Statement of research

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My research interests involve various aspects of algebraic geometry, noncommutative geometry and homological algebra. During my post-doctoral appointment in the past two years, my research has focussed on two different tracks, both of which involved understanding the relation between Hochschild homology and some very familiar algebraic geometric/ geometric results and constructions.

One of the tracks developed out of an effort to explain a more algebraic approach to the Grothendieck Riemann-Roch theorem (for smooth proper complex varieties) which first appeared in a paper of Markarian [10]. The other track involved proving and then generalizing a conjecture of Feigin,Losev and Shoikhet [5] and answering related questions. This conjecture stated that a certain construction via "Topological quantum mechanics" of a linear functional on the top cohomology of a compact complex manifold amounts to integration over that manifold. This too, is linked to yet another approach to the Riemann-Roch Hirzebruch theorem.

Working on these problems has led to my developing interests in areas and problems that I never anticipated working on when I was completing my doctoral thesis. A summary of the research I have done so far follows.

On the nonexistence of certain morphisms from Grassmannian to Grassmannian in characteristic 0.

While working towards my thesis under the guidance of Prof. Madhav Nori, I used some properties of the Big Chern classes to prove results about the nonexistence of certain morphisms between Grassmannians. The kth big Chern class $t_k(V)$ of a vector bundle V over a smooth scheme X is a characteristic class that lives in $\mathrm{H}^k(X, \Omega^{\otimes k})$. These classes first appeared in

Kapranov [7] as far as I know. I was able to equip the space $\oplus_k \mathrm{H}^k(X, \Omega^{\otimes k})$ with the structure of a commutative ring together with Adams operations that arise out of a grading different from the obvious one. Further, $\oplus_k t_k : K(X) \otimes \mathbb{Q} \to \oplus_k \mathrm{H}^k(X, \Omega^{\otimes k})$ was shown to be a ring homomorphism that commutes with Adams operations. This together with the explicit computation of $t_k(Q(r,n))$ was used to prove the main results. Here, $Q(r,n)$ denotes the universal quotient bundle of the Grassmannain $G(r,n)$ of r dimensional quotients of an n dimensional vector space over a field of characteristic 0. Let $[V]$ denote the class of a vector bundle V on a smooth scheme X in $K(X)$, the Grothendieck group of the category of vector bundles on X . The following theorem is one of the main results of my thesis.

Theorem 1. *For any $r \geq 2$, $n \geq 2r+1$, $\psi^p[Q(r,n)] \neq [V]$ for any vector bundle V on $G(r,n)$ for any $p \geq 2$.*

Corollary 1. *There exists no morphism $f : G(r,n) \to G(r,M)$ such that $f^*([Q(r,M)]) = \psi^p([Q(r,n)])$ for any $p \geq 2$,$r \geq 2$ and $n \geq 2r+1$.*

Besides other results on the nonexistence of morphisms between Grassmannians, my thesis contains a result regarding the existence of a relation between the big Chern classes of a vector bundle over a smooth projective variety to its Chern character. This has been superceded in the paper I describe next. A paper [15] containing the main results of my thesis together with their proofs has been submitted for publication.

The big Chern classes and the Chern character.

My first post-doctoral paper [16] is on the Big Chern classes and the Chern character. My interest in the topics that this paper leans on was kindled by a desire to understand the relation between the big Chern classes (that appeared in my thesis) and the Chern character in a more systematic framework. Another motivation for studying Hochschild homology in detail was the appearance of the Eulerian idempotents governing the Hodge decomposition of Hochschild homology (see Loday [9] Section 4.5) in a very crucial "combinatorial" statement in my thesis. This paper realizes the Atiyah class of the tangent bundle of a smooth scheme X (shifted by -1) as the Lie bracket of an explicit complex of $\mathcal{O}_X$-modules having the structure of a Lie algebra object in the category of complexes of $\mathcal{O}_X$-modules. This, together with a computation for this Lie algebra analogous to the computation of the inverse exponential map is then used to prove an explicit formula for the big Chern classes in terms of the components of the Chern character.

Theorem 2.

$$t_k(E) = ch_k(E) \circ \pi + \sum_{l<k} ch_l(E) \circ \Psi_{kl} \ .$$

Here, $\pi : \oplus_k T_X^{\otimes k}[-k] \to \oplus_k \wedge^k T_X[-k]$ is the standard projection. Ψ_{kl} is the component in $\mathrm{END}_{T[-1]}(k,l)$ of an element Ψ of the PROP $\mathrm{END}_{T[-1]}$ where $\mathrm{END}_{T[-1]}(n,m) = \mathrm{Hom}_{\mathrm{D}^+(X)}(T_X^{\otimes n}[-n], T_X^{\otimes m}[-m])$. $\mathrm{D}^+(X)$ denotes the derived category of bounded above complexes of $\mathcal{O}_X$-modules with coherent cohomology. The element Ψ of $\mathrm{END}_{T[-1]}$ is given by an explicit formula that uses the $d(exp^{-1})$ computation described earlier.

This paper also provides a more explicit of the following theorem of Roberts and Willerton [22]. Recall that $T_X[-1]$ has the structure of a Lie algebra in $\mathrm{D}^+(X)$ (see Kapranov [7]).

Theorem 3. *The complex $D^\bullet_{poly}(X)$ of polydifferential operators on X equipped with Hochschild co-boundary is the universal enveloping algebra of $T_X[-1]$ in $D^+(X)$.*

This, together with the fact that the Atiyah class of a vector bundle equips it with the structure of a module over the Lie algebra $T_X[-1]$ in $\mathrm{D}^+(X)$ enables us to interpret the Chern character of a vector bundle E as the "character of the representation E of the Lie algebra $T_X[-1]$ of $\mathrm{D}^+(X)$".

Theorem 2 extends what I did in my thesis in two ways - the new formula is much more explicit and proven in a more algebraically structured framework . It also does not require the scheme X to be projective. A derived category version of the $d(exp^{-1})$ computation was first done by Markarian in his preprint [10] (also see [11] for a newer version of [10]). This paper has appeared in International Journal of Mathematics.

The relative Riemann-Roch theorem from Hochschild homology.

My next paper [17] attempts to explain the rest of how Markarian proves the relative Riemann-Roch theorem starting from the $d(exp^{-1})$ calculation mentioned in the previous paragraph. Doing this by carefully following and elaborating upon Markarian's arguments yields an answer to a conjecture of Caldararu (in his papers [2],[3]) about what the Mukai pairing Caldararu defines on Hochschild homology descends to in Hodge cohomology. Let $\langle , \rangle$

denote the Mukai pairing on Hochschild homology defined by Caldararu [2]. Then,

Theorem 4. *If $a \in HH_i(X)$ and if $b \in HH_{-i}(X)$, then*

$$\langle a, b \rangle = \langle I_{HKR}(a)\sqrt{td_X}, I_{HKR}(b)\sqrt{td_X} \rangle_M \ .$$

Here, $\langle , \rangle_M$ is a pairing on Hodge cohomology related to but different from the Mukai pairing defined by Caldararu [3]. It however, satisfies the adjointness property one expects from a Mukai pairing. This theorem is actually a corollary of a theorem which first appeared in Markarian's paper [10] whose proof my paper attempts to explain. The relative Riemann-Roch theorem is a consequence of Theorem 4 and the fact that the Chern character with values in Hochschild homology defined by Caldararu [2] commutes with proper push-forwards. This paper has been submitted for publication.

Some notes on the Feigin-Losev-Shoikhet integral conjecture.

An attempt to understand the relation between the approach to the Riemann-Roch theorem by Markarian on one hand and the "local Riemann-Roch theorems of Nest and Tsygan (see [12],[13] and [14]) on the other led to my paper [18] on the integral conjecture of B.Feigin, A. Losev and B. Shoikhet (FLS) [5]. In their preprint [5], FLS construct a linear functional on the 0-th completed Hochschild homology of the sheaf $\mathrm{Diff}(\mathcal{E})$ of holomorphic differential operators on a holomorphic vector bundle $\mathcal{E}$ over a compact complex manifold X. On the other hand, the 0th completed Hochschild homology of $\mathrm{Diff}(\mathcal{E})$ is isomorphic to the top cohomology $H^{2n}(X, \mathbb{C})$ of X with complex coefficients. It was conjectured that the linear functional $I_{\mathcal{E}}$ on $H^{2n}(X, \mathbb{C})$ thus obtained is the integral over X. FLS proved this in their preprint under the assumption that $\mathcal{E}$ has nonzero Euler characteristic. This paper of mine shows that

Theorem 5. *$I_{\mathcal{E}}$ is independent of $\mathcal{E}$.*

One side of the argument doing this is essentially a Morita equivalence argument. The integral conjecture is thus proven for any compact complex manifold admitting at least on vector bundle with nonzero Euler characteristic - in particular for compact complex algebraic varieties. A similar linear functional can be constructed on the $2i$-th completed cyclic homology of $\mathrm{Diff}(\mathcal{E})$ for any $i \geq 0$ - the latter yields linear functionals $I_{\mathcal{E},2i,2k}$ on $H^{2n-2k}(X, \mathbb{C})$ for any $k \leq i$. FLS had conjectures about $I_{\mathcal{E},2,0}$ and $I_{\mathcal{E},2,2}$. This paper of mine shows that

Theorem 6. *$I_{\mathcal{E},2i,0} = I_{\mathcal{E}}$ and that $I_{\mathcal{E},2i,2k}$ vanishes for all $k > 0$.*

This is available on the Arxiv at math.QA/0612298 and has appeared in Journal of Noncommutative Geometry.

Integration over complex manifolds via Hochschild homology.

My paper [19] with the above title further develops my work on the integral conjecture by showing (using some heat kernel techniques) that the FLS construction mimics the following natural behaviour of the integral of a top-dimensional smooth form : If U is an open complex disc with inclusions into two compact complex manifolds X and Y, and if f is a top-dimensional smooth form compactly supported on U, then $\int_U f = \int_X f = \int_Y f$ ($\int_Z$ denotes the integral over Z). As a result, it becomes easy to prove the integral conjecture for arbitrary compact complex manifolds. Moreover, the FLS construction can now be generalized to complex manifolds that are not necessarily compact. The construction of $I_{\mathcal{E}}$ for compact complex manifolds generalizes to yield a linear functional $I_{\mathcal{E}}$ on $H_c^{2n}(X, \mathbb{C})$, where H_c^* is cohomology with compact supports. Further,

Theorem 7. *$I_{\mathcal{E}} = \int_X$ for any complex manifold X and any (holomorphic) vector bundle $\mathcal{E}$ on X.*

Similarly, one generalizes the corresponding constructions in the compact case to obtain linear functionals $I_{\mathcal{E},2i,2k}$ on $H_c^{2n-2k}(X, \mathbb{C})$.

Theorem 8. *$I_{\mathcal{E},2i,0} = \int_X$ and $I_{\mathcal{E},2i,2k} = 0$ for any $k > 0$.*

Recall that a holomorphic differential operator D on a vector bundle $\mathcal{E}$ on a compact complex manifold X induces endomorphisms D_i on $\mathrm{H}^i(X, \mathcal{E})$ for each i. The super-trace $\mathrm{str}(D)$ of D is the alternating sum $\sum_{i=0}^{i=n} (-1)^i \mathrm{tr}(D_i)$ of the traces of the D_i. D also induces a 0-cycle on a complex computing the 0-th completed Hochschild homology of Diff(E). It therefore gives rise to a class $[D]$ in $\mathrm{H}^{2n}(X, \mathbb{C})$. Another consequence of Theorem 7 is a different proof of the following "super-trace theorem" of Marcus Engeli and Giovanni Felder [4].

Theorem 9 (Engeli-Felder)**.** *If D is a global holomorphic differential operator on a holomorphic vector bundle $\mathcal{E}$ on a compact complex manifold X, then*

$$str(D) = \int_X [D] \ .$$

This preprint is also available on the Arxiv at arxiv:0707.4528 and has been accepted for publication by Journal of Noncommutative Geometry.

The Mukai pairing and integral transforms in Hochschild homology.

My preprint [20] with the above title was motivated by an effort to answer some of the questions I have mentioned in my future research plans.

Let X be a smooth proper scheme over a field of characteristic 0. Let $\mathrm{perf}(X)$ denote the DG-category of left bounded perfect injective complexes of $\mathcal{O}_X$-modules. Following [26], one can construct a "natural" pairing on the Hochschild homology of $\mathrm{perf}(X)$, and hence, on the Hochschild homology $\mathrm{HH}_\bullet(X)$ of X. Denote this pairing by $\langle\ ,\ \rangle_{\mathrm{nat}}$. On the other hand, Caldararu constructs the Mukai pairing on Hochschild homology in [2]. When X is Calabi-Yau, the Mukai pairing arises from the action of the class of a genus 0 Riemann surface with two incoming and no outgoing boundaries in $\mathrm{H}_0(\mathcal{M}(2,0))$ on the algebra of closed states of a version of the B-Model on X. Denote the Mukai pairing in Hochschild homology by $\langle\ ,\ \rangle_M$. Let $\vee$ denote the involution on $\mathrm{HH}_\bullet(X)$ corresponding via the Hochschild-Kostant-Rosenberg map to the involution on $\mathrm{H}^\bullet(X)$ acting by multiplication by $(-1)^p$ on the direct summand $\mathrm{H}^q(X, \Omega_X^p)$. The main result in this paper is the following theorem.

Theorem 10. *If $a \in HH_i(X)$ and $b \in HH_{-i}(X)$, then*

$$\langle b^\vee, a\rangle_M = \langle a, b\rangle_{nat} \ .$$

If $\Phi \in \mathrm{perf}(X \times Y)$, Φ gives rise to an integral transform from $\mathrm{perf}(X)$ to $\mathrm{perf}(Y)$. This gives rise to a "natural" integral transform Φ_*^{nat} from $\mathrm{HH}_\bullet(X)$ to $\mathrm{HH}_\bullet(Y)$. On the other hand, Caldararu has a construction of the integral transform Φ_*^{cal} from $\mathrm{HH}_\bullet(X)$ to $\mathrm{HH}_\bullet(Y)$. Theorem 10 enables me to prove

Theorem 11.

$$\Phi_*^{nat} = \Phi_*^{cal} \ .$$

In other words, Caldararu's integral transform on Hochschild homology coincides with the "natural" one.

Let $\mathrm{ch}^i : \mathrm{K}_i(X) \to \mathrm{HH}_i(X)$ denote the sheafification of the Dennis trace map (see Nest-Tsygan [14]). This is a generalization of the Chern character from $\mathrm{K}(X)$ to $\mathrm{HH}_0(X)$ defined by Caldararu [2]. Let $\mathrm{ch}^i : K_i(X) \to \oplus_j \mathrm{H}^{j-i}(X, \wedge^j \Omega)$ be the Hochschild-Kostant-Rosenberg isomorphism composed with ch^i. Theorems 10 and 11 lead to the following "higher" analog of the Hirzebruch Riemann-Roch theorem.

Theorem 12. *Let $f : X \to Y$ be a smooth proper morphism between proper schemes X and Y. Let Z be a smooth quasi-compact scheme. Then,*

$$(f \times id)_*(ch^i(\alpha)\pi_X^* td(T_X)) = ch^i((f \times id)_*(\alpha))\pi_Y^* td(T_Y)$$

for any $\alpha \in K_i(X \times Z)$.

This preprint is available on the Arxiv at arxiv:0805.1760. It has been submitted for publication.

A genralized Hirzebruch Riemann-Roch theorem.

My most recent preprint [21] proves a generalization of the Hirzebruch Riemann- Roch theorem equivalent to the Cardy condition described in [2]. This is done using an earlier result [17] that explicitly describes what the Mukai pairing in [2] descends to in Hodge cohomology via the Hochschild-Kostant-Rosenberg map twisted by the root Todd genus. This preprint has been submitted for publication.

Future research plans.

I am interested in working on further developments of my work so far on the integral conjecture and related topics. A specific project that I am currently working on is to use the methods of [18],[19] to obtain a new proof of as well as further generalizations of the index theorem for elliptic pairs by P. Schapira and J.-P. Schneiders [24].

I am also interested in examining whether my methods in [18],[19] would help shed light on questions like the question about a Lefschetz number formula for algebraic differential operators on orbifolds that Giovanni Felder and Xiang Tang ask in [6].

Another project is to use the explicit computation of Caldararu's Mukai pairing at the level of Hodge cohomology to obtain Hirzebruch Riemann-Roch type results in for higher K-theoretic analogs of the Chern character. Theorem 12 is a special case of the following conjecture that I am still working on. The general case of this conjecture is much more complicated than Theorem 12.

Conjecture 1. *If $f : X \to Y$ is a proper morphism of smooth complex varieties, then*

$$f_*(ch^i(x).td_X) = ch^i(f_*x)td_Y \ .$$

It is also known (see [1],[23]) that if X is a smooth scheme with an ample line bundle, then

$$\mathrm{D}^b_{\mathrm{coh}}(X) \simeq \mathrm{D}^{\mathrm{perf}}(A)$$

for some compact homologically smooth dg-algebra A (also see [25]). I am interested in identifying the Lie algebra structure corresponding to $T_X[-1]$ in an explicit "combinatorial" manner in $\mathrm{D}^{\mathrm{perf}}(A)$. This seems related to the following question of Victor Ginzburg, which I would like to understand in depth and answer.

Question 1. *What are the non-commutative analogs of the Eulerian idempotents that govern the Hodge decomposition of Hochschild homology of a commutative algebra ?*

Developments in this direction are bound to lead to a deeper, understanding of the isomorphism

$$\mathrm{D}^b_{\mathrm{coh}}(X) \simeq \mathrm{D}^{\mathrm{perf}}(A) \ .$$

They would also shed new light on topics that I have worked on in [16].

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