Multiscale Analysis on bounded domains with restricted interpolation points

Alex Townsend
Supervised by Holger Wendland

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Overview

1. Introduction to the Multilevel algorithm
2. Motivation
3. Previous Results
4. Extended Convergence Results
5. Numerical Simulations
6. Future Work

Details are in:
Multilevel analysis in Sobolev spaces on bounded domains with restricted data points. *(In Preparation)*
by Townsend & Wendland

Interpolating a target function

INPUT

$\Omega \subseteq \mathbb{R}^d$ bounded Lipschitz domain.
Continuous target function $f \in \mathcal{H}^\tau(\Omega)$

Multilevel Algorithm

$f_0 = 0$, $e_0 = f$
for $j = 1, 2, \ldots, n$
Compute s_j such that $s_j(x) = e_{j-1}(x)\ \forall x \in X_j$
$f_j = f_{j-1} + s_j$
$e_j = e_{j-1} - s_j$
end

OUTPUT

Interpolant to f and an idea of how well we did.

Notation

- For any data set $X = \{x_1, \ldots, x_N\} \subset \Omega$ then the **fill distance** is

$$h_{X,\Omega} := \sup_{x \in \Omega} \min_{1 \leq i \leq n} \|x - x_i\|_2$$

  This is the parameter we use to state all convergence orders.
- Choose sequence of quasi-uniform data sets $X_1, X_2, \ldots, X_n \subset \Omega$ to have decreasing fill distance.

Definition (Compactly Supported Radial Function)

A function $\Phi : \mathbb{R}^d \mapsto \mathbb{R}$ such that $\Phi(x) = \phi(||x||_2)$ for $\forall x \in \mathbb{R}$ with a continuous $\phi : [0, \infty) \mapsto \mathbb{R}$ and $\phi(t) = 0$ for $t \geq 1$.

- s_j is formed by linear combination of scaled and translated compactly supported radial basis functions.

Further Notation

That means,

$$s_j(x) = \sum_{i=1}^{|X_j|} \alpha_i^{(j)} \Phi_{\delta_j}(x - x_i)$$

where

$$\Phi_{\delta_j}(\cdot) = \delta_j^{-d} \Phi\left(\frac{\cdot}{\delta_j}\right)$$

By interpolation conditions we get $s_j(x_k) = e_{j-1}(x_k)\ \forall x_k \in X_j$.

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By interpolation conditions we get $s_j(x_k) = e_{j-1}(x_k)\ \forall x_k \in X_j$.
This corresponds to the symmetric, positive definite linear system

$$A_{X,\Phi}\alpha^{(j)} = e_{j-1}|_{X_j}$$

Compactly supported radial basis functions $\Rightarrow A_{X_j,\Phi}$ is sparse.

Multilevel Motivation

- For compactly supported RBFs [Schaback, 1997]:

$$\text{High accuracy} \longleftrightarrow \text{low efficiency}$$

- Each level is fast but low accuracy. Get high accuracy by working on finer levels!
- Interpolation matrices have good conditioning.
- Capture large scale variation on coarse level and details on finer levels.

Multiscale analysis in Sobolev spaces on bounded domains

Theorem (Wendland, 2010)

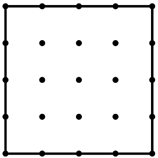
Let $\Omega \subset \mathbb{R}^d$ be Lipschitz bounded domain and a sequence of denser data sets $X_1, X_2 \cdots \subset \Omega$. Further, let Φ be a compactly supported radial basis function with reproducing Hilbert space equivalent to $\mathcal{H}^\tau(\mathbb{R}^d)$. Then if the target function $f \in \mathcal{H}^\tau(\Omega)$, $\tau > d/2$ then the multilevel algorithm converges with

$$\|e_n\|_{L_2(\Omega)} \leq C h_n^{\tau-\epsilon} \|f\|_{\mathcal{H}^\tau(\Omega)}$$

for some constants C and $\epsilon < \tau$.

Problem: Support of interpolant overlaps the domain boundary.

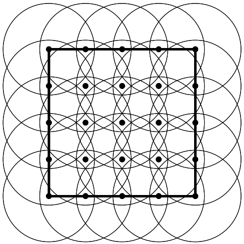
Support is not contained on domain



Support overlapping the boundary causes three main problems:

- Large point-wise error when on cracked domains.
- Unable to enforce boundary conditions of the interpolant.
- Makes Galerkin methods for solving PDEs hard to analyse.

Support is not contained on domain



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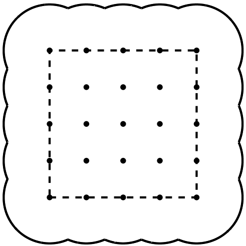

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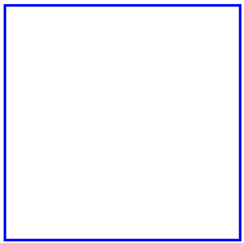
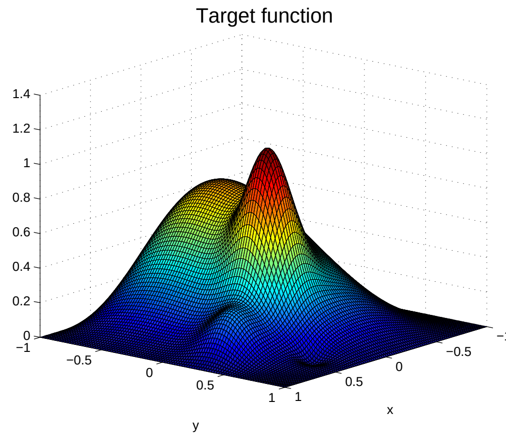
Multilevel with restricted interpolation points in action





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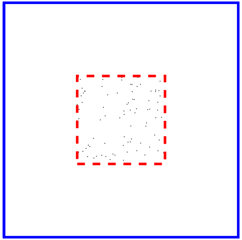

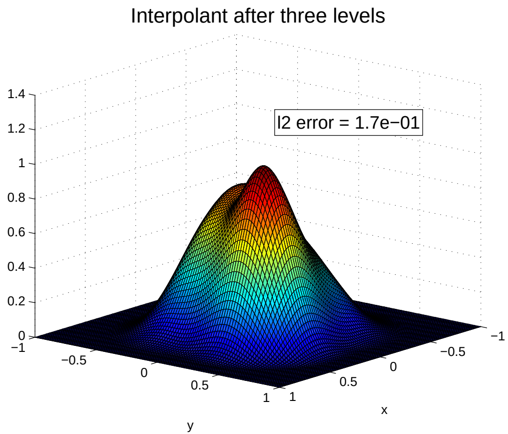

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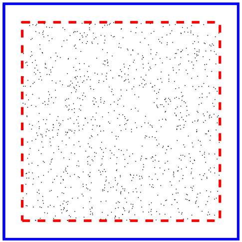

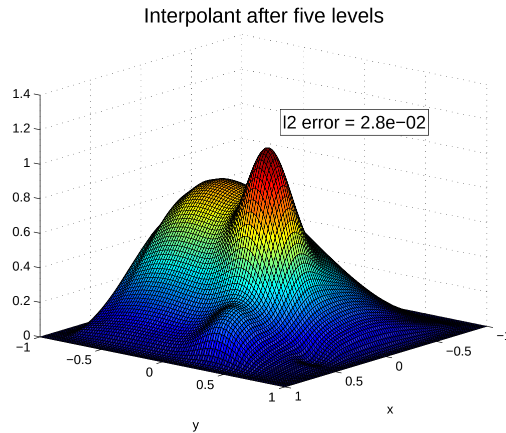

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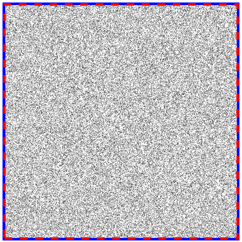
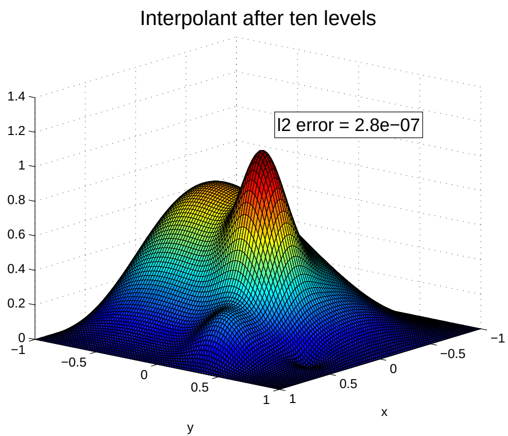


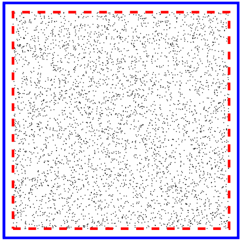


Restricting Interpolation points

Definition (δ-interior of a domain)

Given $\delta > 0$ and a bounded domain $\Omega \subset \mathbb{R}^d$ the δ-*interior* of Ω is

$$\Omega_\delta = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \delta\}$$



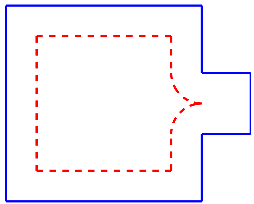
- Want to do $\|e_j\|_{L_2(\Omega)} \leq \|e_j\|_{L_2(\Omega_{\delta_j})} + \|e_j\|_{L_2(\Omega \setminus \Omega_{\delta_j})}$
- For the region near the boundary, no existing theory.
- If the δ-interior domain has smooth boundary we can use previous ideas.

Smoothing interior domain boundary

Problem

Ω is a Lipschitz domain $\not\Rightarrow \Omega_\delta$ is a Lipschitz domain.

Things that can go wrong − Cusp forms



Interweave δ-interior domains with interior cone domains:

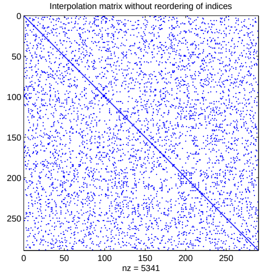
$$K_1 \subseteq \Omega_{\delta_1} \subseteq K_2 \subseteq \Omega_{\delta_2} \subseteq \ldots \subseteq K_n \subseteq \Omega_{\delta_n}$$

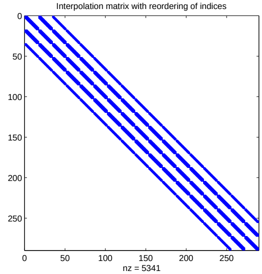
For each level j,

$$\|e_j\|_{L_2(\Omega)} \leq \|e_j\|_{L_2(K_j)} + \|e_j\|_{L_2(\Omega \setminus K_j)}$$

Bounding error near the boundary

Spying on the interpolation matrix





- Make the interpolation matrix banded.
- Estimate convexity of $p \mapsto \left\|A^{-1}\right\|_p$
- With quasi-uniform data set $X_j \subset \Omega_{\delta_j}$ and $\delta_j = \nu h_{X_j,\Omega_{\delta_j}}$ we have

$$\left\|A^{-1}_{X_j,\Phi}\right\|_\infty \leq C\delta_j^d$$

Getting Convergence

- Allows us to bound interpolants,

$$\|s_f\|_{L_\infty(\Omega)} \leq C \|f\|_{L_\infty(\Omega)}$$

and error at each level,

$$\|e_n\|_{L_\infty(\Omega)} \leq D(1+C)^n \|f\|_{\mathcal{H}^\tau(\Omega)}$$

- Using **Conditional Brownian Motion in rapidly exhaustible domains** [Falkner, 1987] gives

$$\mathrm{Vol}\,(\Omega \setminus K_n) \leq C\delta_n$$

Hence,

$$\|e_n\|_{L_2(\Omega\setminus K_n)} \leq C\delta_n^{1/2} \|e_n\|_{L_\infty(\Omega)} \leq C(1+C)^n\delta_n^{1/2} \|f\|_{\mathcal{H}^\tau(\Omega)}$$

New convergence result

Theorem

Let $\Omega \subset \mathbb{R}^d$ be Lipschitz bounded domain and a sequence of denser quasi-uniform data sets $X_1 \subset \Omega_{\delta_1}, X_2 \subset \Omega_{\delta_2} \ldots$. Further, let Φ be a compactly supported radial basis function with reproducing Hilbert space equivalent to $\mathcal{H}^\tau(\mathbb{R}^d)$. Then if the target function $f \in \mathcal{H}^\tau(\Omega)$, $\tau > d/2$ then the multilevel algorithm converges with

$$\|e_n\|_{L_2(\Omega)} \leq \left(C_1 h_n^{\tau-\epsilon_1} + C_2 h_n^{1/2-\epsilon_2} \right) \|f\|_{\mathcal{H}^\tau(\Omega)}$$

for some constants C_1, C_2, ϵ_1 and ϵ_2.

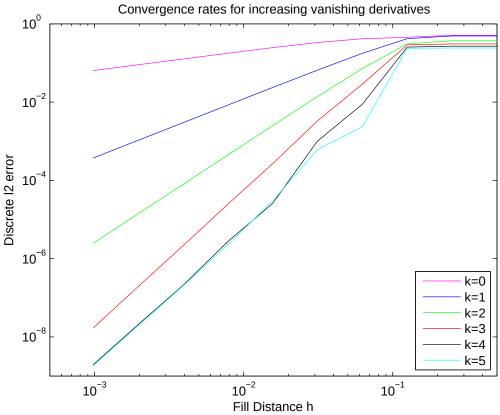
If we have $l \in \mathbb{N}$, $l < \tau - d/2$ vanishing derivatives of the function on the boundary. Then,

$$\|e_n\|_{L_2(\Omega)} \leq \left(D_1 h_n^{\tau-\epsilon_3} + D_2 h_n^{l+1/2-\epsilon_4} \right) \|f\|_{\mathcal{H}^\tau(\Omega)}$$

Apply the multilevel algorithm with Wendland's radial basis function $\phi_{2,1} \in \mathcal{H}^{2.5}(\Omega)$ to

$$f_k(x) = (\sin(\pi x)\sin(\pi y))^k$$

on the domain $\Omega = [-1, 1]^2$.



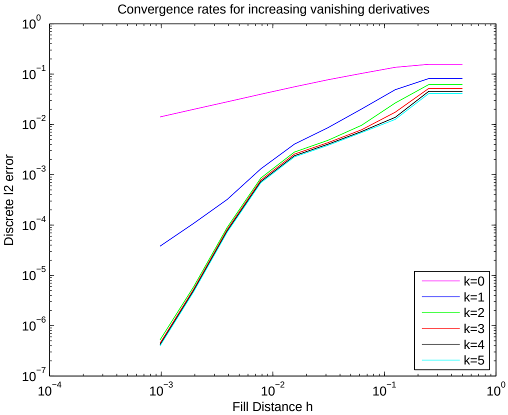
k	l2order	expected
0	0.499	0.50
1	1.51	1.50
2	2.50	2.50
3	3.47	2.50
4	3.40	2.50
5	3.42	2.50

Numerical Simulations

Apply the multilevel algorithm with Wendland's radial basis function $\phi_{2,1} \in \mathcal{H}^{2.5}(\Omega)$ to

$$f_k = (1 - x^2)^k (1 - y^2)^k (tanh(100(x - y)) + 1)/9$$

on the domain $\Omega = [-1, 1]^2$.



Level	l2	l2 order	CG
1	$5.19e-02$	0.00	1
2	$5.19e-02$	0.00	1
3	$1.74e-02$	1.57	5
4	$7.82e-03$	1.16	37
5	$4.32e-03$	0.85	55
6	$2.56e-03$	0.76	59
7	$7.81e-04$	1.71	60
8	$8.16e-05$	3.26	58
9	$5.35e-06$	3.93	54
10	$4.49e-07$	3.57	59

Future Directions

- Convergence Results for variations on the multilevel algorithm.
- Multilevel algorithm backwards.
- Galerkin Methods for solving PDEs.
- Optimising constants.