A fast Chebyshev–Legendre transform using an asymptotic formula

Alex Townsend
MIT
(Joint work with Nick Hale)

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Introduction

What is the Chebyshev–Legendre transform?

- Suppose we have a degree $N-1$ polynomial. It can be expressed in the Chebyshev or Legendre polynomial basis:

$$p_{N-1}(x) = \sum_{k=0}^{N-1} c_k^{leg} P_k(x) \qquad \Longleftrightarrow \qquad p_{N-1}(x) = \sum_{k=0}^{N-1} c_k^{cheb} T_k(x)$$

- The Chebyshev–Legendre transform:

$$c_0^{leg}, \ldots, c_{N-1}^{leg} \quad \underset{\text{inverse transform}}{\overset{\text{forward transform}}{\rightleftarrows}} \quad c_0^{cheb}, \ldots, c_{N-1}^{cheb}. \qquad \mathcal{O}(N(\log N)^2 / \log\log N)$$

- Applications in:
  - Convolution [Hale & T., 14]
  - Legendre-tau spectral methods
  - QR of a quasimatrix [Trefethen, 08]
  - best least squares approximation

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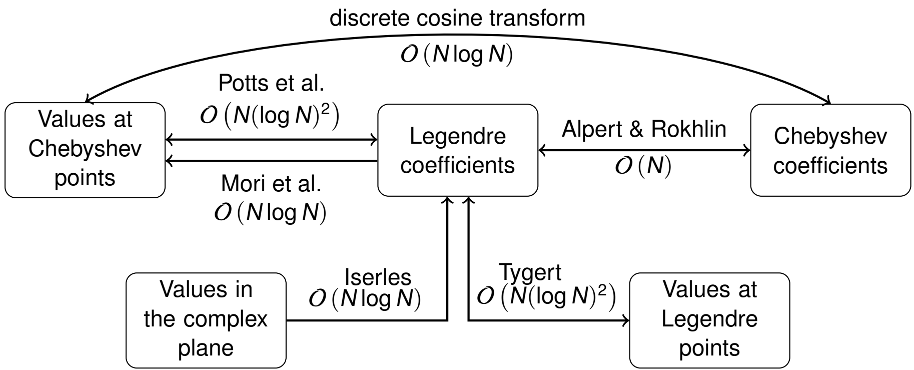


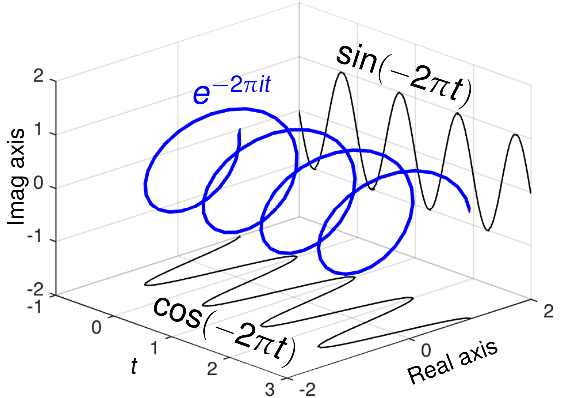
New features of our algorithms

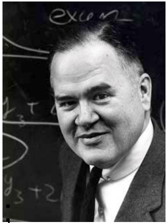
- FFT-based approach
- Analysis-based
- No precomputation

The FFT computes the DFT in $\mathcal{O}(N \log N)$ operations:

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-2\pi i n k/N}, \qquad 0 \leqslant k \leqslant N-1.$$


; James Cooley

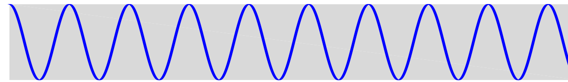



; John Tukey

Introduction

Many special functions are trigonometric-like

Trigonometric functions
$\cos(\omega x), \quad \sin(\omega x)$



Chebyshev polynomials
$T_n(x)$

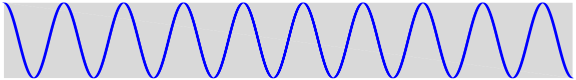
Legendre polynomials
$P_n(x)$

Bessel functions
$J_\nu(z)$

Airy functions
$Ai(x)$

Also, Jacobi polynomials, Hermite polynomials, cylinder functions, etc.

Trigonometric functions
cos(ωx),   sin(ωx)



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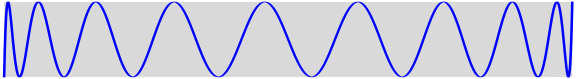

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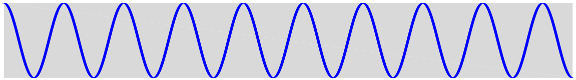

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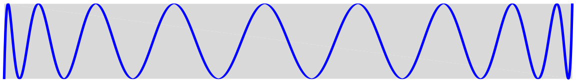

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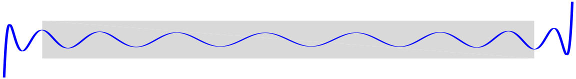

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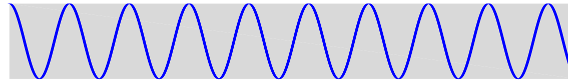

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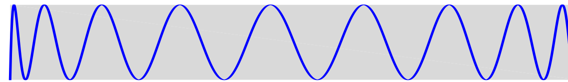

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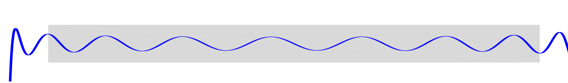

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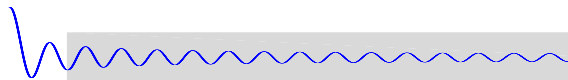

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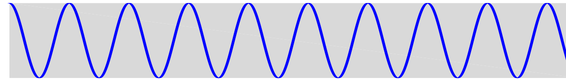

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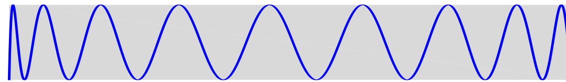

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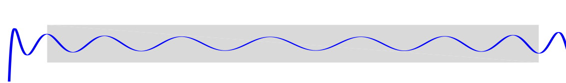

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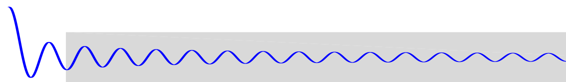

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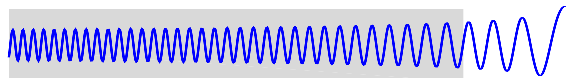

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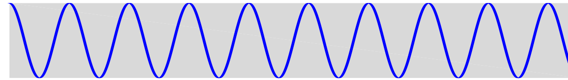

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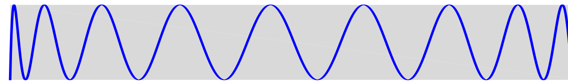

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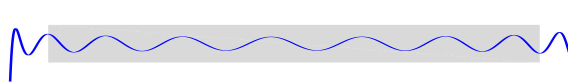

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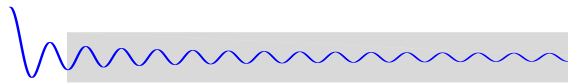

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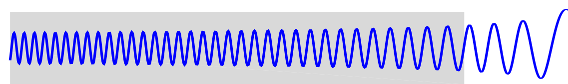

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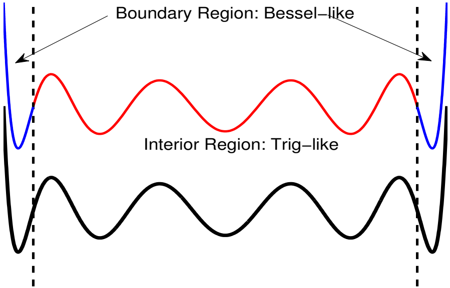
Legendre polynomials

$$P_n(\cos\theta) = C_n \sum_{m=0}^{M-1} h_{m,n} \frac{\cos\left((m+n+\frac{1}{2})\theta - (m+\frac{1}{2})\frac{\pi}{2}\right)}{(2\sin\theta)^{m+1/2}} + R_{n,M}(\theta)$$

$$C_n = \sqrt{\frac{4}{\pi}} \frac{\Gamma(n+1)}{\Gamma(n+3/2)}, \qquad h_{m,n} = \begin{cases} 1, & m = 0, \\ \prod_{j=1}^{m} \frac{(j-1/2)^2}{j(n+j+1/2)}, & m > 0. \end{cases}$$

Thomas Stieltjes



Numerical pitfalls of an asymptotic expansionist

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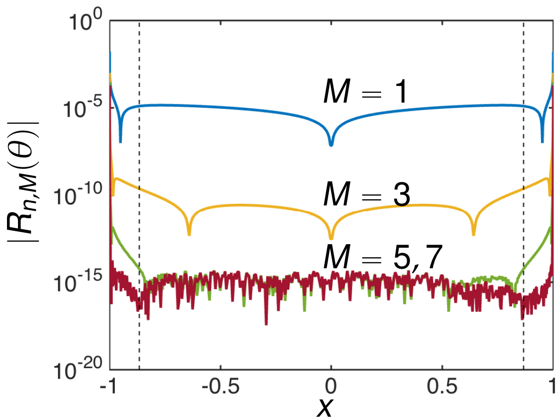

$n = 1{,}000$

Fix M. Where is the asymptotic expansion accurate?

Computing the Chebyshev–Legendre transform

The transform comes in two parts

The Chebyshev–Legendre transform naturally splits into two parts:

$$c_0^{leg}, \ldots, c_{N-1}^{leg} \underset{}{\overset{\text{This bit}}{\rightleftharpoons}} p_{N-1}(x_0^{cheb}), \ldots, p_{N-1}(x_{N-1}^{cheb}) \underset{\text{DCT}}{\overset{\text{IDCT}}{\rightleftharpoons}} c_0^{cheb}, \ldots, c_{N-1}^{cheb}$$

Task: Compute the following matrix-vector product in quasilinear time?

$$\mathbf{P_N}\underline{c} = \begin{pmatrix} P_0(\cos\theta_0) & \ldots & P_{N-1}(\cos\theta_0) \\ \vdots & \ddots & \vdots \\ P_0(\cos\theta_{N-1}) & \ldots & P_{N-1}(\cos\theta_{N-1}) \end{pmatrix} \begin{pmatrix} c_0^{leg} \\ \vdots \\ c_{N-1}^{leg} \end{pmatrix}, \qquad \theta_k = \frac{k\pi}{N-1}$$

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Computing the Chebyshev–Legendre transform

Asymptotic expansions as a matrix decomposition

The asymptotic expansion

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gives a matrix decomposition (sum of diagonally scaled DCTs and DSTs):

$$\mathbf{P}_N = \sum_{m=0}^{M-1}\left(D_{u_m}\mathbf{C}_N D_{Ch_m} + D_{v_m}\begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathbf{S}_{N-2} & 0 \\ 0 & 0 & 0 \end{bmatrix} D_{Ch_m}\right) + \mathbf{R}_{N,M}.$$

$$\mathbf{P_N} = \mathbf{P}_{\mathbf{N}}^{\mathrm{ASY}} + \mathbf{R}_{N,M}$$

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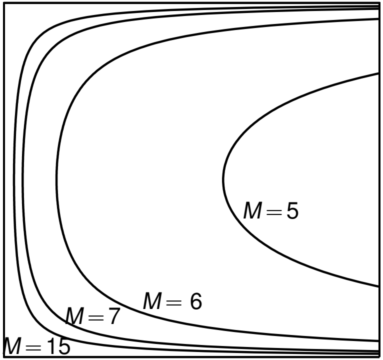
$$\mathbf{P}_\mathbf{N} = \mathbf{P}_\mathbf{N}^{\mathrm{ASY}} + \mathbf{R}_{N,M}$$

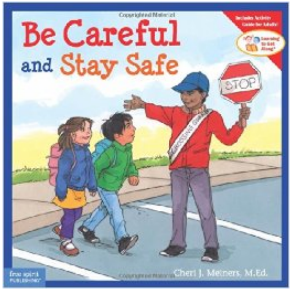
Error curve: $|R_{n,M}(\theta_k)| = \epsilon$

Fix M. Where is the asymptotic expansion accurate?

$\mathbf{R}_{N,M} =$



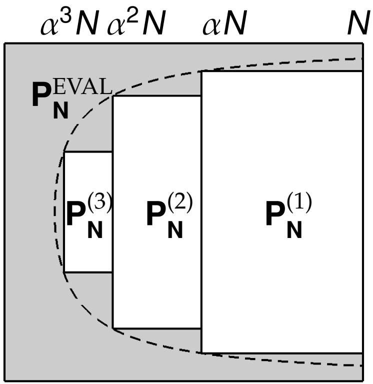
$$|R_{n,M}(\theta_k)| \leqslant \frac{2C_n h_{M,n}}{(2\sin\theta_k)^{M+1/2}}$$



Computing the Chebyshev–Legendre transform

Partitioning and balancing competing costs

$\mathbf{P_N} =$



Theorem

The matrix-vector product $\underline{f} = \mathbf{P_N}\underline{c}$ can be computed in $\mathcal{O}(N(\log N)^2/\operatorname{loglog} N)$ operations.

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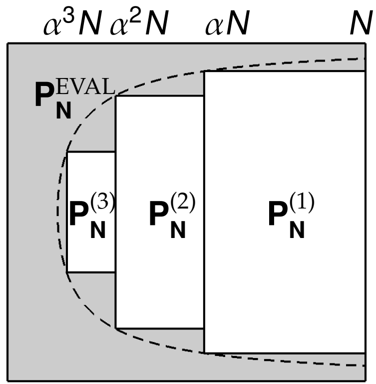

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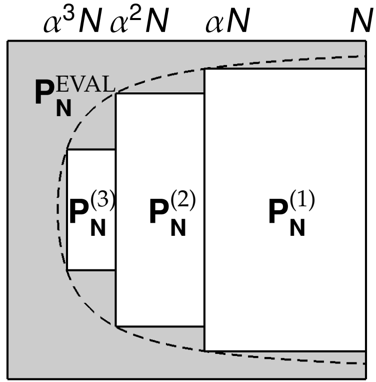

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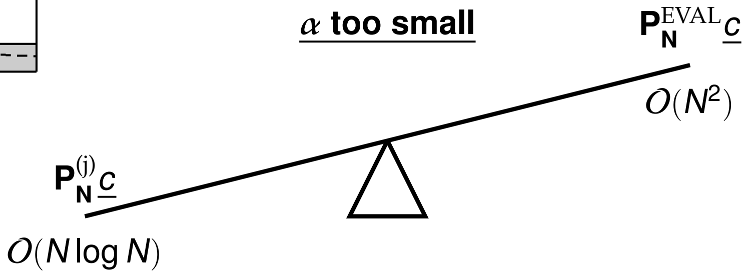

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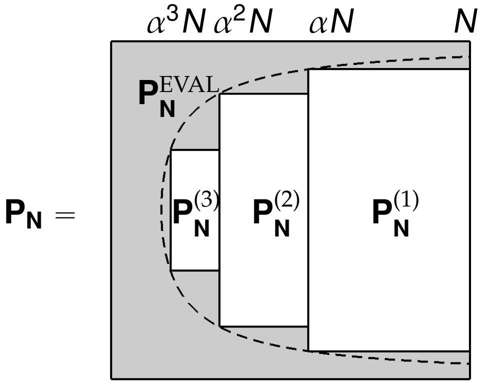

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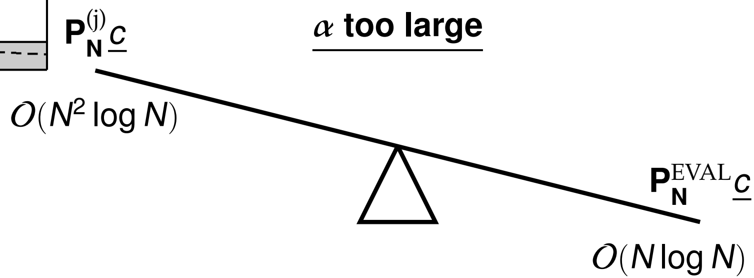

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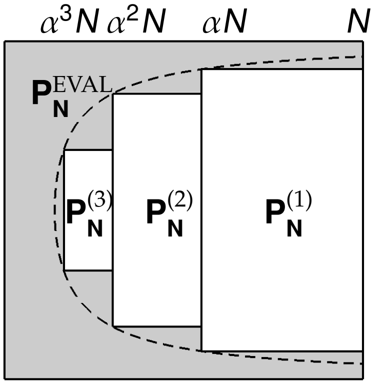

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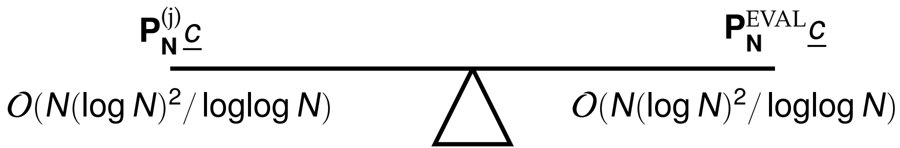
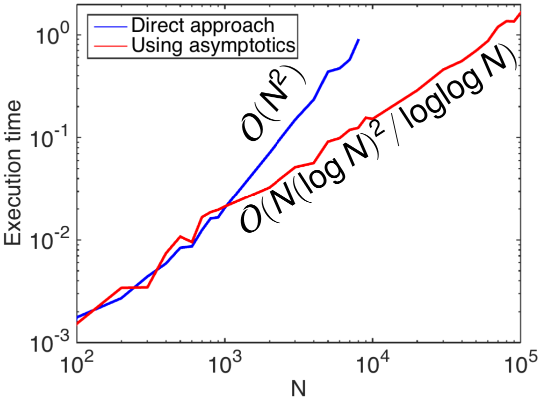
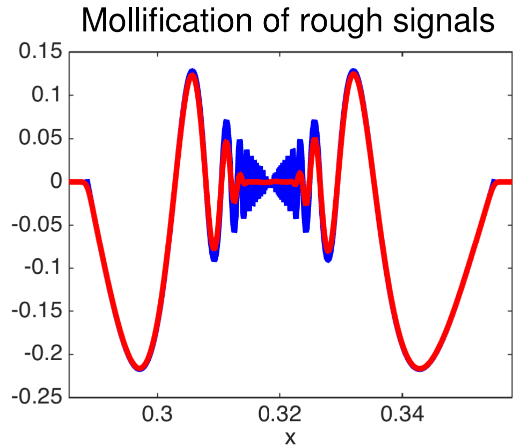
$\boldsymbol{\alpha} = \mathcal{O}(\mathbf{1}/\log \mathbf{N})$





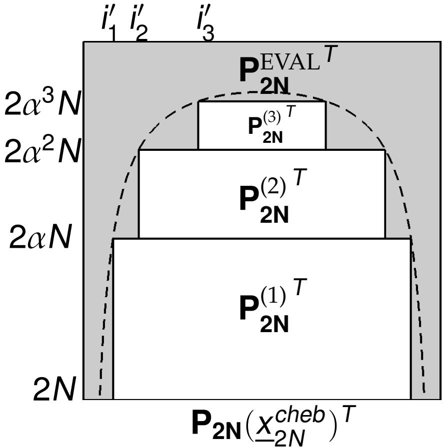


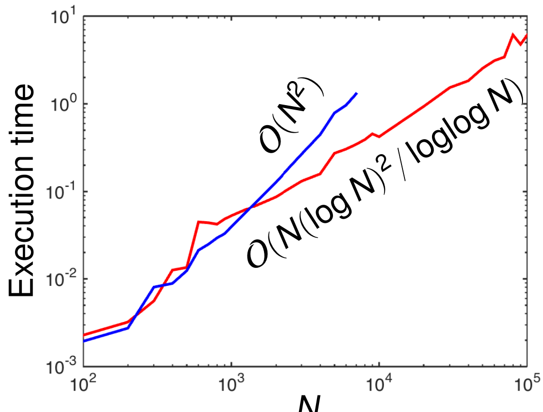
No precomputation.

$$\int_{-1}^{1} P_n(x) e^{-\mathrm{i}\omega x} dx = \mathrm{i}^m \sqrt{\frac{2\pi}{-\omega}} J_{m+1/2}(-\omega)$$

The integral formula for Legendre coefficients gives the following relation:

$$\underline{c}_N^{leg} = \left[I_{N+1} \mid \mathbf{0}_N \right] D_{\underline{s}_{2N}} \mathbf{P_{2N}}(\underline{x}_{2N}^{cheb})^T D_{\underline{w}_{2N}} \mathbf{T_{2N}}(\underline{x}_{2N}^{cheb}) \begin{bmatrix} I_{N+1} \\ \mathbf{0}_N \end{bmatrix} \underline{c}_N^{cheb},$$



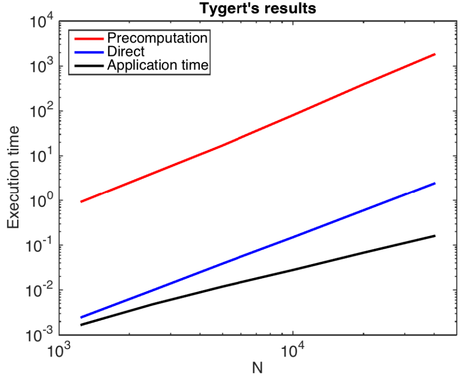


Future work
Fast spherical harmonic transform

Spherical harmonic transform:

$$f(\theta,\phi) = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} \alpha_l^m P_l^{|m|}(\cos\theta) e^{im\phi}$$

[Mohlenkamp, 1999], [Rokhlin & Tygert, 2006], [Tygert, 2008]



A new generation of fast transforms with no precomputation.

Thank you

Thank you

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