MATH 7390 SPRING 2022 – SYMPLECTIC RESOLUTIONS

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Contents

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1. A first example: the Springer resolution of the nilpotent cone

...this was in the first couple lectures on the Canvas page and I should write it up here as well at some point.

Part 1. HyperKähler manifolds and hyperKähler reduction

2. Leadup to Kähler and hyperKähler manifolds

The manifolds of interest in this course are in some sense quaternionic, but this sense is slightly delicate. To tease apart the multiple notions, we recall a classic theorem of Riemannian geometry.

Given a connected, simply connected manifold M with Riemannian metric g and a point $m \in M$, we obtain a connected **holonomy group** $\Gamma \leq SO(T_m M)$ by translating vectors along loops from m to m, using the Levi-Civita connection derived from the metric. There are various degenerate possibilities when M is a product, or a symmetric space G/K, but aside from these there are only a few possibilities [Be53]:

- $SO(n)$ itself
- $U(n)$, on Kähler manifolds M^{2n} like $\mathbb{CP}^n$
- $SU(n)$, on Calabi-Yau Kähler manifolds M^{2n}
- $U(n,\mathbb{H})$[1], on "hyperKähler manifolds" M^{4n}
- $U(n,\mathbb{H}) \cdot U(1,\mathbb{H})$, on quaternionic manifolds like $\mathbb{HP}^n$
- G_2, for $\dim M = 7$
- $Spin(7)$, for $\dim M = 8$

The manifolds of interest in this course will be hyperKähler (though we will not end up studying them Riemannianly). What are these?

2.1. Kähler manifolds.

On a Kähler manifold M each real tangent space is given a complex structure, i.e. a notion of multiplying by i; perversely this section of $End_{\mathbb{R}}(TM)$ is usually called J (and satisfies $J^2 = -Id$). Its compatibility with the Riemannian structure is the condition that $J_m \in O(T_m M)$. It should also define a complex, not almost complex, structure; one way to state this is that (M, J) should be locally diffeomorphic to $\mathbb{C}^{n/2}$ in a J-equivariant way.

Together, g and J define on each $T_m M$ a Hermitian structure, whose imaginary part is a symplectic structure $\omega(\vec{v},\vec{w}) := g(\vec{v}, J\cdot\vec{w})$.

2.2. HyperKähler manifolds.

One's likely first guess about how to extend this definition to something quaternionic is correct: we should have three complex structures I, J, K satisfying $IJ = K = -JI$. They end up with three associated real symplectic structures $\omega_I, \omega_J, \omega_K$.

A first thing to note is that this defines a whole S^2 worth of complex structures, $aI+bJ+cK$ where $a^2+b^2+c^2 = 1$, i.e. these particular three aren't picked out.

A second is that the complex-valued symplectic form $\omega_J + i\omega_K$ is I-linear, i.e. M can be regarded as a *complex symplectic* manifold. In fact this is how we will usually view it, neglecting the other complex and symplectic structures.

2.3. Quaternionic manifolds.

On the space of $n \times 1$ matrices over $\mathbb{H}$ we have a left action of $GL_n(\mathbb{H})$ commuting with a right action of $GL_1(\mathbb{H})$, which is to say, the $GL_n(\mathbb{H})$-action is $\mathbb{H}$-linear w.r.t. the right $\mathbb{H}$-module structure.

In the Berger classification, the $U(n,\mathbb{H}) \cdot U(1,\mathbb{H})$ holonomy on a $4n$-manifold includes this right action, which is to say, is *not* linear w.r.t. this right $\mathbb{H}$-module structure. One way to think of this is that there are no global sections I, J, K of $End(TM)$, well-defined from point to point – rather, they get mixed up during holonomy.

[1] This is often denoted $Sp(n)$, making it indistinguishable notationally from the group of transformations of $\mathbb{R}^{2n}$ with its symplectic form. All that these groups have in common is their complexification and we do not feel that that is enough cause to suffer this confusion.

The most basic example of a quaternionic manifold is $\mathbb{HP}^n$, or even $\mathbb{HP}^1 \cong S^4$. We can already see that this cannot be hyperKähler or even Kähler, as it has no H^2 but compact symplectic manifolds must have nontrivial H^2.

Unfortunately (?), our hyperKähler manifolds will essentially never be compact.

3. Symplectic reduction vs. Geometric Invariant Theory reduction

Let (M, g, J, ω) be a Kähler manifold, with the action of a compact Lie group K preserving any two (and therefore all three) of these structures.

3.1. **GIT quotients.** Let R be an $\mathbb{N}$-graded $\mathbb{C}$-algebra, and $M = \operatorname{Proj} R$. In the simplest case R is generated in degrees 0 and 1, and more generally, if R is Noetherian then for some $D \gg 0$ the Veronese subring $\oplus_{n\in\mathbb{N}} R_{nD}$ (which always has the same Proj) will be generated in degrees 0 and 1. So to some extent we can reduce to this case.

The $\mathbb{Z}$-grading is equivalent to a $\mathbb{C}^\times$ action on R, and the nonnegativity of the grading means that $R \to R_0$ is a ring homomorphism. With it we can define $\operatorname{Proj} R = (\operatorname{Spec} R \setminus \operatorname{Spec} R_0)/\mathbb{C}^\times$.

Being generated in degrees 0 and 1 means that R is a quotient of $\mathbb{C}[x_1^{(0)}, \ldots, x_n^{(0)}, \ell_1^{(1)}, \ldots, \ell_m^{(1)}]$ whose Proj is $\mathbb{C}^n \times \mathbb{CP}^{m-1}$; our M is then a closed subvariety of that. In particular, M comes bearing an ample line bundle $\mathcal{L}$ pulled back from $\mathbb{CP}^{m-1}$.

If K acts on M, it does not follow that K's action can be lifted to $\mathcal{L}$, nor is the extension unique. Nonuniqueness is easy to come by; consider $R = \mathbb{C}[\ell]$ and $K = S^1$, where there are $\mathbb{Z}$ many choices of S^1-weight for ℓ, but only one action on $M = pt$. Nonexistence is a little trickier, but consider the action of $SO(3) \cong SU(2)/\pm I$ on $\mathbb{CP}^1 = \operatorname{Proj} \mathbb{C}[\ell_1, \ell_2]$ derived from the action of $SU(2)$ on $\mathbb{C}[\ell_1, \ell_2]$. Since the $\pm I \leq SU(2)$ acts faithfully on $\mathcal{O}(1)$ over $\mathbb{CP}^1$, the action doesn't descend to the quotient $SO(3)$.

For a much subtler nonexistence example, let M be an elliptic curve e.g. $\operatorname{Proj} \mathbb{C}[x, y, z]/\langle x^3 + y^3 + z^3\rangle$, and $K = S^1$ rotating M in some direction. Since K is acting complexly, the action extends to an action of $\mathfrak{k}^{\mathbb{C}}$, which in turn extends to a complexification G of K. One might initially guess $G = \mathbb{C}^\times$, but that does not act on M algebraically (proof: the stabilizer of a point is infinite discrete hence not a subscheme), so cannot be derived from an action of G on R. The other option is $G \cong M$, which *does* act on M algebraically, but cannot be derived from an action of G on the graded ring R because any map $G \to GL(R_m)$ must be constant by Liouville.

There are two sources of nonuniqueness, if we decide that the story begins with K and (M, J). One is the choice of R, or nearly the same thing, of an ample line bundle $\mathcal{L}$ over M. (One can recover R_n in large degrees n as $\Gamma(M;\ \mathcal{O}(n))$, which is as good as can be expected since $\operatorname{Proj} R$ only depends on R's behavior in large degrees.) The other is the lift of the action to $\mathcal{L}$. If we consider two different actions $k \cdot_1 \vec{v}$ vs. $k \cdot_2 \vec{v}$, we can construct a third $k \cdot \vec{v} := k^{-1} \cdot_1 k \cdot_2 \vec{v}$, which leaves M alone but twists each fiber, hence defines a homomorphism $K \to GL_1(\mathcal{L}|_m)$ for each $m \in M$. The set of such homomorphisms is discrete, and is in fact a single point if K is semisimple.

In Geometric Invariant Theory, one *fixes* the action of K on $\mathcal{L}$, or equivalently on (some Veronese of) R. At that point (by the theory of linear actions of compact Lie groups) the action extends to one of an affine-algebraic complexification G of K, preserving J but not g

or ω. Then we define the **GIT quotient**

$$M//G := \text{Proj } (R^G)$$

That's an algebraic definition; what's going on geometrically?

If we worked with Specs, we would have a G-invariant map $\pi\colon \operatorname{Spec} R \to \operatorname{Spec} R^G$, which seems like the right thing to have. Trying to descend that to the Projs we run into a problem;

$$(\operatorname{Spec} R \setminus \operatorname{Spec} R_0) \not\to (\operatorname{Spec} R^G \setminus \operatorname{Spec} R_0^G)$$

because $\pi(\operatorname{Spec} R_0)$ may not land inside $\operatorname{Spec} R_0^G$. Define the **unstable locus**

$$M^{us} := (\pi^{-1}(\operatorname{Spec} R_0^G) \setminus \operatorname{Spec} R_0)/\mathbb{C}^\times \subseteq M$$

so that we can descend π to a G-invariant map

$$\pi\colon M \setminus M^{us} \to M//G$$

The notation is **very** misleading, as M^{us} depends very much on the lift of the action of G to R. Consider $R = \mathbb{C}[x, y, \ell]$ where $G = \mathbb{C}^\times$ acts on the three variables with weights $1, -1, n$. Then $\ker(R \to R_0) = \langle \ell \rangle$, so $\operatorname{Proj} R = (\{(x, y, \ell)\} \setminus \{\ell = 0\})/\mathbb{C}^\times \cong \{(x, y)\}$. We will therefore identify $M = \mathbb{C}^2$.

- $n > 0$. $R^G = \mathbb{C}[xy, y^n \ell]$, and the kernel of its map to R_0^G is $\langle y^n \ell \rangle$. The map $\operatorname{Spec} R \to \operatorname{Spec} R^G$ takes $(x, y, \ell) \mapsto (xy, y^n \ell)$, so $\pi^{-1}(\operatorname{Spec} R_0^G)$ is $\{(x, y, \ell) : y^n \ell = 0\}$. Meanwhile, the kernel of $R \to R_0$ is $\langle \ell \rangle$, so $\pi^{-1}(\operatorname{Spec} R_0^G) \setminus \operatorname{Spec} R_0 = \{(x, y, \ell) : \ell \neq 0, y^n = 0\}$. Dividing by $\mathbb{C}^\times$, we get $M^{us} = \{y^n = 0\} \subset \mathbb{C}^2$. The map $M \setminus M^{us} \to \operatorname{Proj} R^G \cong \mathbb{C}$ exactly divides by the $\mathbb{C}^\times$ action.
- $n < 0$. Now $R^G = \mathbb{C}[xy, x^{-n} \ell]$ and all works the same except that $M^{us} = \{x^{-n} = 0\}$.
- $n = 0$. This is trickier. $R^G = \mathbb{C}[xy, \ell]$, and the kernel of $R^G \to R_0^G$ is $\langle \ell \rangle$. Consequently, $M^{us} = \emptyset$ – there are no unstable points. The map $\operatorname{Proj} R \to \operatorname{Proj} R^G$ is simply $(x, y) \mapsto xy$, which *mostly* divides by $\mathbb{C}^\times$, but there are two non-closed orbits that end up glued together with the orbit in their closure, $\vec{0}$.

When the action on the stable set has only finite stabilizers (so, not like $\vec{0}$ in the above), the reduction is called **stable**. In this case nice things happen, like, the map $M \setminus M^{us} \to M//G$ is onto and exactly divides by G. ***** Better look that up in GIT ***** (Note that this depends on G being reductive, i.e., complexification of compact. For an example where the map isn't onto, consider $SL_2//$ the unipotent group, where the ordinary quotient is the quasi-affine $\mathbb{C}^2 \setminus \vec{0}$.)

Here is a typical example. Consider PGL_2 acting on $(\mathbb{CP}^1)^n$, so the quotient should be somehow n points up to Möbius inversion. Every ample line bundle on $\mathbb{CP}^1$ is of the form $\mathcal{O}(a)$ for some $a > 0$, and on the product we take $\mathcal{O}(a_1) \boxtimes \cdots \boxtimes \mathcal{O}(a_n)$. Then the stability condition turns out to be, a subset $S \subseteq [n]$ of the points is allowed to collide only when $\sum_{i \in S} a_i \leq \frac{1}{2} \sum_{i \in [n]} a_i$. There is a nice map to this "polygon space" (name to be explained soon), first studied by Deligne-Mostow, from $\bar{M}_{0,n}$.

3.2. **Symplectic quotients.** On the GIT side, we needed to extend the action beyond M to an ample line bundle over M. We now take a moment to discuss the corresponding extension on the symplectic side, which is the "Hamiltonian" condition. For the moment we drop $J, \mathfrak{g}$ and just consider K a connected Lie group acting on the real symplectic manifold (M, ω).

Since K preserves ω, we have a Lie algebra homomorphism $\mathfrak{k} \to \mathfrak{symp}(\mathsf{M}, \omega)$ to the symplectic vector fields (those $\vec{v}$ such that the Lie derivative $\mathcal{L}_{\vec{v}}\omega = 0$). This latter fits into an exact sequence of Lie algebras

$$0 \to H^0(M) \xrightarrow{c} (C^\infty(M), \{,\}) \xrightarrow{H} \mathfrak{symp}(M, \omega) \to H^1(M) \to 0$$

where

- H^0, H^1 are abelian Lie algebras,
- the central extension c is the inclusion of constant functions,
- the map H associates the **Hamiltonian vector field**, $f \mapsto H_f := \omega^{-1}(df)$, and
- $\{,\}$ is the **Poisson bracket** $\{f, g\} := \omega(H_f, H_g)$.

When we try to invert H, we takes a symplectic vector field $\vec{v}$ to a 1-form $\omega(\vec{v}, \bullet)$, and the condition that $\vec{v}$ be symplectic is exactly that the 1-form is closed. We then try to invert d, and the measure of failure is the cohomology class of that 1-form – which accounts for the final map to $H^1(M)$.

We can then ask whether $\mathfrak{k} \to \mathfrak{symp}(M, \omega)$ factors through $(C^\infty(M), \{,\})$. The first obstruction is the $H^1(M)$ – the vector fields may not be **Hamiltonian**, meaning, in the image $\mathit{Ham}(M, \omega)$ of H. This issue arose in the example of S^1 acting on an elliptic curve.

Of course one can sidestep that one by assuming $H^1(M) = 0$, or, that $\mathfrak{k} = \mathfrak{k}'$ because then its map to the abelian Lie algebra $H^1(M)$ must be trivial. However we enforce it, at that point we meet a canonical central extension

$$\begin{array}{ccccccccc} 0 & \to & H^0(M) & \hookrightarrow & \mathfrak{k}' & \twoheadrightarrow & \mathfrak{k} & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & H^0(M) & \hookrightarrow & (C^\infty(M), \{,\}) & \twoheadrightarrow & \mathit{Ham}(M, \omega) & \to & 0 \end{array}$$

of $\mathfrak{k}$. This extension can be nontrivial, as in the famous case of $K = M = T^*\mathbb{R}^n$ acting on itself by translation, where the extension is the Heisenberg group. Another fascinating example comes with $LK := \mathit{Map}(S^1, K)$ acting on $LK/K \cong \Omega K := \mathit{Map}_\bullet(S^1, K)$ for K a compact group, where ΩK has an easily guessed symplectic structure; in this case the extension is the "affine Lie group".

If $\mathfrak{k}$ is finite-dimensional semisimple, then its central extensions are necessarily trivial, i.e. the map automatically factors. Even when the action doesn't factor, we can simply decide that $\mathfrak{k}'$ was the Lie algebra we wanted from the beginning.

So assume hereafter that we have a Lie algebra homomorphism $\mathfrak{k} \to C^\infty(M)$. We can rewrite this first as $\mathfrak{k} \times M \to \mathbb{R}$ (linear in the $\mathfrak{k}$ argument), then, as $M \to \mathfrak{k}^*$. This map, denoted usually by Φ (or occasionally by μ), is called a **moment map** for the action and the action is called **Hamiltonian**. It is K-equivariant[2] and Poisson, which characterizes it.

Having discussed nonexistence, we now comment on nonuniqueness. The difference between two different lifts $\mathfrak{k} \to C^\infty(M)$ maps to 0 in $\mathfrak{symp}(M, \omega)$, hence, hits the locally constant functions by the exactness. Put another way, $\Phi_1 - \Phi_2$ is locally constant on M, and takes values in the perp of the commutator subalgebra.

The **symplectic reduction** of (M, ω) by the compact group K is defined as

$$M//K := \Phi^{-1}(0)/K$$

[2]If one picks only a linear lift $\mathfrak{k} \to C^\infty(M)$, then the resulting Φ is not K-equivariant, leading to a theory of "nonequivariant moment maps". The existence of the central extension $\mathfrak{k}'$ above shows that this is a wholly unnecessary and indeed unnatural theory; just let Φ take values in $(\mathfrak{k}')^*$.

As before, the notation is extremely misleading, in that it doesn't visibly depend on the choice of Φ.

Consider again the example of $\mathsf{PGL}(2)$ acting on $\prod_{i=1}^n(\mathbb{CP}^1, \mathcal{O}(a_i))$, but now we consider it symplectically, as $SO(3)$ acting on $M = \prod_{i=1}^n(S^2$ with area $a_i)$. The moment map exists uniquely (since $SO(3)$ is semisimple) and turns out to be "endpoint", if we interpret M as a space of n-step paths in $\mathbb{R}^3$ starting at the origin. Then $M//K$ is the space of n-step polygons with edge-lengths $a_1, \ldots, a_n$, considered up to rotation.

3.3. **Playing with moment maps.** It is easy to calculate that for $G \xrightarrow{\beta} H \circlearrowright (M, \omega)$, we can compute Φ_G as $\beta^* \circ \Phi_H$ where β^* is the induced map $\mathfrak{h}^* \to \mathfrak{g}^*$. An important case is $\beta : G \to G^2$ the diagonal inclusion, with which one can determine that the moment map for a product action is the sum of the two individual moment maps.

Recall that a compact connected Lie group G is of the form $(G' \times Z(G)_0)/\Gamma$ where G' (the commutator subgroup) is semisimple, $Z(G)_0$ is abelian and connected hence a torus, and Γ is finite and central. If $\dim Z(G)_0 = d$, then there is a d-dimensional space of possible G-equivariant maps $pt \to \mathfrak{g}^*$, and every one of them serves as a choice of moment map.

Fix $\lambda \in (\mathfrak{g}^*)^G$ and let G act on a point with moment map $\rho : pt \to -\lambda$ (the minus to be explained in a moment). Let $\Phi : M \to \mathfrak{g}^*$ be a moment map, and $\Phi' : M \times pt \to \mathfrak{g}^*$ the sum of the two moment maps. Then

$$\Phi'^{-1}(0) \cong \Phi^{-1}(\lambda)$$

and we use this to define $M//_\lambda G$. There is then plenty to study about how $M//_\lambda G$ changes as one changes λ, in particular, passing through critical values of Φ.

3.4. **Comparing the two setups.** The notations $M//K, M//G$ suggest that the two reductions should match, in some sense. While we won't prove this theorem, we will at least get to the point of stating it precisely.

Notationally, each begins with a group (K or its complexification G) acting on a space M. On the algebraic side, we insist on realizing M as $\operatorname{Proj} R$, or what is almost the same, picking an ample line bundle $\mathcal{L}$ over M. On the symplectic side, we pick a symplectic form ω on M. In a first attempt to relate these, we note that $c_1(\mathcal{L})$ lives in $H^2(M; \mathbb{Z})$, whereas ω is a lift of the cohomology class $[\omega]$ living in $H^2(M; \mathbb{R})$.

In our second attempt, we put a Hermitian connection α on $\mathcal{L}$ (one with holonomy in $U(1)$), and demand that its curvature be ω. Now when $X \in \mathfrak{k}$ acts on M, the lift of the action defines a vector field on $\mathcal{L}$, which we can pair against the connection 1-form α. The result is a function on $\mathcal{L}$ that descends to M, i.e. we recover the comoment map $\mathfrak{k} \to C^\infty(M)$. So we have found further parallelism between the lift of the action (on the GIT side) and the moment map (on the symplectic side).

This "Hermitian connection on the prequantum line bundle" idea in fact lets us soup up most of the diagram of Lie algebras from before, to groups:

$$\begin{array}{ccccccc} & & & & & & K \\ & & & & & & \downarrow \\ 1 & \to & \mathrm{Aut}_0(\mathcal{L}, \alpha) & \to & Aut(\mathcal{L}, \alpha) & \to & Symp(M, \omega) \end{array}$$

The middle group is the space of maps $\mathcal{L} \to \mathcal{L}$ that fit into a commuting triangle with the projections to M, such that the pullback of α is α. Any such map induces a map from M to itself that preserves the curvature of α, hence is a symplectomorphism. The kernel

$\mathrm{Aut}_0(\mathcal{L}, \alpha)$ is bundle automorphisms preserving α, hence, a single circle's worth of freedom over each component of M. I am not sure how to describe the cokernel.

3.5. Comparing the two reductions.

Theorem 3.1 (Kirwan/Kempf–Ness). *Let $M \subseteq \mathbb{P}V$ be a smooth complex projective variety, with an action of a compact connected Lie group K on R. So $M = \mathrm{Proj}\, R$ for R some quotient of $\mathrm{Sym}(V^*)$, and the action of K on R extends to that of its complexification G. Assume K preserves a fixed Hermitian form on V, inducing the Fubini-Study Hermitian metric on $\mathbb{P}V$ and on M. As explained above, the action of K on (M, ω) is Hamiltonian with a moment map Φ.*

Then $\Phi^{-1}(0) \subseteq M \setminus M^{us}$, and the induced composite map $\Phi^{-1}(0)/K \to (M \setminus M^{us})/K \to M//G$ is a homeomorphism.

The relation between $M \setminus M^{us}$ and $\Phi^{-1}(0)$ can be made more precise, in either of two ways.

(1) $m \in M^{us} \iff (K^{\mathbb{C}} \cdot m) \cap \Phi^{-1}(0) = \emptyset$
(2) M^{us} is the "attracting set of 0" for gradient flow w.r.t. the almost Morse function $|\Phi|^2$.

A Morse function has no eigenvalues zero of its Hessians. A Morse-Bott function is allowed to have some zero eigenvalues, but only along the critical sets (not more). A Morse-Bott-Kirwan function (like $|\Phi|^2$) is allowed to have yet more zero eigenvalues, but only along the attracting sets, not the repelling sets. In particular the negative of a Morse-Bott function is Morse-Bott, but of a Morse-Bott-Kirwan need not be Morse-Bott-Kirwan.

3.6. Affine quotients.

Need to talk about the map from projective quotient to affine quotient...

define the **core** in the conical case and point out that it's a deformation retract

4. An important example: toric varieties associated to polytopes

Let ω be the imaginary part of the standard Hermitian form on $\mathbb{C}^n$, and let $T \leq U(n)$ be the diagonal unitary matrices. The moment map for this simple action is simple:

$$\Phi : (x_1, \ldots, x_n) \mapsto (|x_1|^2, \ldots, |x_n|^2)$$

There is not much to say here as regards reduction. The moment polytope is the positive orthant, and reduction in there (i.e. $\Phi^{-1}(\lambda)/T^n$) gives a point. The action of $T^{\mathbb{C}}$ has one orbit for each of the 2^n faces of the orthant, and Φ gives the correspondence.

If we take $S^k \leq T^n$, then $\mathbb{C}^n//_\lambda S$ carries a Hamiltonian action of T/S. What does it look like?

We are considering fibers of the map $\mathbb{R}^n_{\geq 0} \hookrightarrow \mathbb{R}^n \cong \mathfrak{t}^* \twoheadrightarrow \mathfrak{s}^*$. It is easy to see that every $(n-k)$-polyhedron P with n facets can be constructed as such a fiber. Since we want $\mathfrak{s}$ to integrate to a torus, we need each facet of P to be perpendicular to some integer vector. The resulting $M := \mathbb{C}^n//_\lambda S$ is of dimension $n-k$, bears the action of the $(n-k)$-torus T/S, and has $\Phi_{T/S}$-moment polytope P.

It is easy to understand the unstable set – the union of those coördinate spaces in $\mathbb{C}^n$ whose corresponding faces in $\mathbb{R}^k_{\geq 0}$ don't intersect the λ fiber of $\mathbb{R}^n_{\geq 0} \to \mathfrak{s}^*$. Check this out in the $n = 2$, $\dim S = 1$ example we worked out in §3.1.

It is again the case that the orbits of $(T/S)^{\mathbb{C}}$ on M are in correspondence with the faces of P, with $\Phi_{T/S}$ giving the correspondence.

These varieties M are cases of **toric varieties**. In the general definition, one asks that T acts on M with a dense orbit, and that M be normal. Not all authors demand the normality; it is there in order to be able to say that the combinatorics of P determines M. (Otherwise one can e.g. put arbitrarily bad cusps at $0, \infty \in \mathbb{P}^1$ and still have the moment polytope be an interval.)

Not every toric variety can be obtained this way, i.e., from a polyhedron. For a nonexample, start with an octahedron P, and imagine stretching its top vertex and bottom vertex out to edges, resulting in P'. Then $TV(P)$ and $TV(P')$ are isomorphic once once removes the corresponding subvarieties (two points vs. two $\mathbb{P}^1$s, respectively). Now glue $TV(P)$ minus its bottom point to $TV(P')$ minus its top $\mathbb{P}^1$, and the resulting toric variety can't come from a polytope.

Of course there are entire books on toric varieties, and here is one reason why. For fixed S but generic λ, the resulting polytope is "simple", meaning that the minimum number of facets meet at each vertex. The toric variety is "rationally smooth" (its links have the rational homology of spheres), so its rational cohomology satisfies Poincaré duality and even hard Lefschetz. The dual polytope is "simplicial" (every face a simplex), and every simplicial polytope arises this way (up to combinatorial equivalence). Then one can, and Stanley did, use the properties of $H^*(TV(P); \mathbb{Q})$ to restrict the f-vector $(\ldots, f_i := \#\{i\text{-dimensional faces}\}, \ldots)$ of the polytope.

5. HyperKähler reduction

Let $G_{\mathbb{R}}$, a compact Lie group (that would be confusing to call K), act on $(M, g, I, J, K, \omega_I, \omega_J, \omega_K)$, preserving all the structures. (In fact it is enough to preserve the metric g [Besse, ???].) Let Φ_I, Φ_J, Φ_K be moment maps for the three choices of symplectic form. The easiest place to find such a special setup is on $\mathbb{H}^n \cong T^*\mathbb{C}^n$, where $\mathbb{C}^n$ is a unitary representation of $G_{\mathbb{R}}$.

In this setup, the analogue of symplectic reduction is

$$M///G_{\mathbb{R}} := \left(\bigcap_{a \in \{I,J,K\}} \Phi_a^{-1}(0) \right) \Big/ G_{\mathbb{R}}$$

(Perhaps it should have four slashes, since the dimension goes down by $4 \dim G_{\mathbb{R}}$, but three is standard.) This suggests that one should sum the three moment maps together into one map, and Nick Proudfoot informs me that the natural target for that map is $(\mathrm{Im}\, \mathbb{H}) \otimes \mathfrak{g}_{\mathbb{R}}$.

Recall that $\omega := \omega_J + i\omega_K$ is a *holomorphic* (not Kähler) symplectic form on the complex manifold (M, I). Similarly, $\Phi := \Phi_J + i\Phi_K \colon M \to \mathfrak{g}$ (the complexification) is a moment map for the action of the *complex* Lie group G. Since that only puts two of the moment maps together, we only get to rewrite as

$$\Phi^{-1}(0)//G_{\mathbb{R}} \qquad \text{symplectic reduction using } (\omega_I, \Phi_I)$$

At that point (and assuming ω_I is the curvature of a connection on an ample line bundle) we can use Kirwan/Kempf-Ness to rewrite **in an entirely complex algebraic way:**

$$\Phi^{-1}(0)//G \qquad \text{GIT quotient}$$

Since this is the way we will most often be working with these, we spell out the steps.

(1) On M, we have an algebraic action of the complex group G preserving an algebraic symplectic form ω, with an algebraic moment map $M \to \mathfrak{g}^*$. Take the zero level set, a subvariety in M.
(2) The action of G on there usually isn't free. Use the additional choice of ω_I to determine a "stable set".
(3) Take the quotient of that by G. If the reduction involves properly semistable points, we may have to Hausdorffify afterward. (We will not always be able to avoid this.)

6. Hypertoric varieties

We start by following [PW].

Fix a d-dimensional torus T and a homomorphism $\rho : T^n \twoheadrightarrow T$, whose coördinates induce n integral vectors $a_i \in \mathfrak{t}$. Let $\mathcal{A}$ be the **central hyperplane arrangement** $\bigcup_{i=1}^n a_i^\perp \subseteq \mathfrak{t}^*$, and $\widetilde{\mathcal{A}} := \bigcup_i \{\vec{v} \in V \colon \langle a_i, \vec{v}\rangle = c_i\}$ a **simplification** of it, meaning that the intersection of any k hyperplanes is codimension k (when nonempty).

From this we define two $2d$-dimensional **hypertoric varieties**

$$\mathfrak{M}(\mathcal{A}) := T^*\mathbb{C}^n///_0 \ker(\rho) \qquad \mathfrak{M}(\widetilde{\mathcal{A}}) := T^*\mathbb{C}^n///_{\vec{c}} \ker(\rho)$$

each bearing the action of $T \cong T^n/\ker(\rho)$. The condition on $\vec{c}$ ensures that $\mathfrak{M}(\widetilde{\mathcal{A}})$ is an orbifold. The natural map $\mathfrak{M}(\widetilde{\mathcal{A}}) \to \mathfrak{M}(\mathcal{A})$ is an orbifold resolution of singularities.

This $T^*\mathbb{C}^n$ contains 2^n natural Lagrangian copies of $\mathbb{C}^n$, whose individual reductions (when nonempty) are d-dimensional toric varieties X_P, whose polytopes P are the facets in the hyperplane arrangement. Assume for convenience that there is a compact facet Q, and use it to make a choice of $\mathbb{C}^n \subseteq T^*\mathbb{C}^n$, and of toric subvariety $X_Q \subseteq \mathfrak{M}(\mathcal{A})$. We can define an extra *non-symplectic* circle action acting with weight 0 on these coördinates and weight 1 on the complementary coördinates. On the reduction, this circle induces a retraction of $\mathfrak{M}(\mathcal{A})$ to a point and of $\mathfrak{M}(\widetilde{\mathcal{A}})$ to $\bigcup_{Q \text{ compact}} X_Q$. In particular $H^*(\mathfrak{M}(\widetilde{\mathcal{A}})) \cong H^*\left(\bigcup_{Q \text{ compact}} X_Q\right)$ and likewise if we extend to equivariant cohomology and/or K-theory.

In fact the choice of $\vec{c}$ lives in $\mathbb{R}^3 \otimes \mathfrak{t}^*$, within which the non-simplifications form a set of real codimension 3. As such the cohomology is canonically isomorphic for different $\vec{c}$. (If one is only interested in real $\vec{c}$ (where the walls become codim 1), then one can include the complex conjugation action, and the Z_2-equivariant K-theory is indeed sensitive to the actual hyperplane arrangement [HP].) In particular we should be able to give a formula for its Betti numbers independent of $\vec{c}$.

Let Δ be the simplicial complex on $[n]$ where $S \subseteq [n]$ is a face if the corresponding $\{a_i : i \in S\}$ are linearly independent, called the **matroid complex**. The **f-vector** has $f_i := \#\{S \in \Delta : \#S = i\}$, and from it we define the **h-polynomial** by

$$\sum_i h_i q^i = \sum_i f_i\, q^i (1-q)^{d-i}$$

The h-polynomial is a specialization of the two-variable "Tutte polynomial" of the matroid, the universal invariant of matroids obeying a certain recurrence.

While the h-polynomial is definable in this way for any simplicial complex, it rarely has nonnegative coefficients. One situation (including this) in which it does is when the simplicial complex is shellable. In the case of a matroid realizable over $\mathbb{Q}$, as here, the h-polynomial has a topological interpretation:

Theorem 6.1. [BD, HS] *This h-polynomial is the Poincaré polynomial of $\mathfrak{M}(\widetilde{\mathcal{A}})$ (where q has degree 2, insofar as $\mathfrak{M}(\widetilde{\mathcal{A}})$ has only even-dimensional cohomology).*

For an example, consider two parallel lines L, M crossing two other lines P, Q in $\mathbb{R}^2$. Depending on whether P, Q cross between L, M or not, the core of the hypertoric variety is either $\mathbb{P}^2 \cup_{\mathrm{pt}} \mathbb{P}^2$ with Poincaré polynomial $2(1+q+q^2)-1$ or $\mathbb{P}^2 \cup_{\mathbb{P}^1} F_1$ with Poincaré polynomial $(1+q+q^2)+(1+q)^2-(1+q)$ (which is of course the same). The corresponding matroid complex has faces $\emptyset, L, M, P, Q, LP, LQ, MP, MQ, PQ$ with h-polynomial $(1-q)^2+4q(1-q)+5q^2 = 1+2q+2q^2$.

In [PW] they give a similar but more subtle calculation of the intersection Poincaré polynomial for the singular variety $\mathfrak{M}(\mathcal{A})$, in terms of the "broken circuit complex" of the matroid. Then the Decomposition Formula applied to the affinization map recovers (a specialization of) the Kook-Reiner-Stanton convolution formula for Tutte polynomials.

Part 2. Hilbert schemes of points in the plane

- fixed points $\leftrightarrow$ partitions
- isotropy action at tangent spaces, moment polytopes for $n \leq 4$
- affinization is Hilbert-Chow map to $(\mathbb{C}^2)^n/S_n$

Part 3. Nakajima quiver varieties

We follow [?] fairly closely in this part.

The definition is simple enough, now that we have set up hyperKähler reduction. Let $\Gamma = (V, E, V \to \{\boxed{\text{framed}}, \text{gauged}\})$ be a directed graph or **quiver** with a two-coloring[3] on the vertices. There are three more structures on the vertices:

(1) a **dimension vector** $\vec{d}\colon V \to \mathbb{N}$, that we will sometimes separate into $\vec{d}^{\text{framed}}$ and $\vec{d}^{\text{gauged}}$,
(2) a **complex moment map level** $\lambda_{\mathbb{C}}\colon V_{\text{gauged}} \to \mathbb{C}$ which we will usually take to be 0 (in which case we omit it from the notation), and
(3) a **real moment map level** $\lambda_{\mathbb{R}}\colon V_{\text{gauged}} \to \mathbb{R}$ which we will usually take to be all positive (in which case we omit it from the notation).

Then the **(Nakajima) quiver variety** is defined as

$$\mathcal{M}\left(\Gamma, \vec{d}, \lambda_{\mathbb{C}}, \lambda_{\mathbb{R}}\right) := \left(T^* \prod_{E} \mathrm{Hom}(\mathbb{C}^{d_{t(e)}}, \mathbb{C}^{d_{h(e)}})\right) /\!/\!/\!/_{\lambda_{\mathbb{C}} \oplus \lambda_{\mathbb{R}}} \prod_{V_{\text{gauged}}} GL(\mathbb{C}^{d_v})$$

[3]Contrary to some definitions of two-coloring, there is no prohibition on adjacent vertices having the same color.

where $\lambda_{\mathbb{R}}, \lambda_{\mathbb{C}}$ are interpreted as realified/complexified 1-dimensional characters of $\prod_{V_{\text{gauged}}} GL(\mathbb{C}^{d_v})$. This variety bears an action of the **flavor group** $\prod_{V_{\text{framed}}} GL(\mathbb{C}^{d_v})$.

We now perform a sequence of reductions of the description.

- Note that $T^* \mathrm{Hom}(V, W) \cong \mathrm{Hom}(V, W) \times \mathrm{Hom}(W, V)$, using the trace pairing. So why do we orient Γ? The answer is that the orientation enters only *very* briefly in defining the sign of the symplectic form on $\mathrm{Hom}(V, W) \times \mathrm{Hom}(W, V)$, and reversing an edge gives a canonically isomorphic variety. As such people often don't bother orienting the quiver. At the very least we can insist that each gauged-$\boxed{\text{framed}}$ edge points gauge$\to\boxed{\text{framed}}$.
- Any edge between $\boxed{\text{framed}}$ vertices just multiplies the space before, and after, by a vector space. So we lose very little if we insist that no edges connects $\boxed{\text{framed}}$ to $\boxed{\text{framed}}$.
- We can replace any $\boxed{\text{framed}}$ vertex connecting to m different gauged vertices (with multiplicity potentially, if we don't forbid having multiple edges between vertices) by m separate copies, each attached to a single gauged vertex. Specifically, the space being quotiented and its group action don't change at all.
- Finally, if we add or remove vertices v with $d_v = 0$ the presentation again doesn't change. So it is common to assume that there is indeed a framed vertex attached to every gauged vertex. At that point the quiver is completely determined by the subgraph of gauged vertices and is called a **Nakajima quiver**.

  Graphically, we will usually erase all framed vertices that have $d_v = 0$.

On a Nakajima quiver we can regard $\vec{d}^{\text{framed}}, \vec{d}^{\text{gauged}}, \lambda_{\mathbb{R}}, \lambda_{\mathbb{C}}$ as living in the same vector space (since the vertices are in bijection). This vector space comes with coördinates with which we can define the dot product $\langle , \rangle$.

There is a trick due to Crawley-Boevey, where each $\to \boxed{n}$ is replaced by n parallel edges to $\boxed{1}$, and the $\boxed{1}$ vertices are all amalgamated into one. At that point it no longer matters if we turn the unique framed vertex into a gauged vertex. In this way one can embed the theory of quiver varieties with framed vertices into that of quiver varieties with only gauged vertices. This won't be very useful for us – in particular, we will want to relate different quiver varieties with the same quiver and this messes up the quiver – so we won't perform this reduction except for one argument in the next subsection.

One way to understand linear representations of a group G is to consider G as a category $\mathcal{C}$ with one object $*$ where $\mathrm{Aut}(*) = G$, and look at functors $\mathcal{C} \to$ **Vec**. (G-intertwiners are then natural transformations of these functors.) It is then possible, but not often done, to consider the moduli space of G-actions on a *fixed* vector space. What we have on hand is a directed graph Γ instead of a category, but there is a forgetful functor **Cat** $\to$ **Digraph** and a left adjoint Free : **Digraph** $\to$ **Cat** taking a directed graph to its category of directed paths under concatenation. Then the moduli spaces we are considering are functors from $\mathsf{Free}(\Gamma) \to$ **Vec** where the map on objects is fixed. This nicely foreshadows the situation where we fix the map on framed objects but not on gauged.

6.1. **Stability conditions.** Fix a quiver (for us the Nakajima quiver), and consider representations of this quiver. We can define a **subrepresentation** $S \le R$ as a choice of subspace as each vertex, such that the linear maps ϕ_e take $\phi_e(S_{t(e)}) \le S_{h(e)}$. Given our choice $\lambda_\mathbb{R}$ of real moment map level, define the **slope** of $S \leq R$ as $\mathtt{slope}_{\lambda_\mathbb{R}}(S) := \langle \lambda_\mathbb{R}, \overrightarrow{\dim}_{\text{gauged}} S\rangle \Big/ \langle (1,1,\ldots,1), \overrightarrow{\dim}_{\text{gauged}} S\rangle$. A representation R is called **semistable** resp. **stable** if for every nonzero proper subrepresentation, $\mathtt{slope}_{\lambda_\mathbb{R}}(S) \leq \mathtt{slope}_{\lambda_\mathbb{R}}(R)$ resp. $\mathtt{slope}_{\lambda_\mathbb{R}}(S) < \mathtt{slope}_{\lambda_\mathbb{R}}(R)$. The main result of [K] – in the unframed case – is that these slope inequalities define GIT-(semi)stability in the quotient defining the quiver varieties. (Since we are defining the variety using a GIT quotient, we need $\lambda_\mathbb{R}$ integral, so we refer to it as $\lambda_\mathbb{Z} = \lambda_\mathbb{R}$.) When a representation R is properly semistable, it comes naturally filtered by the subrepresentations with maximal slope, and R gets identified with $\mathtt{gr}\ R$ under the quotient by GIT S-equivalence.

To understand the case of framed quivers, we make use of the Crawley-Boevey trick, amalgamating all the framed vertices into one $\boxed{1}$. However, the character $\lambda_\mathbb{Z}$ acting at the gauged vertices needs to act as minus $\sum_{\nu \text{ gauged}} d_\nu \lambda_\mathbb{Z}(\nu)$ at the framed vertex in order for this larger action to have semistable points at all. *** **Ought to check that** *** The whole representation R then has slope 0, and there are two kinds of subrepresentations: those involving the Crawley-Boevey vertex, and those not.

If we now assume $\lambda_\mathbb{R} > 0$ in each entry, then those representations have slope < 0 and > 0, respectively. Hence, **a representation is stable iff it has no subrepresentations supported on the gauged vertices.**

Sometimes it will be slightly more convenient to work with $\lambda_\mathbb{R} < 0$. Then instead of forbidding proper subrepresentations contained in the kernel of the maps gauged$\to\boxed{\text{framed}}$, we forbid proper subrepresentations containing the image of the maps $\boxed{\text{framed}}\to$gauged. (The two concepts are related by dualizing *all* the maps.)

6.2. **Main example:** $\boxed{n} \longleftarrow a_d \longleftarrow \ldots \longleftarrow a_1$. ...did this in class...

6.3. **Another example: the Jordan quiver.** The Jordan quiver is a single vertex with a self-loop, so called because its moduli space of representations is indexed by Jordan canonical forms. The corresponding Nakajima quiver has a framed vertex that we will only give dimension $\boxed{1}$, and a gauged vertex labeled n. The complex moment map level set is

$$\{(X \in \mathrm{End}(\mathbb{C}^n), Y \in \mathrm{End}(\mathbb{C}^n), i \in \mathrm{Hom}(\mathbb{C}, \mathbb{C}^n), j \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C})\colon [X,Y] + ji = 0\}$$

The 1×1 matrix ij has $\mathrm{Tr}(ij) = \mathrm{Tr}(ji) = -\mathrm{Tr}([X,Y]) = 0$, hence is zero; therefore the rank ≤ 1 matrix ji is nilpotent (of order ≤ 2).

In this example we'll take $\lambda_\mathbb{R} < 0$ (on the one gauged vertex), which is to say, the stability condition says that $i(1)$ must generate $\mathbb{C}^n$ as a $\mathbb{C}\langle X, Y\rangle$-module.

Now we observe a result of Guralnick from '79 (slicker proof by Rudakov given in [EG, 12.7]), that $\mathtt{rank}([X,Y]) \leq 1$ implies that X, Y can be made simultaneously upper triangular. It is obviously enough to show that X, Y preserve a nontrivial subspace, then to use induction. First, replace X by X minus an eigenvalue without loss of generality. If $\ker X \leq \ker[X,Y]$ then $\forall \vec{v} \in \ker X, XY\vec{v} = (YX + [X,Y])\vec{v} = \vec{0}$ so $\ker X$ is such a subspace. (It might be all of $\mathbb{C}^n$ but then the result is easy.) Otherwise $\exists \vec{z} \in \ker X \setminus \ker[X,Y]$. Then $\vec{0} \neq [X,Y]\vec{z}$ therefore spans the assumed-1-dimensional $\mathrm{Im}([X,Y])$. But now $[X,Y]\vec{z} = XY\vec{z} - YX\vec{z} = XY\vec{z} \in \mathrm{Im}(X)$, so $\mathrm{Im}([X,Y]) \leq \mathrm{Im}(X)$. Meanwhile $YX\vec{w} = (XY - [X,Y])\vec{w} \in \mathrm{Im}(X) + \mathrm{Im}([X,Y]) = \mathrm{Im}(X)$, so $\mathrm{Im}(X)$ is Y-invariant, and serves as the desired subspace. Whew!

Now that ji is strictly upper triangular, there must be some m such that i vanishes in the left m entries, j vanishes in the bottom $n-m$ entries. But then $\mathbb{C}^m \ni i(1)$ would be a proper $\mathbb{C}\langle X, Y\rangle$-submodule, contrary to the stability condition. Hence $m = n$, so $j = 0$, so $[X, Y] = 0$.

Finally, we want $GL(n)$-invariant quantities associated to (X, Y, i), so we consider the kernel of $\mathbb{C}[X, Y] \to \mathbb{C}^n$, $p(X, Y) \mapsto p(X, Y) \cdot i(1)$. This defines a point in $\mathrm{Hilb}_n(\mathbb{C}^2)$, and that is in fact the quiver variety.

6.4. **Dimension.**

Proposition 6.2. *Let* C *be the* **Cartan matrix** *of* Γ*, namely* $2I$ *minus the adjacency matrix. Let* $\vec{v} = \vec{d}^{gauged}, \vec{w} = \vec{d}^{framed}$ *since this is usual in the literature. Then*

$$\dim \mathcal{M}(\Gamma, \vec{d}) = 2(\langle \vec{v}, \vec{w}\rangle - \langle \vec{v}, (C - I)\vec{v}\rangle)$$

Proof.

$$\begin{aligned}
\dim \mathcal{M}(\Gamma, \vec{d}) &= \dim \left(T^* \prod_{i\in V} \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{w_i}) \times T^* \prod_{E} \mathrm{Hom}(\mathbb{C}^{v_{t(e)}}, \mathbb{C}^{v_{h(e)}}) \right) - 2 \dim \prod_{i\in V} GL(v_i) \\
&= \sum_{i\in V} 2v_i w_i + \sum_{e\in E} 2v_{t(e)}v_{h(e)} - 2\sum_{i\in V} v_i^2 = 2\left(\sum_{i\in V} v_i w_i - \left(\sum_{i\in V} v_i^2 - \sum_{e\in E} v_{t(e)}v_{h(e)}\right)\right) \\
&= 2(\langle \vec{v}, \vec{w}\rangle - \langle \vec{v}, (C - I)\vec{v}\rangle)
\end{aligned}$$

□

Assume Γ has no loops (though it may have cycles). There is an action r_α of the corresponding Coxeter group on the space of dimension vectors, replacing a gauge label D at vertex α by sum(its neighbors, including the attached framed vertex)-D. There is a corresponding action on the spaces $\{\lambda_\mathbb{R}\}, \{\lambda_\mathbb{C}\}$ (with no input from the framed vertices).

Theorem 6.3. [N] *Let* α *be a gauged vertex with no self-loops. Then*

$$\mathcal{M}(\Gamma, r_\alpha \cdot \vec{d}, r_\alpha \cdot \lambda_\mathbb{C}, r_\alpha \cdot \lambda_\mathbb{R}) \cong \mathcal{M}(\Gamma, \vec{d}, \lambda_\mathbb{C}, \lambda_\mathbb{R})$$

In particular, this gives an easy way to prove a quiver variety is empty – reflect its dimension vector until some dimension goes negative.

6.5. **Circle fixed points.**

We consider the action of the circle $\delta\colon t \mapsto \prod_{v \text{ framed}} \mathrm{diag}(1^d t^e)$ on $Gr(k, \mathbb{C}^{d+e})$. It is easy to prove that a k-plane W is δ-invariant iff $W = (W \cap \mathbb{C}^d) \oplus (W \cap \mathbb{C}^e)$. The cleanest statement involves the disconnected scheme of subspaces of *all* dimensions in $(d + e)$-space: it says the direct sum map

$$Gr(*, \mathbb{C}^d) \times Gr(*, \mathbb{C}^d) \xrightarrow{\oplus} Gr(*, \mathbb{C}^{d+e})$$

is exactly the inclusion of the δ-fixed point set.

That was a warmup for the analogous situation with quiver varieties. If $\vec{d}, \vec{e}$ are two dimension vectors for the same colored quiver, then there is a natural map

$$\mathcal{M}(\Gamma, \vec{d}, \lambda_\mathbb{C}, \lambda_\mathbb{R}) \times \mathcal{M}(\Gamma, \vec{e}, \lambda_\mathbb{C}, \lambda_\mathbb{R}) \quad \xrightarrow{\oplus} \quad \mathcal{M}\left(\Gamma, \vec{d} + \vec{e}, \lambda_\mathbb{C}, \lambda_\mathbb{R}\right)$$

whose image is invariant under, and indeed a fixed-point component for, the flavor group circle $\delta\colon t \mapsto \prod_{v \text{ framed}} \mathrm{diag}(1^{d_v} t^{e_v})$. Working in the opposite direction, write $\vec{f}^{\text{framed}}$ as

$\vec{d}^{\text{framed}} + \vec{e}^{\text{framed}}$ and define the same circle δ. To best organize its fixed-point components, we define the disconnected **quiver scheme**

$$\mathcal{M}\left(\Gamma, \vec{f}^{\text{framed}}, \lambda_{\mathbb{C}}, \lambda_{\mathbb{R}}\right) := \coprod_{\vec{f} \text{ with fixed } \vec{f}^{\text{framed}}} \mathcal{M}\left(\Gamma, \vec{f}^{\text{framed}}, \lambda_{\mathbb{C}}, \lambda_{\mathbb{R}}\right)$$

and observe that

$$\mathcal{M}\left(\Gamma, \vec{d}^{\text{framed}}, \lambda_{\mathbb{C}}, \lambda_{\mathbb{R}}\right) \times \mathcal{M}\left(\Gamma, \vec{e}^{\text{framed}}, \lambda_{\mathbb{C}}, \lambda_{\mathbb{R}}\right) \quad \stackrel{\oplus}{\to} \quad \mathcal{M}\left(\Gamma, \vec{f}^{\text{framed}}, \lambda_{\mathbb{C}}, \lambda_{\mathbb{R}}\right)^{\delta}$$

is an isomorphism.

Let us consider the example of $\boxed{\text{n}} \leftarrow * \leftarrow \ldots \leftarrow *$ with d gauged vertices, ...

7. Cherkis bow varieties

Type A quiver varieties are a subfamily of a larger family of *bow varieties*. We describe these now, following [RS] and [NT].

7.1. Brane diagrams and contingency tables.

In [RS] they start with "brane diagrams" consisting of

- "NS5-branes" / denoted V
- "D5-branes" \ denoted U

with natural numbers labeling "D3-branes" in between, e.g.

$$/2\backslash 2/2\backslash 4/3/3/4\backslash 3/2\backslash 2\backslash$$

(plus silent 0s attached to the ends when necessary). To this we associate two "margin vectors":

- $r_i :=$ the number below the ith / minus the number above, plus # of \s above
- $c_j :=$ the number below the ith \ minus the number above, plus # of /s above

where "above" and "below" mean left and right for /s, the reverse for \s. In the example above these r, c vectors are

$$r = (2, 1, 1, 2, 3, 2) \qquad c = (5, 2, 2, 0, 2)$$

(note that for a general brane diagram they need not be nonnegative).

Lemma 7.1.
- $\sum_{i=1}^{n} r_i = \sum_{j=1}^{m} c_j$, *where* $m = \#/$ *and* $n = \#\backslash$.
- *If we reflect the brane diagram up-down, changing \s with /s, the new* r', c' *have* $r'_i = m - c_i$, $c'_j = n - r_j$. *Call this the* $3d$ **mirror** *of the brane diagram.*

Proof. If we decrement a number, it changes the totals as follows

	$\Delta \sum_i r_i$	$\Delta \sum_j c_j$
$/ - /$	0	0
$/ - \backslash$	-1	-1
$\backslash - /$	$+1$	$+1$
$\backslash - \backslash$	0	0

which lets us reduce to the all-0s case. Then, each side is computing an inversion number, the number of pairs of a \ somewhere left of a /. □

In particular we can recode the brane diagram as a rectangle with rows and columns labeled by r and c respectively, plus a path from NW to SE (or, an English partition) whose $\downarrow, \rightarrow$ steps indicate $/, \backslash$ respectively.

In [RS] they make the crucial Assumption 2.4 about r, c: there should exist $0,1$-matrices with these given row and column sums. (Eventually this will be equivalent to: the corresponding bow variety has T-fixed points.) This assumption has two trivial consequences – r, c should be nonnegative, and lemma 7.1 must hold. Combinatorially it is well understood:

Theorem 7.2 (Gale/Ryser '57). *Assuming that r has been sorted into decreasing order, then there exist such $0,1$-matrices iff $\sum_{i=1}^{k} r_i \leq \sum_{i=1}^{n} \min(k, c_i)$ for each k.*

If one keeps the same r, c but changes the partition by one box, the resulting change in the brane diagram

$$a \backslash b / c \quad \leftrightarrow \quad a/(a+c+1-b)\backslash c$$

is called a **Hanany-Witten transition**. It is not obvious (or true) that the new brane diagram still has nonnegative integers, but Assumption 2.4 implies this.

There is an evident $3d$ mirror symmetry for the $0,1$-matrices – transpose and exchange 0s with 1s.

7.2. **The bow variety of a brane diagram.** For each "D3-brane" U, i.e. a number p in the diagram, and each variety below, we will have a Hamiltonian action of $GL(p)$.

7.2.1. *The triangle variety of a $\backslash$-brane.* To a segment $p \backslash q$ in a brane diagram, we associate a Lie group

$$G := \left\{ \begin{array}{cc} & \begin{array}{ccc} p & 1 & q \end{array} \\ \begin{array}{c} p \\ 1 \\ q \end{array} & \begin{bmatrix} g & a & A \\ 0 & 1 & b \\ 0 & 0 & h \end{bmatrix} \end{array} \right\} \text{ with } \mathfrak{g}^* \cong \left\{ \begin{array}{cc} & \begin{array}{ccc} p & 1 & q \end{array} \\ \begin{array}{c} p \\ 1 \\ q \end{array} & \begin{bmatrix} B_U & * & * \\ a_U & * & * \\ A_U & b_U & -B_{U'} \end{bmatrix} \end{array} \right\}$$

the identification given using the trace form. The $*$s mean that we have *modded out* by that subspace – these live naturally in a quotient space. The coadjoint action is given by conjugation (note that this descends, of course, to the quotient space).

On that quotient space $\mathfrak{g}^*$, squaring descends to a well-defined map

$$\begin{array}{cc} & \begin{array}{ccc} p & 1 & q \end{array} \\ \begin{array}{c} p \\ 1 \\ q \end{array} & \begin{bmatrix} B_U & * & * \\ a_U & * & * \\ A_U & b_U & -B'_U \end{bmatrix} \end{array} \quad \mapsto \quad \begin{array}{cc} & \begin{array}{ccc} p & 1 & q \end{array} \\ \begin{array}{c} p \\ 1 \\ q \end{array} & \begin{bmatrix} * & * & * \\ * & * & * \\ A_U B_U + a_u b_u - B_{U'} A_U & * & * \end{bmatrix} \end{array}$$

which is of course equivariant under the coadjoint action.

Theorem 7.3. *The subvariety $\{M \in \mathfrak{g}^* : SW_{q \times p}(M^2) = 0\}$ is the closure of a coadjoint orbit of G, hence is symplectic.*

Proof. We can use the action of g, h to reduce A_U to a (rectangular) diagonal matrix of r 1s then $\min(p, q) - r$ 0s. We can then use the

First we recall that on all matrices, the equations "M^2 is upper triangular" form a regular sequence, so any subset of them gives a complete intersection. Hence our subvariety has codimension pq.

It now suffices □

One can identify this vector space (without the M^2 condition) with the dual of the Lie algebra of matrices transpose to these, thereby endowing it with a Poisson structure, and in [NT] they observe that the subvariety is Poisson. The coadjoint action is

I suspect that it is in fact the closure of a coadjoint orbit, hence singular symplectic.

The moment map extracts the $B_u, -B_{u'}$ matrices. Note that there is an additional action from the $GL(1)$ in the middle.

> *Question.* Is there a version in which the 1 is replaced with a larger dimension? This may relate to the quiver variety subcase, discussed below, where the natural flavor group is not just the torus. Probably this would require some modification to the Hanany-Witten story. Also one might expect the $0, 1$-matrices to admit larger numbers.

Theorem 7.4.

7.2.2. *The two-way variety of a /-brane.* To p/q in a brane diagram, we associate the symplectic vector space $T^*\mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^p) \cong \mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^p) \times \mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^p)$, much as we would in the quiver variety construction.

The $GL(p), GL(q)$ moment maps compute the two composites, one negated.

7.2.3. *All together.* To construct the bow variety, take the product over all 5-branes (both / and \), impose the vanishing of the moment map (the sum of the individual moment maps), and GIT quotient by $\prod_u GL(U)$ (which involves a stability condition I suppose we could discuss).

Each $GL(p)$ moment map is the sum of the (two) individual moment maps, hence these conditions give in the various cases

$/p/$ $\circlearrowright = \circlearrowleft$, much as in the construction of (type A) quiver varieties
$/p\backslash$ $\circlearrowright = B_u$
$\backslash p/$ $B_u = \circlearrowleft$
$\backslash p\backslash$ $B_u = B_{u'}$ This is "why" there's a minus in the triangle variety definition.

Theorem 7.5. *If two bow varieties are related by a Hanany-Witten transition, then they are isomorphic as varieties.*

7.3. **The torus action and fixed points.** ... Each \-brane contributes a circle action. This has fixed points iff Assumption 2.4 holds, in which case, the fixed points are in bijection with "binary contingency tables" with the given row and column sums.

7.4. **A_d quiver varieties.** Given an A_d Nakajima quiver with dimension vector

$$\begin{array}{ccccccc} \boxed{w_1} & & \boxed{w_2} & & \cdots & & \boxed{w_d} \\ \downarrow & & \downarrow & & \cdots & & \downarrow \\ v_1 & \to & v_2 & \to & \cdots & \to & v_d \end{array}$$

associate the brane diagram and margin vectors

$$\begin{array}{c|ccccccccc} & / & \nu_1 \overbrace{\backslash \nu_1 \backslash \cdots \backslash}^{w_1} \nu_1 & / & \nu_2 \overbrace{\backslash \nu_2 \backslash \cdots \backslash}^{w_2} \nu_2 & / & \cdots & / & \nu_d \overbrace{\backslash \nu_d \backslash \cdots \backslash}^{w_d} \nu_d & / \\ \hline r & \nu_1 & & \substack{\nu_2-\nu_1 \\ +w_1} & & \substack{\nu_3-\nu_2 \\ +w_1+w_2} & \cdots & & & -\nu_d + \sum w_i \\ c & & d^{\nu_1} & & (d-1)^{\nu_2} & & & & 1^{\nu_d} & \end{array}$$

where the number of \s in each group – not the number of ν_i, which is one more! – is the framing dimension $\boxed{w_i}$.

In particular r is weakly decreasing. Conversely, given a pair (c, r) with r weakly decreasing and satisfying Assumption 2.4, the corresponding bow variety is a quiver variety [RS, Theorem 5.4].

> *Question.* If r is weakly decreasing, we can permute c and not change the variety, using Nakajima reflection isomorphisms. Is this also true for arbitrary r, and, are there Nakajima-like isomorphisms when we permute r? (Obviously the contingency tables don't mind.) That'd be pretty boring, then all the bow varieties would be isomorphic to type A quiver varieties.

7.4.1. *A basic example.* The cotangent bundle $T^*Gr(k, n)$ arises as the A_1 quiver variety $\mathcal{M}(\boxed{n} - k)$ with brane diagram and r, c vectors

$$\begin{array}{c|ccccccccccc} & / & k & \backslash & k & \backslash & k & \backslash & \cdots & \backslash & k & / \\ \hline r & k & & & & & & & & & & n-k \\ c & & & 1 & & 1 & & 1 & & 1 & & \end{array}$$

Part 4. Slodowy slices

Recall that if G acts on M, then G acts Hamiltonianly on T^*M, with the moment map

$$\Phi : \quad (m, \vec{v}) \ \mapsto \ (X \mapsto \langle X_m, \vec{v} \rangle)$$

where X_m is the restriction to m of the vector field on M induced by $X \in \mathfrak{g}$. In the case that G acts transitively on M, so $\{X_m : X \in \mathfrak{g}\} = T_m M$, the product map $T^*M \xrightarrow{Id \times \Phi} M \times \mathfrak{g}^*$ is injective, and even a closed embedding.

We apply this to the case $M = G/P$, obtaining a **Springer map** $T^*(G/P) \to \mathfrak{g}^*$. This map is proper, since $T^*(G/P) \xrightarrow{Id \times \Phi} G/P \times \mathfrak{g}^* \to \mathfrak{g}^*$ is the closed embedding followed by a proper projection. Also, it is equivariant with respect to the non-symplectic action that dilates the fibers of $T^*(G/P)$, while scaling $\mathfrak{g}^*$. In particular the image is a cone.

With more work one can show that G acts on that cone with a dense orbit. A **nilpotent orbit** of G on $\mathfrak{g}^*$ is defined to be one with $\vec{0}$ in its closure. Not every nilpotent orbit is the image of some $T^*(G/P)$, and for those who are, the map Φ is not necessarily birational. *** **Example?** ***

8. Slices

If $X \subseteq M$ is a subvariety and $x \in X_{reg}$ is a smooth point of both, call $S \subseteq M$ a **slice** to X at x if

- $x \in S$ (of course), and is a smooth point of S

- T_xS is a complement to T_xX inside T_xM
- X and S are invariant under a circle action $\mathbb{C}^\times \circlearrowright M$, x is a fixed point, and is attractive in S. In particular this implies that S is smooth and that $X \cap S = \{x\}$.

If Y sits between X and V and is invariant under the circle action, we call $Y \cap S$ the **slice to X at x inside Y**. The difference is that Y is not assumed smooth at x. However, since S is transverse to X at x it is transverse also to Y, and the locus in Y where S meets it nontransversely is (1) closed (2) retracts to x (3) doesn't contain x, hence is empty.

For a first example, let $M = G/B$ and $X = X_v := \overline{B_-vB}/B$, and $x = vB/B$. Then $X_\circ^v := BvB/B$ serves as a slice, with any regular dominant coweight in T (e.g. $\check\rho$) contracting it. Then for $Y = X_w$ with $w \leq v$, the intersection $X_w \cap X_\circ^v$ is a slice at vB/B to X_v inside X_w. This generalizes readily to the case $M = G/P$.

8.1. The Slodowy slice.

Let $\mathcal{O}_\mu = G \cdot e$ be a "nilpotent orbit" in $\mathfrak{g}$, meaning, an orbit invariant under dilation. The tangent space $T_e\mathcal{O}_\mu$ is $[\mathfrak{g}, e] = (\mathrm{ad}\ e) \cdot \mathfrak{g}$.

By the Jacobson-Morozov theorem any such e arises as the image of $e \in \mathfrak{sl}_2$ under a Lie algebra homomorphism $\phi : \mathfrak{sl}_2 \to \mathfrak{g}$, unique up to conjugation by $C_G(e)$. In any rep V of $\mathfrak{sl}_2$, the image of e is complementary to the kernel of f, both are h-invariant, and the kernel of f has only nonnegative weights. We can therefore decompose $\mathfrak{g}$ as $[\mathfrak{g}, e] \oplus Z_{\mathfrak{g}}(f)$ and

- the slice S is $e + Z_{\mathfrak{g}}(f)$,
- the $\mathbb{C}^\times$ action comes from $\phi(T_{SL(2)})$, but twisted by dilation^{-2}, in order to make e a fixed point. In particular all the weights on $Z_{\mathfrak{g}}(f)$ are ≤ -2, making e attractive.

8.1.1. *The Slodowy slice as a symplectic reduction.*

For a pair $\mathcal{O}_\lambda \supseteq \mathcal{O}_\mu$ of nilpotent orbit closures, we then get a slice $\mathcal{O}_\lambda \cap (e + Z_{\mathfrak{g}}(f))$. Its dimension is obviously the difference of the two, hence even, but it is less obvious that it is symplectic. We will obtain the slice as a symplectic *reduction* of $\mathcal{O}_\lambda$, following [GG] and [BK].

In fact these statements can be upgraded:

- S is naturally a Poisson manifold (quantized in [GG], generalizing work of [P]),
- it can be obtained from $\mathfrak{g}^*$ as a Poisson reduction, and
- its intersection with $\mathcal{O}_\lambda$ is a symplectic leaf.

Start from the $\mathfrak{sl}_2$-triple $e, f, h \in \mathfrak{g}$, and pick a scaling $\Phi : \mathfrak{g} \to \mathfrak{g}^*$ of the Killing form s.t. $\langle \Phi(e), f\rangle = 1$. Decompose $\mathfrak{g} = \bigoplus_n \mathfrak{g}_n$ into weight spaces for $\mathrm{ad}\ h$, so $e \in \mathfrak{g}_2$. In particular this defines a Levi $L := G^h$ and opposed parabolics $P_\pm \geq L$.

Define an antisymmetric form ω on $\mathfrak{g}_{-1}$ by $\omega(X, Y) := \langle \Phi(e), [X, Y]\rangle$. Since the pairing of $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ is perfect, and by $\mathfrak{sl}_2$-theory the map $\mathrm{ad}\ e : \mathfrak{g}_{-1} \to \mathfrak{g}_1$ is an isomorphism, this pairing is nondegenerate.

Choose a Lagrangian subspace $\ell \leq \mathfrak{g}_{-1}$, with which to define a Lie algebra

$$\mathfrak{n}_\ell := \ell \oplus \bigoplus_{n \leq -2} \mathfrak{g}_n \qquad \leq \mathrm{rad}(P_-)$$

with Lie group N_ℓ.

Theorem 8.1. [GG] *The action map $N_\ell \times S \to \mathfrak{g}^*$ gives an isomorphism to $\Phi(e) + \mathfrak{n}_\ell^\perp$. Hence $S \cong \mathfrak{g}^*//_{\Phi(e)} N_\ell$.*

The principal ingredient in the proof is the contracting circle action; as in the proof (from the beginning of the section) that S is smooth, one checks that any failure of the map to be $1:1$ or to be onto would be visible near $\Phi(e)$, but is not.

If we now have a Springer resolution T^*G/P of $\mathcal{O}_\lambda$, then we can use $(T^*G/P)//N_\ell$ to resolve the slice $\mathcal{O}_\lambda//N_\ell$.

9. Type A and the Mirković-Vilonen slice.

In the type A case, we take e to be a direct sum of nilpotent Jordan blocks of decreasing size. A single block of size $n+1$ corresponds to the representation $\mathfrak{sl}_2 \circlearrowright \mathrm{Sym}^n(\mathbb{C}^2)$ with basis $\left\{\frac{x^a}{a!}y^{n-a}\right\}$, where $f = y\frac{d}{dx}$ has $f \cdot \frac{x^a}{a!}y^{n-a} = \frac{x^{a-1}}{(a-1)!}y^{n-a+1}$. Meanwhile $e = x\frac{d}{dy}$ has $e \cdot \frac{x^a}{a!}y^{n-a} = (n-a)(a+1)\frac{x^{a+1}}{(a+1)!}y^{n-a-1}$. In matrices, for $n = 4$,

$$e = \begin{bmatrix} 0 & 4\cdot 1 & & & \\ & 0 & 3\cdot 2 & & \\ & & 0 & 2\cdot 3 & \\ & & & 0 & 1\cdot 4 \\ & & & & 0 \end{bmatrix} \qquad f = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & 1 & 0 & \\ & & & 1 & 0 \end{bmatrix} \qquad Z_{\mathfrak{sl}_5}(f) = \begin{bmatrix} 0 & & & & \\ b & 0 & & & \\ c & b & 0 & & \\ d & c & b & 0 & \\ e & d & c & b & 0 \end{bmatrix}$$

Then S, in the $3+2$ case, looks like

$$\begin{bmatrix} a & 1 & 0 & 0 & 0 \\ b & a & 1 & d & 0 \\ c & b & a & e & d \\ f & 0 & 0 & h & 1 \\ g & f & 0 & i & h \end{bmatrix}$$

In each $p \times q$ block, there are $\min(p, q)$ many free parameters, giving constant diagonals starting from the SW corner. Then in the diagonal blocks, there is one more constant diagonal of 1s.

Inspired by affine Grassmannian concerns, Mirković-Vilonen define a different slice

$$\begin{bmatrix} \cancel{a} & 1 & 0 & 0 & 0 \\ \cancel{b} & \cancel{a} & 1 & \cancel{d} & 0 \\ c & b & a & e & d \\ \cancel{f} & 0 & 0 & \cancel{h} & 1 \\ g & f & 0 & i & h \end{bmatrix} =: \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ c & b & a & e & d \\ 0 & 0 & 0 & 0 & 1 \\ g & f & 0 & i & h \end{bmatrix}$$

where the free parameters appear only along the bottom line of each block, but one must take care to remember that if a block is wider than it is tall, then the number of parameters equals the height.

They use this to isomorph type A_{n-1} quiver varieties with (resolutions of) type A_N Slodowy slices (for $N \gg n-1$). Fix gauge dimensions $v_\bullet$ and framed dimensions $\boxed{d_\bullet}$, and define

- $m = \sum_{i=1}^{n-1} \boxed{d_i}$, $N = \sum_{i=1}^{n-1} i \boxed{d_i}$
- $\lambda_i^T := d_i + \ldots + d_{n-1}$ a partition of N and dominant weight of GL_n
- $a_i := v_{n-1} + \sum_{j=1}^{n-1}(d_j - (Cv)_j)$ a weight of V_{λ^T}

- $\mu = (\mathrm{sort}\ a)^T$

Part 5. Slices in the affine Grassmannian

10. The affine Grassmannian

We start from an unlikely source: K a compact group. Let the **loop group** LK denote the space of smooth maps $S^1 \to K$ with pointwise multiplication, i.e. $(\gamma\delta)(t) := \gamma(t)\delta(t)$. This has a subgroup $\Omega K := \mathrm{Map}_\bullet(S^1, K)$ of **based loops**, where $\gamma(1) = 1$. However it will be most fruitful to view it as a *quotient* LK/K by the subgroup of **constant loops**.

Since ΩK is a group, we can identify each tangent space with the tangent space $T_1\Omega K = \mathrm{Map}_\bullet(S^1, \mathfrak{k})$ at the identity. Using the Killing form κ on $\mathfrak{k}$ and the measure on S^1, we can define an interesting bilinear form on $T_1\Omega K$:

$$(\vec{\gamma}, \vec{\delta}) := \int_{\theta \in S^1} \kappa\left(\vec{\gamma}(\theta), \left(\frac{d}{d\theta}\vec{\delta}\right)(\theta)\right) d\theta$$

Using integration by parts, one can show that this is antisymmetric. It is fairly obvious that it is LK-invariant, which is enough to show that it is closed. It take a little more analytic work to show that it is nondegenerate, i.e. defines a symplectic form on ΩK. Indeed, using the Killing form one can also define a Riemannian metric on ΩK, and from the two together one can define a complex structure!

Given the transitive symplectic action one might hope that ΩK is a coadjoint orbit of LK, but this is not quite true – one needs to centrally extend LK to the *affine Lie group*. However, we are more interested in the action of S^1 rotating the circle (which is included in the "extended affine Lie group"). Circle rotation acts on LK in a transparent way, preserving the subgroup K of constant loops, but looks a little weird in ΩK coördinates:

$$(t \cdot \gamma)(s) = \gamma(ts)\gamma(t)^{-1}$$

Theorem 10.1. *This action of S^1 on ΩK is Hamiltonian, with moment map*

$$\Phi_{S^1} \colon\ \gamma \ \mapsto\ \int_{S^1} |\gamma'|^2\ d\theta$$

The fixed points of the action (or equivalently, the critical points of the moment map) are the group homomorphisms $\gamma \colon S^1 \to K$. In particular there is one fixed-point component for each dominant coweight $\lambda \in \mathfrak{t}_{\mathbb{Z}}$, and it is a generalized flag manifold $K/\mathrm{Stab}_K(\lambda)$. The index (number of negative weight spaces) of this action is $2\mathrm{ht}(\lambda)$, where $\mathrm{ht}(\lambda)$ is the height of λ expressed in the basis of fundamental coweights.

Proof. □

For λ a dominant coweight, let $t^\lambda \in \Omega T \subseteq \Omega K$ denote the one-parameter subgroup, and $K \cdot t^\lambda \cong K/\mathrm{Stab}_K(\lambda)$ its K-orbit.

This **energy functional** (though in quantum mechanics, it should really be called the "action functional") makes sense on the based loop space of any Riemannian manifold, and is the one for which Morse invented Morse theory. Specifically on compact groups, it is the one for which Bott invented Morse-Bott theory (which can handle these nonisolated fixed points). Let $s \cdot \gamma$, $s \in \mathbb{R}_+$, denote the flow of the Riemannian gradient. Then define

$$\mathrm{Gr}^\lambda := \{\gamma \in \Omega K \colon \lim_{s \to \infty} s \cdot \gamma \in K \cdot t^\lambda\}$$

as the Morse-Bott stratum. By the theorem, this is a finite-dimensional *complex* submanifold. (Soon we will make it into a variety!) The corresponding limits $s \to -\infty$ are finite-codimensional.

The **affine Grassmannian** $\mathsf{Gr} \subseteq \Omega K$ (or Gr_G, where $G = K^{\mathbb{C}}$) is defined as $\coprod_\lambda \mathsf{Gr}^\lambda$. By Morse-Bott theory, it is a deformation retract of ΩK, but as an inductive union of finite-dimensional complex varieties, it is also an "ind-scheme", and people do algebraic geometry with it.

11. Gr as a homogeneous space

In finite dimensions the complex generalized flag variety H/P has a compact presentation $H_{\mathbb{R}}/(P \cap H_{\mathbb{R}})$, where $P \cap H_{\mathbb{R}}$ is necessarily a "Levi subgroup" of the maximal compact subgroup $H_{\mathbb{R}} \leq H$. Reversing this, if a compact group K acts on K/L preserving a complex structure, we can extend the action to $K^{\mathbb{C}}$. Soon we will attempt this with LK acting on ΩK.

Another familiar construction in finite dimensions is the Białynicki-Birula decomposition of G/P w.r.t. a circle action S. If we decompose $\mathfrak{g} = \oplus_{m \in \mathbb{Z}} \mathfrak{g}_m$ into S-weight spaces, the zero weight space $\mathfrak{g}_0$ is the Lie algebra of the Levi $L := C_G(S)$, and the sum of the nonnegative (resp. nonpositive) weight spaces is the Lie algebra of a parabolic Q (resp. Q_-). Then L acts on the S fixed points, and

$$(G/P)^S = \coprod_{w \in W_Q \backslash W / W_P} \big(LwP/P \cong L/(L \cap w \cdot P) \big)$$

The corresponding B-B strata are exactly the Q orbits

$$G/P = \coprod_{w \in W_Q \backslash W / W_P} QwP/P$$

with the opposite B-B decomposition being the Q_- orbits.

In the case at hand we have LK acting on LK/K preserving a complex structure, so we want to reverse-engineer the above compact picture to define a complex group action. Our first guess for this group is $LG = \mathrm{Map}(S^1, G)$ where $G = K^{\mathbb{C}}$ but this isn't as algebraic as we might want – for example, its Lie algebra is not the direct sum of its S^1-weight spaces. The next idea is $\mathrm{Map}(\mathbb{C}^\times, G) = G(\mathbb{C}[z^\pm])$, the $\mathbb{C}[z^\pm]$-valued points of the group scheme; for $G = GL_n$ this is exactly $GL_n(\mathbb{C}[z^\pm])$. This group has a weird map det to the units in $\mathbb{C}[z^\pm]$, which is $\{cz^m : c \in \mathbb{C}^\times, m \in \mathbb{Z}\}$. It would be nicer to take G(a field), so we move on to our final answer, $G(\mathbb{C}((z)))$ where $\mathbb{C}((z))$ is the field of Laurent series.

Let's try to use the usual approach to Lie groups – decompose the Lie algebra into root spaces, intersect the set of roots with a half-space to get positive roots, figure out which of those aren't sums to get simple roots. The decomposition is easy: $\{\mathfrak{g}_\beta \otimes z^k\}$ and $\{\mathfrak{t} \otimes z^k\}$ (so, sadly, the root spaces are not all 1-dimensional as in the finite-dimensional case). We have an obvious half-space, $\mathfrak{g}[[z]]$, which is not generic enough to give a Borel but does determine a parabolic with Levi $\mathfrak{g}z^0$, the complexification of the Lie algebra of the subgroup group K of constant loops.

On a homogeneous space H/P there is a "big cell", the free open orbit through the basepoint of the group $\mathrm{Rad}(P_-)$. In the case of $H/P = \mathsf{Gr}$ this P_- is $G(\mathbb{C}[z^{-1}])$ and $\mathrm{Rad}(P_-)$ is the kernel of the (weird-sounding) map $P_- \to G$ taking $z^{-1} \mapsto 1$. Note that in the case that H is disconnected, the cell lives in the zeroth component of H/P.

In type A we make use of a nice picture of $GL_n(\mathbb{C}((z)))$ as **affine** (infinite periodic) matrices,

$$\sum_i M_i z^i \qquad \mapsto \qquad \begin{bmatrix} & & \ddots & & & \\ \cdots & & M_{-1} & M_0 & M_1 & M_2 & \cdots \\ & & \cdots & M_{-1} & M_0 & M_1 & \cdots \\ \text{eventually } 0 & & & & & \ddots & \end{bmatrix} \qquad M_i \in Mat_n(\mathbb{C})$$

One of its nice aspects: the outer automorphism that shifts the whole matrix one step Southeast is a manifest symmetry. The **Iwahori (Borel) subgroup** $I \leq GL_n(\mathbb{C}((z)))$ is defined as corresponding to the upper triangular matrices, in which $M_{<0} = 0$ and $M_0 \in B(\mathbb{C})$; this subgroup is shift-invariant. In the parabolic subgroup $P = GL_n(\mathbb{C}[[z]])$ one only asks $M_{<0} = 0$. Meanwhile, in $Rad(P_-)$ one asks that $M_{>0} = 0$ and $M_0 = Id$.

The map $GL_n(\mathbb{C}((z))) \to Gr$, in these affine matrix coördinates, is slightly subtle: start with the column span of columns $\ldots, -2, -1, 0$, where row i indexes the basis element $e_{i \bmod n} \otimes z^{-\lfloor i/n \rfloor}$ of $\mathbb{C}^n \otimes \mathbb{C}((z))$. That is only countable-dimensional, and doesn't yet actually contain $z^{N \gg 0}\mathbb{C}[[z]]^n$, so we add that in for large N (it doesn't matter which once it's large enough). Put differently, M maps to the unique smallest lattice L containing the span of columns $1-n, \ldots, 0$. If M is the permutation matrix $m_{ij} = \delta_{i,f(j)}$ of a permutation $f : \mathbb{Z} \to \mathbb{Z}$, $f(i+n) = f(i) + n$, then its lattice contains $e_{f(j) \bmod n} \otimes z^{-\lfloor f(j)/n \rfloor}$, $j = 1-n, \ldots, 0$. There are two ways to standardize f without changing its lattice: insist that $f(i) \equiv i \bmod n\ \forall i$, or insist that $f(1) > f(2) > \ldots > f(n)$.

To an affine matrix M and matrix entry (i,j), we can associate the rank of the submatrix with NE corner (i,j). By the condition $M_{\ll 0} = 0$, this rank is finite. These ranks are invariant under left and right multiplication by the Iwahori I, and serve collectively to index the $I \times I$ orbits. If $j \equiv 0 \bmod n$, then the rank is invariant under right multiplication by P. If $i \equiv 1 \bmod n$, then the rank is invariant under left multiplication by P.

The shift matrix $\Psi_{ij} := \delta_{i+1,j}$ is periodic, and Ψ^k defines an element in the kth component of Gr_{GL_n}; in particular $\Psi^{kn} \mapsto z^k\mathbb{C}[[z]]^n$. This operator normalizes the Iwahori, so takes I-orbits in Gr_k to I-orbits in Gr_{k+1}. However, only Ψ^n normalizes P.

If λ is a partition and $\#\lambda = kn$, then

$$Rad(P_-) \cdot \Psi^{kn} = \left\{ \begin{bmatrix} & & \ddots & & & & & \\ \cdots & & M_{-1} & M_0 & \cdots & M_{k-1} & Id & 0 & \cdots \\ & & \cdots & M_{-1} & M_0 & M_1 & \cdots & & \\ \text{eventually } 0 & & & & & \ddots & & & \end{bmatrix} \right\}$$

and its intersection with $\overline{Gr^\lambda}$

12. The varieties $\overline{Gr^\lambda}$ in type A, and their resolutions

At this point we want a picture of $G(\mathbb{C}((z)))/G(\mathbb{C}[[z]])$, at least in the case $G = GL_n$. That is, we want a space with a $GL_n(\mathbb{C}((z)))$-action, and a point with stabilizer exactly $GL_n(\mathbb{C}[[z]])$; then we'll use the orbit.

We warm up to it by considering $GL_n(\mathbb{Q})/GL_n(\mathbb{Z})$. Of course $GL_n(\mathbb{Q})$ acts on $\mathbb{Q}^n$, but the stabilizer of a single vector is matrices whose first column (except for m_{11}) is zero. Instead we consider the space of subgroups of $\mathbb{Q}^n$, and note that $\mathbb{Z}^n$ has stabilizer $GL_n(\mathbb{Z})$.

Theorem 12.1. *Let* $\mathsf{L} \le \mathbb{Q}^n$ *be a subgroup. Then the following are equivalent:*

(1) $\mathsf{L} = g \cdot \mathbb{Z}^n$ *for some* $g \in GL_n(\mathbb{Q})$ *(not unique).*
(2) *There exist* $A, B \in \mathbb{N}_+$ *such that* $AL \le \mathbb{Z}^n \le B^{-1}L$.
(3) $L \cap \mathbb{Z}^n$ *is of finite index in each.*

Proof. (1) $\implies$ (2): take A, B big enough to clear out the denominators of g and g^{-1}.

(2) $\implies$ (1): AL is a subgroup of $\mathbb{Z}^n$ and via Smith normal form (which uses an element of $GL_n(\mathbb{Z})$ to change the basis of $\mathbb{Z}^n$), one can take it to be generated by $\mathbb{N}$-multiples of part of the standard basis. The condition $\mathbb{Z}^n \le BL$ shows that the entire basis must be used, i.e. the Smith normal form is a diagonal matrix with $\mathbb{N}_+$-entries, and thereby lies in $GL_n(\mathbb{Q})$.

The other connection is similar. □

If $M \le L, \mathbb{Z}^n$ is of finite index in each (say $M = L \cap \mathbb{Z}^n$, or $M = AL$) then there is a positive rational number to associate: $\frac{\#(\mathbb{Z}^n/M)}{\#(L/M)}$. If $L = g \cdot \mathbb{Z}^n$, then this number is $\pm \det g$. Really, we should see $\det g$ as only giving a well-defined element of $\mathbb{Q}^\times/\mathbb{Z}^\times$.

Now we finally return to $GL_n(\mathbb{C}((z)))/GL_n(\mathbb{C}[[z]])$. In order to apply the Smith normal form tricks above, we need to consider not just subspaces but $\mathbb{C}[[z]]$-submodules. Then

$$Gr_{GL_n} := \{L \le \mathbb{C}((z))^n \text{ a } \mathbb{C}[[z]]\text{-submodule} : L \cap \mathbb{C}[[z]]^n \text{ has finite } \mathbb{C}\text{-codim in } L, \mathbb{C}[[z]]^n\}$$

Define the **index** of L as the difference in these codimensions. It is an integer, which we may view as an element of $\mathbb{C}((z))^\times/\mathbb{C}[[z]]^\times$.

This decomposition $Gr_{GL_n} = \coprod_{d \in \mathbb{Z}} (Gr_{GL_n})_{\text{index } d}$ can be seen in the $\Omega U(n)$ picture as deriving from $\pi_1(U(n)) \cong \mathbb{Z}$. In particular if we replace $U(n)$ by $SU(n)$ or $PU(n)$ we get very similar Grs: $Gr_{SL_n} \cong (Gr_{GL_n})_{\text{index } 0}$, $Gr_{PGL_n} \cong \coprod_{d \in [0,n)} (Gr_{GL_n})_{\text{index } d}$.

The lattices $L \le \mathbb{C}[[z]]^n$ have an additional structure: to L we can associate the nilpotent action of $z \circlearrowright (\mathbb{C}[[z]]^n/L)$, and extract a partition $\lambda \vdash \dim(\mathbb{C}[[z]]^n/L)$ of at most n rows. This association is obviously $GL_n(\mathbb{C}[[z]])$-invariant. Moreover, via Smith normal form arguments, the partition associated to t^λ is λ. Hence

$$Gr^\lambda := \{L \in Gr \colon L \le \mathbb{C}[[z]]^n, JCF(z \circlearrowright CC[[z]]^n/L) = \lambda\}$$

is exactly the $GL_n(\mathbb{C}[[z]])$-orbit of t^λ.

If we replace L by zL, the associated partition adds a full column. So to any $L \in Gr$, we can associate a dominant coweight $(\lambda_1 - B, \lambda_2 - B, \ldots, \lambda_n - B) \in \mathbb{Z}^n$ where B is chosen large enough that $z^B L \le \mathbb{C}[[z]]^n$. With this, we get a full decomposition

$$Gr = \coprod_{\lambda \in \mathbb{Z}^n} Gr^\lambda$$

which is the algebro-geometric picture of the Morse-Bott decomposition of $\Omega U(n)$ from before.

Fix $\lambda \in \mathbb{Z}^n$, so $t^\lambda \in [z^{\lambda_1}\mathbb{C}[[z]]^n, z^{\lambda_n}\mathbb{C}[[z]]^n]$. This gives an embedding

$$Gr^\lambda \hookrightarrow Gr\left(\sum_i (\lambda_1 - \lambda_i),\ (z^{\lambda_n}\mathbb{C}[[z]]^n)/(z^{\lambda_1}\mathbb{C}[[z]]^n)\right)$$

We'd like to know that the closure of Gr^λ in Gr can be computed inside this finite-dimensional projective variety, but, we can't embed this Grassmannian into Gr. However, if we consider its closed subvariety consisting of $\mathbb{C}[z]$-*submodules* of $(z^{\lambda_n}\mathbb{C}[[z]]^n)/(z^{\lambda_1}\mathbb{C}[[z]]^n)$, *that* embeds into Gr (taking a space to its preimage in $z^{\lambda_n}\mathbb{C}[[z]]^n$). Hence $\overline{\mathsf{Gr}^\lambda}$ is a projective variety.

Two special cases worth focusing on: $\lambda_1 = \ldots = \lambda_n$, in which case Gr^λ is the point $\{z^{\lambda_1}\mathbb{C}[[z]]^n\}$, and $\lambda_1 = \ldots = \lambda_k = 1 + \lambda_{k+1} = \ldots = 1 + \lambda_n$, in which case $\mathsf{Gr}^\lambda \cong \mathsf{Gr}(n - k, z^{\lambda_n}\mathbb{C}[[z]]^n / z^{\lambda_1}\mathbb{C}[[z]]^n)$ is already closed. These are the only cases when $\mathsf{Gr}^\lambda = \overline{\mathsf{Gr}^\lambda}$ (and in fact, are the only ones for which $\overline{\mathsf{Gr}^\lambda}$ is smooth).

This analysis gives us a slightly different description of $\overline{\mathsf{Gr}^\lambda}$, more in line with the Hilbert scheme from before. Let $R = \mathbb{C}[x]$, and let $M_\lambda := \bigoplus_{i=1}^n R/\langle x^{\lambda_1 - \lambda_i}\rangle$. Then consider quotients M of the free module R^n which

- are torsion modules, so the support in $\operatorname{Spec} R = \mathbb{A}^1$ is finite
- more specifically have support at 0, so the action extends to $\mathbb{C}[[x]]$
- $\dim x^k M \leq \dim x^k M_\lambda$ for each $k \in [0, \lambda_1 - \lambda_n]$, or
- $\operatorname{codim}_M x^k M \geq \operatorname{area}(\text{left } k \text{ columns of } \lambda)$ for each $k \in [0, \lambda_1 - \lambda_n]$.

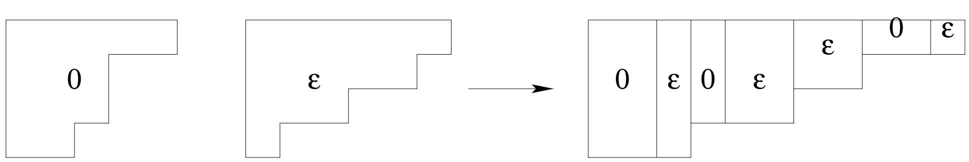
This is a subvariety of a "Quot scheme", a variant of the Hilbert scheme (which parametrizes quotient modules of R not R^n).

While it will not be central to our topic – symplectic resolutions – we can't help but mention some of the results that place these varieties at the center of geometric representation theory.

Theorem 12.2 (The Geometric Satake Correspondence, part I)**.** *There is a natural* ${}^L G$*-action on* $IH^*(\overline{\mathsf{Gr}^\lambda})$*, making it the irreducible representation* V_λ*.*

The proof is a sort of recurrence relation – instead of defining the ${}^L G$-action on any one fixed $IH^*(\mathsf{Gr}^\lambda)$, one defines a tensor product on a category of perverse sheaves on Gr, and then uses Tannaka-Krein duality to say that such a tensor category must be the representations of some group. (Recognizing the group to be ${}^L G$ is the easy part.) This tensor product structure was first defined in the unpublished (!) work [G95], using the multiplication on the topological group ΩK.

The Quot scheme picture gives a more algebro-geometric approach, due to Beilinson-Drinfel′d. Let $R = \mathbb{C}[x]$ again but consider torsion quotients of R^n supported at 0 and ε. Fix partitions λ, μ and label the columns of λ "0" and the columns of μ "ε". List the columns together, in decreasing order of size, with 0s left of εs, e.g.



Consider quotients M of R^n of dimension $|\lambda| + |\mu|$, such that for each column i,

$$\operatorname{codim}_M \left(\prod_{j \leq i} (x - \text{ label on } j) \cdot M \right) \geq |\text{left } i \text{ columns in } \lambda + \mu|.$$

For each $\varepsilon \neq 0$, this space is $\overline{\mathsf{Gr}^\lambda} \times \overline{\mathsf{Gr}^\mu}$, but for $\varepsilon = 0$, it is $\overline{\mathsf{Gr}^{\lambda+\mu}}$. (Beilinson-Drinfel′d describe it with the entire Gr, giving a degeneration of $\mathsf{Gr} \times \mathsf{Gr}$ to Gr!) One then uses this family and the "nearby cycles sheaf" construction to construct a sheaf on $\overline{\mathsf{Gr}^{\lambda+\mu}}$ whose global sections are $IH(\overline{\mathsf{Gr}^\lambda}) \otimes IH(\overline{\mathsf{Gr}^\mu})$. The result is a tensor product structure on the category

$\mathsf{Perv}(\mathsf{Gr})$ of "perverse sheaves on Gr w.r.t. the $\{\mathsf{Gr}^\lambda\}$ stratification", and ${}^L G$ is the group of natural automorphisms of the "fiber functor" $\mathsf{Perv}(\mathsf{Gr}) \to$ **Vec**.

12.1. **Generalized Bott-Samelson varieties.** Let $w = (w_1, \dots, w_m)$ be a sequence of Weyl group elements for some Lie (or even Kac-Moody) group G, and define

$$BS^w := B \times^B \overline{Bw_1B} \times^B \overline{Bw_2B} \times^B \cdots \times^B \overline{Bw_mB}/B$$

where $\times^B$ means to multiply, then divide by the diagonal B-action. (It is not a fiber product, and so, should not be denoted $\times_B$.)

If we project out the last factor, we get a bundle map $BS^{w_1,\dots,w_m} \twoheadrightarrow BS^{w_1,\dots,w_{m-1}}$ with fiber the Schubert variety $X^{w_m} := \overline{Bw_mB}/B$. Hence BS^w is an iterated bundle of Schubert varieties, so, is a projective variety of dimension $\sum_{i=1}^m \ell(w_i)$. In the case actually considered by Bott-Samelson (and later given algebro-geometric structure by Demazure and Hansen), each w_i is a simple reflection, so each $\overline{Bw_iB}/B \cong \mathbb{P}^1$, and the Bott-Samelson *manifold* is an iterated $\mathbb{P}^1$-bundle.

Define a monoidal product $*$, called the "nil-Hecke" / "monoidal" / "Demazure" / "greedy" product, on W by $\overline{B(w * v)B} = \overline{BwBvB}$. (One of the many combinatorial ways to think about this product is to pick reduced words for w and v, and find maximal reduced subwords of their concatenation. Any two will have the same product, $w * v$. One thing to notice is the subadditivity $\ell(w * v) \leq \ell(w) + \ell(v)$.) There is another natural map on these generalized Bott-Samelson varieties,

$$BS^{w_1,\dots,w_i,w_{i+1},\dots,w_m} \twoheadrightarrow BS^{w_1,\dots,w_i * w_{i+1},\dots,w_m}$$

By composing these maps suitably, and concatenating, we get an *embedding*

$$BS^w \hookrightarrow \prod_{i=0}^m X^{w_1 * \cdots * w_i} \subseteq (G/B)^{m+1}$$

with image the tuples $(F_0, \dots, F_m) \in (G/B)^{m+1}$ s.t. $F_0 = B/B$, $(F_i, F_{i+1}) \in \overline{G_\Delta \cdot (B/B \times X^{w_i})}$ $\forall i$. In geometric language, a point is a tuple of flags starting from the base flag and with "distance" $\leq w_i$ from one flag to the next. In even more flowery language, one can imagine "polygonal paths in G/B" where the ith step has "length" $\leq w_i$.

In the applications of Demazure/Hansen, w is a reduced word of simple reflections, so that BS^w is smooth and the map to X^{*w} is birational. In this way the Bott-Samelson provides a (rather large) resolution of singularities. (The application of Bott-Samelson was entirely different – they take w to be a reduced word for w_0 the longest element of a finite Weyl group, so that $X^{*w} = G/B$. They prove that $H^*(G/B) \to H^*(BS^w)$ is injective, and characterize the image, in work predating Borel's presentation of $H^*(G/B)$.)

We now go one step further in generalization, and choose a parabolic $P \geq B$ of our group G – which we rename H because soon it will be G's loop group. We probably want P/B finite-dimensional (but not H/B). Let $w_i \in W_P \backslash W_H / W_P$, and define

$$BS^w := P \times^P \overline{Pw_1P} \times^P \overline{Pw_2P} \times^P \cdots \times^P \overline{Pw_mP}/P$$

Most things stay the same: BS^w is an iterated bundle of $\overline{Pw_iP}/P$s (which are finite-dimensional if P/B is), and $W_P \backslash W_H / W_P$ acquires a monoidal product.

We now focus on the H/P we care about: $G(\mathbb{C}((z)))/G(\mathbb{C}[[z]])$. Then

$$W_P \backslash W_H / W_P \cong W_G \backslash (W_G \ltimes \Lambda)/W_G \cong \Lambda/W_G \cong \Lambda_+$$

is the dominant coweights. The building blocks $\overline{Pt^\lambda P}/P$ are the $\overline{Gr^\lambda}$. Many miracles occur now:

Theorem 12.3. (1) *The monoidal product $*$ on Λ_+ is just addition.*
(2) *It is additive, not just subadditive; in particular the map $\beta\colon BS^{(\lambda_i)_{i=1}^m} \to \overline{Gr^{\sum \lambda_i}}$ is* always *birational.*
(3) *Even better than birational, it's semismall.*
(4) *The strata of $\overline{Gr^\lambda}$ (over each of which β is a bundle) are simply connected.*

Theorem 12.4 (The Geometric Satake Correspondence, part II)**.** *If we apply the Decomposition Theorem to this map, we get the decomposition $\bigotimes_{i=1}^m V_{\lambda_i} \cong \bigoplus_{\mu \leq \sum \lambda_i} H^{top}(\beta^{-1}(\mu)) \otimes V_\mu$ whose ${}^LG$-multiplicity spaces come with (semi)canonical bases indexed by components of the "Satake fiber".*

Let us consider the case that each λ_i is a partition. Then

$$BS^w = \{(L_0 \geq L_1 \geq \ldots \geq L_m) \in Gr^{m+1} \colon L_0 = \mathbb{C}[[z]]^n,\ JCF(L_i/L_{i+1}) \leq \lambda_i\}$$

where JCF means the partition of the the "$\leq$" is in dominance order. This is typically written

$$Gr^{\lambda_1} \tilde{\times} Gr^{\lambda_2} \tilde{\times} \cdots \tilde{\times} Gr^{\lambda_m}$$

which is perhaps somewhat silly notation. When each λ_i is a single column, then this is an iterated bundle of $Gr(|\lambda_i|, n)$s.

12.2. **A resolution in type A.** Let λ be a partition of height $\leq n$ and c_i the height of its ith *column* (i.e. the ith entry of λ^T). Then $\lambda = \sum_{i=1}^m \omega_{c_i}$, and the corresponding generalized Bott-Samelson is

$$\{(L_0 \geq L_1 \geq \ldots \geq L_m) \colon L_0 = \mathbb{C}[[z]]^n,\ \dim(L_{i-1}/L_i) = c_i,\ zL_{i-1} \leq L_i\}$$

where the map to Gr^λ extracts L_m.

There is a key difference between type A and other groups: $\overline{Gr^\omega}$ is smooth for every fundamental ω. Under geometric Satake, this smoothness corresponds to the fundamental irrep of ${}^LG$ being **minuscule**, meaning, has only extremal weights. Outside type A, every group has nonminuscule fundamental representations.

Consider now the case that $\lambda \vdash n$, and consider $Gr^{\lambda-(1,1,\ldots,1)}$ (shifted so as to lie in the 0th component of Gr).

12.3. **The Beilinson-Drinfel′d redux.** Fix partitions λ and μ (though the generalization to tuples is straightforward) and consider triples $L_1 \leq L_2 \leq \mathbb{C}[[z]]^n$ with $L_1 \in \overline{Gr^{\lambda+\mu}}$, $L_2 \in \overline{Gr^\lambda}$, $g \cdot L_1 \in \overline{Gr^\mu}$

...Springer resolution example. Geo Satake in that case...

13. The slices $\mathrm{Gr}^{\lambda}_{\mu}$

14. Another Mirković-Vybornov isomorphism

Part 6. Symplectic resolutions and their deformations

15. Kaledin's theorem

...follow Ben's answer at `https://mathoverflow.net/questions/32069/deformations-of-nakajima-rq=1`

Part 7. Classification results

16. Beauville's theorem

Beauville introduces **symplectic singularities** as normal varieties V bearing an algebraic symplectic form on their regular part V_{reg}, such that for some (equivalently any!) resolution $X \to V$ of singularities, the symplectic form on $X_{reg} = V_{reg}$ extends as a 2-form on X. He does not require, as we will elsewhere, that the resulting form on X be nondegenerate. (This would be a minimality condition on X – any further blowup of X along its boundary gives another resolution, but one on which the form will be degenerate along the boundary.)

For convenience we assume that V is conical. (Beauville instead works point by point and considers the tangent cone.) We recapitulate his proof here.

We recall first the adjunction formula:

Theorem 16.1. *Let X be regular, and let σ be a section of the anticanonical bundle vanishing on $D \cup E$ where D is also regular (and in particular, σ vanishes to order 1 on D). Then there is a section of D's anticanonical bundle vanishing on $D \cap E$.*

In particular, if $\iota : D \to X$ is the inclusion, then the anticanonical class $-k_D$ of D is $\iota^*(-k_X + [D])$.

Proof. Since D is regular inside X, it is Cartier, so locally defined by a function f. If we contract σ with the 1-form "$d(\log f)$" $:= df/f$, we get (again locally) a $(\dim X - 1)$-tensor on $X \setminus D$. In formal coördinates (where D is a hyperplane inside a vector space X) one can check that the resulting tensor extends across X, and, its restriction to X doesn't depend on f. □

Theorem 16.2. [?] *Let V be a conical symplectic singularity such that 0 is an isolated singularity. Then V is the closure of a minimal nilpotent orbit for a unique simple Lie algebra $\mathfrak{g}$.*

Proof. Let X be the blowup of V at 0, so, the total space of $\mathcal{O}(1)$ over the projectivization E of V. Let L be the line bundle on X corresponding to the Cartier divisor E, so $L|_E = \mathcal{O}(1)$.

If $\dim V = 2r$, and ω is the 2-form on X, then $\omega^{\wedge r}$ is a top form vanishing only on E to some order k, hence the canonical class of X is $k[E]$. By the adjunction formula, the canonical class of E itself is L^{-1-k}.

□

17. Namikawa's theorem on Springer spaces

18. Namikawa's finiteness theorem

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Last classes:

- April 18
- April 20
- April 25
- April 27
- May 2
- May 4
- May 9

Bow varieties, maybe Mirković-Vybornov, Kaledin's theorem about deformation to affine, Namikawa's theorems, some statements about symplectic duality. Braverman-Finkelberg-Nakajima.