of $\mathrm{Hom}_{\mathrm{D}^b_{\mathrm{coh}}(X)}(\Delta^* \Delta_* \mathcal{O}_X, S_X)$. This corresponds to the element

$$\bar{\nu} = \Delta_* \nu \circ \eta \in \mathrm{Hom}_{\mathrm{D}^b_{\mathrm{coh}}(X \times X)}(\Delta_* \mathcal{O}_X, \Delta_* S_X)$$

where η is the unit of the adjunction $\Delta^* \dashv \Delta_*$ applied to $\Delta_* \mathcal{O}_X$.

Note that

$$\mathrm{Tr}_X(\nu \circ \hat{\mathrm{ch}}_v(\mathcal{E})) = \mathrm{Tr}_{X \times X}(\bar{\nu} \circ \iota^{\mathcal{E}}(v))(\text{ see the proof of Proposition 1 of [4] for instance})$$

$$= \mathrm{Tr}_X(\pi_{2*}(\pi_1^* \mathcal{E} \otimes \bar{\nu}) \circ v) \ (\text{ by Proposition 1 })$$

$$= \mathrm{Tr}_X(\pi_{2*}(\pi_1^* \mathcal{E} \otimes (\Delta_* \nu \circ \eta)) \circ v) = \mathrm{Tr}_X(\pi_{2*}((\pi_1^* \mathcal{E} \otimes \Delta_* \nu) \circ (\pi_1^* \mathcal{E} \otimes \eta)) \circ v)$$

$$= \mathrm{Tr}_X((\mathcal{E} \otimes \nu) \circ (\pi_{2*}(\pi_1^* \mathcal{E} \otimes \eta)) \circ v)$$

$$= \mathrm{Tr}_X(\nu \circ \mathrm{Tr}_{\mathcal{E}}(\pi_{2*}(\pi_1^* \mathcal{E} \otimes \eta) \circ v))(\text{ by Lemma 2.4 of [1]}).$$

By the non-degeneracy of the Serre duality pairing, it follows that

$$\hat{\mathrm{ch}}_v(\mathcal{E}) = \mathrm{Tr}_{\mathcal{E}}(\pi_{2*}(\pi_1^* \mathcal{E} \otimes \eta) \circ v) \ .$$

Hence,

$$
\begin{aligned}
\mathrm{I}_{HKR}(\hat{\mathrm{ch}}_v(\mathcal{E})) &= \mathrm{I}_{HKR}(\mathrm{Tr}_{\mathcal{E}}(\pi_{2*}(\pi_1^*\mathcal{E}\otimes\eta)\circ v) \\
&= \mathrm{Tr}_{\mathcal{E}}(\pi_{2*}(\pi_1^*\mathcal{E}\otimes\Delta_*\mathrm{I}_{HKR}\circ\eta)\circ v)\ .
\end{aligned}
$$

By Proposition 4.4 of [2],

$$\Delta_*\mathrm{I}_{HKR}\circ\eta = \exp(\alpha)$$

where $\alpha : \Delta_*\mathcal{O}_X \to \Delta_*\Omega_X[1]$ is the universal Atiyah class. It follows that

$$\mathrm{Tr}_{\mathcal{E}}(\pi_{2*}(\pi_1^*\mathcal{E}\otimes\Delta_*\mathrm{I}_{HKR}\circ\eta)\circ v) = \mathrm{ch}_v(\mathcal{E})\ .$$

$\square$

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