Schubert puzzles and R-matrices

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Abstract

We recast the "puzzle" computation of an equivariant Schubert calculus structure constant as a "scattering amplitude", computed from a planar diagram (specifically, dual to the puzzles). Restrictions $[X_w]|_v$ of equivariant Schubert classes can also be interpreted so, and we use this formalism to give an easy proof of the puzzle rule. The key features to check are the "Yang-Baxter" and "bootstrap" invariance under planar isotopies, requiring the extra freedom of the planar diagrams.

Known solutions of the YBE for the groups A_2, D_4, E_6 let us **discover and prove** puzzle formulæ for K_T of Grassmannian/"1-step" flag manifolds (known from [Pechenik–Yong], [Wheeler–Zinn-Justin]), K_T of 2-step (new), and K of 3-step (new). Maulik–Okounkov create YBE solutions ("R-matrices") using quiver varieties, such as T^*(d-step flag manifolds); we spell out the connection for $d = 1$.

Equivariant Schubert classes on GL_n/P and their restrictions.

Let $G = GL_n$ always, $B_\pm$ the upper/lower triangular matrices with intersection T, and $P \geq B_+$ with Levi $\prod_{i=0}^{d} GL(n_i)$. Then GL_n/P is a **d-step flag manifold** and we can index its B_--orbits by words λ with $sort(\lambda) = 0^{n_0} 1^{n_1} \cdots d^{n_d}$.
Let X_λ be the corresponding orbit closure, and $[X_\lambda] \in K_T(GL_n/P)$ its class in T-equivariant K-theory. If $\lambda = sort(\lambda)$ then $X_\lambda = G/P$, $[X_\lambda] = 1$.

We want formulæ for the $c_{\lambda\mu}^{\nu} \in K_T(pt)$ in the expansion $[X_\lambda]\,[X_\mu] = \sum_\nu c_{\lambda\mu}^{\nu} [X_\nu]$. By Kirwan injectivity, it's enough to prove $[X_\lambda]|_\sigma\,[X_\mu]|_\sigma = \sum_\nu c_{\lambda\mu}^{\nu} [X_\nu]|_\sigma$, an equation in $K_T \cong \mathbb{Z}[e^{\pm y_1}, \ldots, e^{\pm y_n}]$.

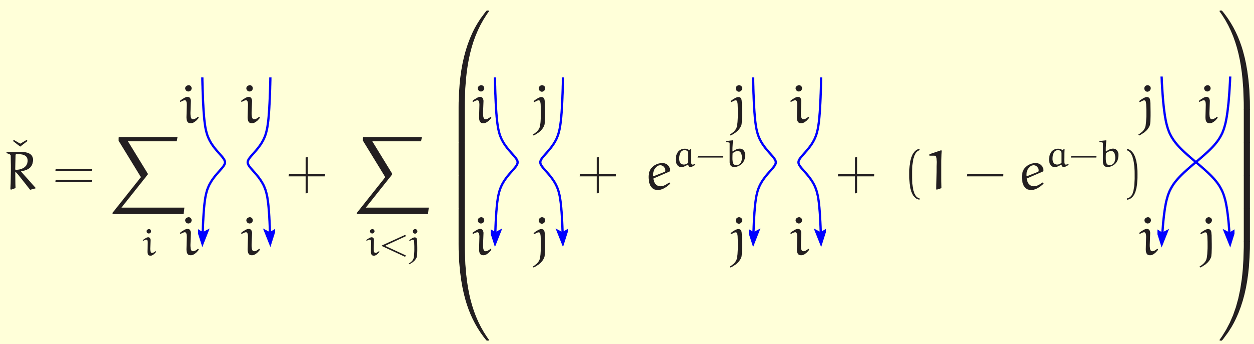
Theorem (AJS/Billey in H_T; Graham/Willems in K_T.) Let Q be a reduced expression for $\sigma \in W^P$. Then $[X_\lambda]|_\sigma$ can be computed as a sum over subwords of Q with Demazure/nil Hecke product (or 0-Hecke product, for H_T^*) equal to λ.

If σ is 321-avoiding, then Q is unique up to (unimportant) commuting moves, and its heap is a skew partition. These hold when $d = 1$ ("Grassmannian permutations are 321-avoiding"), where Q is read from σ's partition [Ikeda-Naruse].

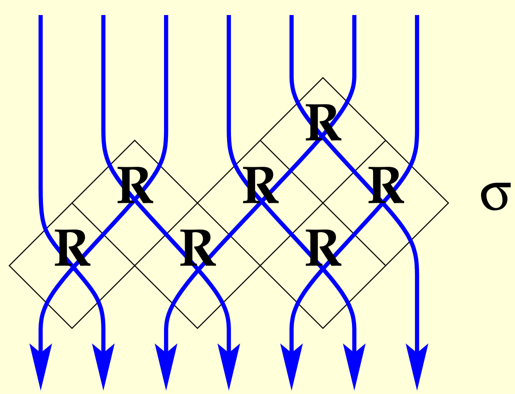
Restrictions to fixed points, as scattering amplitudes.

Let V_a be the vector space with basis $\underset{\downarrow}{0}, \underset{\downarrow}{1}, \ldots, \underset{\downarrow}{d}$, where a is a currently mysterious parameter. Hence the Schubert classes on *all* d-step flag manifolds, taken together, correspond to the tensor basis of $\bigotimes_{i=1}^{n} V_{y_i}$.

Define a very sparse matrix $\check{R} : V_a \otimes V_b \to V_b \otimes V_a$ by specifying only a few of its $(d+1)^4$ entries to be nonzero:



Then $[X_\lambda]|_\sigma$ is the $(\lambda, \mathrm{sort}(\lambda))$ matrix entry in $\prod_Q \check{R} \in \mathrm{End}(\bigotimes_{i=1}^{n} V_{y_i})$, expressed diagramatically as follows:



More general scattering amplitudes.

In the most general setup, we consider edge-colored directed graphs in a disc, with some prescribed lists of colors and of allowed vertices (up to isotopy). Each edge has a parameter, and the vertices may include restrictions on the parameters.

To obtain a number (or rational function) from a graph, which we will call a **scattering amplitude**, we need some more data:

• A vector space with basis for each color.

In Graham/Willems, the only color is the standard rep of $A_d = SL_{d+1}$.

• A tensor in $Hom(\otimes$incoming edges, $\otimes$outgoing edges$)$ for each vertex type, whose matrix entries are functions of the edge parameters.

In Graham/Willems, there is only one kind of vertex, and the in- and outgoing parameters must match up: a, b, a, b.

• For each boundary vertex, a chosen basis element in its vector space.

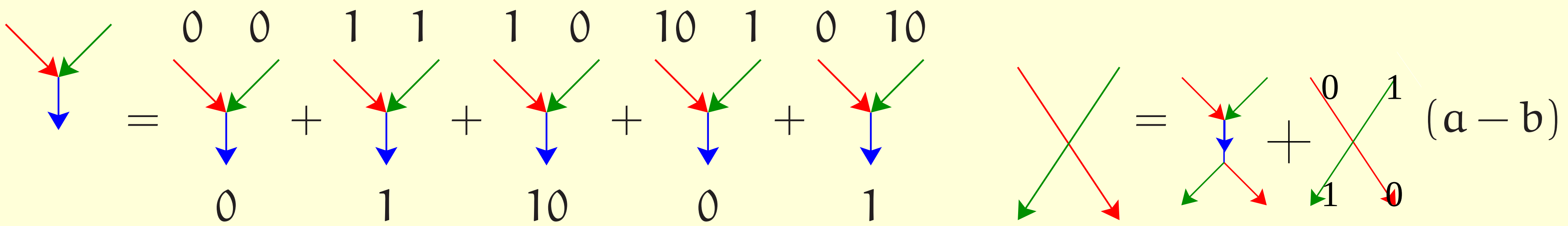
In Graham/Willems, the labels along the bottom are weakly increasing.

The key feature to look for: is the scattering amplitude invariant under isotopies of the graph rel its intersection with the disc? (More about this soon.)

Scattering amplitudes for puzzles: the vertices.

We focus on H_T^* and $d = 1$, where all the salient features are already visible. There are three colors $\mathbb{C}^3$, $\mathbb{C}^3$, and $(\mathbb{C}^3)^*$, irreps of SL_3. (In fact they will extend to irreps of $U_q(\mathfrak{sl}_3[t])$, and the choice of extension involves a parameter.) In all cases the bases are indexed by $\{0, 1, 10\}$.

Then we define three kinds of vertices, two trivalent (one rotated 180° with arrows reversed), and a tetravalent:



On the tetravalent vertex, the parameters must pass through as before; on the trivalent (except inside the tetravalent), all three parameters must match. In both cases the element of $\mathrm{Hom}(\otimes\text{incoming edges}, \otimes\text{outgoing edges})$ will be $U_q(\mathfrak{sl}_3[t])$-equivariant. (The T-equivariance alone suffices to figure out which basis vector corresponds to which of $0, 1, 10$.)

Scattering amplitudes for puzzles: the diagrams.

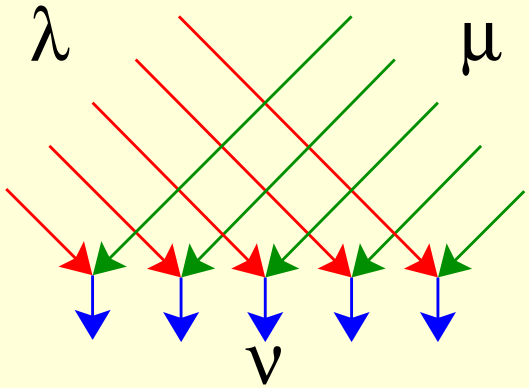
Theorem 1. [K-Tao '03, restated]

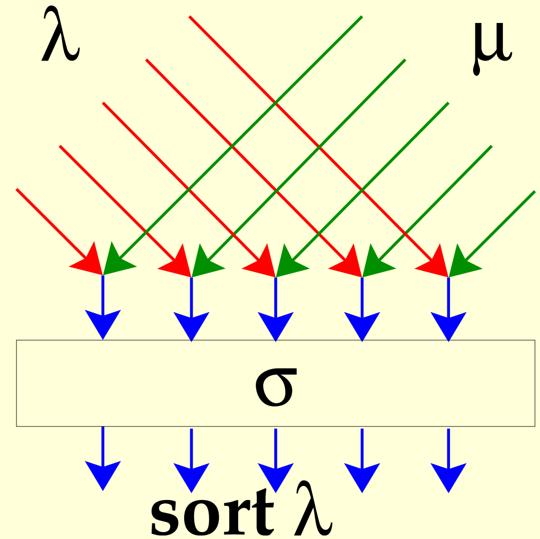
$c_{\lambda\mu}^{\nu}$ is the scattering amplitude of the diagram on the left.

2. (combined with [AJS/Billey])

$\sum_\nu c_{\lambda\mu}^{\nu}[X_\nu]|_\sigma$ is the scattering amplitude of the diagram on the right.

(Note that $sort\ \lambda = sort\ \mu =$ the identity class.)

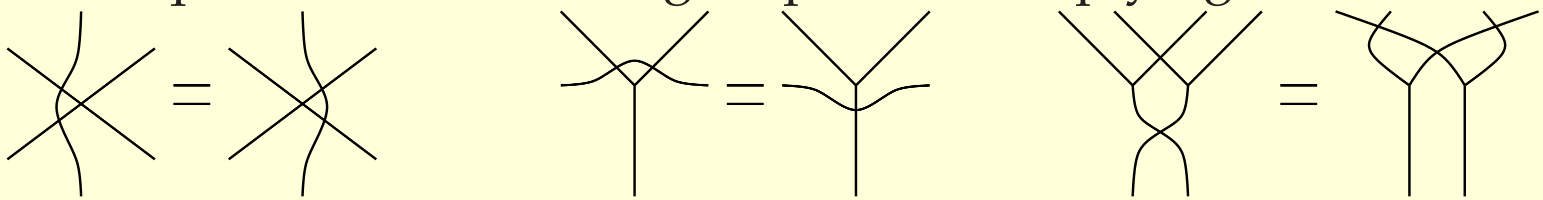




So we've got the RHS of the equation we want to prove, as the scattering amplitude of a single diagram. That suggests that we should manipulate it to get the desired LHS, $[X_\lambda]|_\sigma\,[X_\mu]|_\sigma$.

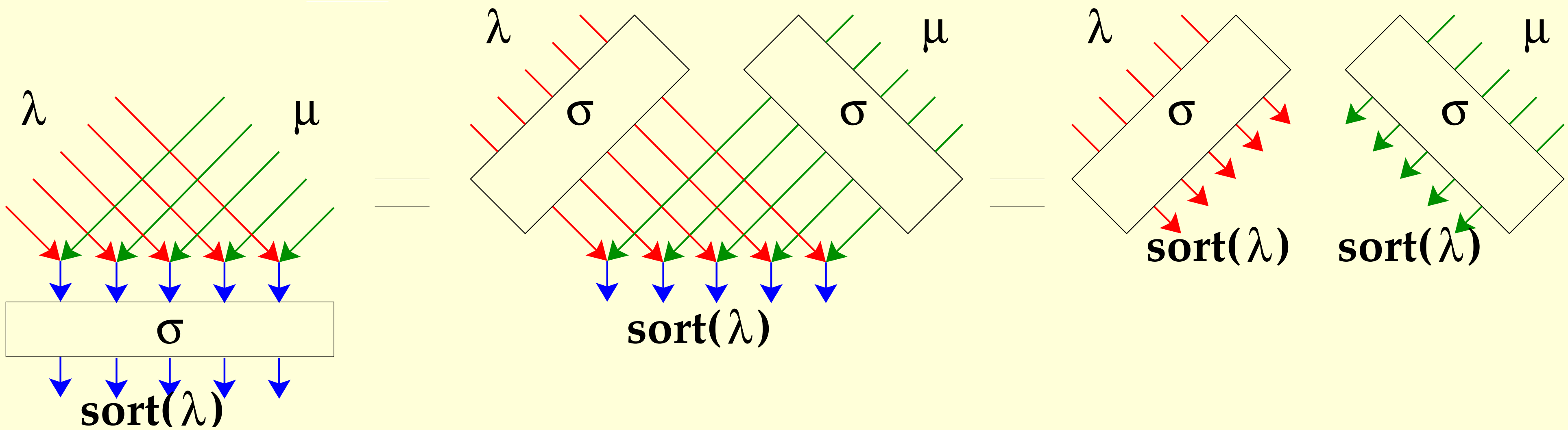
Keys to the proof: The Yang-Baxter and bootstrap equations.

Proposition.
1. With any choice of orientations, colors, and boundary conditions, we have the first two equations on scattering amplitudes, implying the third:



2. If a puzzle has the identity on the bottom, it must also have it on the NW and NE sides, and have scattering amplitude $= 1$.

Hence



so there's our $[X_\lambda]|_\sigma\, [X_\mu]|_\sigma$. Of course proposition #1 above is a big case check.

Sources of solutions to the YBE and bootstrap equations.

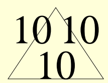
Any minuscule representation V_ω (i.e. all weights extremal) of a Lie algebra $\mathfrak{g}$ extends to its quantized loop algebra $U_q(\mathfrak{g}[z^{\pm}])$, but the extension $V_{\omega,c}$ depends on a choice of parameter c. Then as Drinfel′d and Jimbo observed, the Schur's-lemma-unique (!) map $\check{R} : V_{\omega_1,c} \otimes V_{\omega_2,d} \to V_{\omega_2,d} \otimes V_{\omega_1,c}$ gives a solution to the "trigonometric" YBE (meaning, entries depend only on c/d).

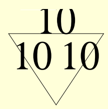
In order to have a trivalent vertex, we need $V_{\omega_1,c} \otimes V_{\omega_2,d}$ to become reducible $\twoheadrightarrow V_{\gamma,e}$, which only happens at special c/d. For our Schubert situation, where we know the ordinary-cohomology specialization should be Z_3-symmetric, we need $Z_3 = \langle \tau \rangle$ to act on $\mathfrak{g}$ and its weight lattice with $\omega_1 = \tau\omega_2 = -\tau^2\gamma$.

Theorem.
$d = 2$. The 8 puzzle edge labels $0, 1, 2, 10, 20, 21, 2(10), (21)0$ now index bases of the three minuscule representations $\mathbb{C}^8$, $spin_+$, $spin_-$ of D_4.
$d = 3$. The 27 labels, including Buch's "three parenthesis rule" labels like $3(((32)1)0)$, now index bases of the minuscule representations $\mathbb{C}^{27}, \mathbb{C}^{27}, (\mathbb{C}^{27})^*$.

These turn out to be easy to guess from the known/conjectured puzzle rules, from two considerations: each puzzle piece/trivalent vertex should be T_G-equivariant (essentially Buch's theory of "auras"), and (for minusculeness) the T-weights associated to edge labels should have the same norm.

Degenerating – or not – the standard $\check{R}$-matrices.

Already at $d = 1$ the $\check{R}$-matrix $\mathbb{C}^3_a \otimes \mathbb{C}^3_b \to \mathbb{C}^3_b \otimes \mathbb{C}^3_a$ has matrix entries we don't see in H^* puzzles: 



If we include only the first, we get K-theory (Buch/Tao); only the second, we get K-theory in the dual basis [Wheeler–ZJ].

Theorem (foreshadowing) 1. If one includes *both* pieces (with factor $+1$ not -1), the resulting puzzles compute the *co*product structure constants of CSM classes under $Gr(k, n) \overset{\Delta}{\hookrightarrow} Gr(k, n) \times Gr(k, n)$.
2. If one gives those pieces independent weights α, β, the resulting algebra is still commutative associative!

Interesting as those are, this says that the standard $\check{R}$-matrix is not quite computing K_T. To "fix" it we rescale various basis vectors by powers of $q^{\pm}$, and let $q \to 0$ (similar to, but not quite the same as, the crystal limit).

Theorem. For $d = 1, 2$ this works great and gets us K_T puzzles.
For $d = 3$ certain matrix entries go to ∞ as $q \to 0$, but we can suppress those by first specializing to the nonequivariant case, which is why we only get K- (and H-)puzzles, not K_T (or H_T). To do K requires 151 new puzzle pieces.
For $d = 4$ we actually have a nice group E_8 and three representations, $\mathfrak{e}_8 \oplus \mathbb{C}$, but alas, even nonequivariance doesn't save $q \to 0$ this time.

Cotangent bundles as quiver varieties.

An A_d **quiver variety** $\mathcal{M}(\vec{h}, \vec{w})$ is associated to two "dimension vectors" $(h_1, \ldots, h_d), (w_1, \ldots, w_d) \in \mathbb{N}^d$, and is a moduli space of representations of the doubled quiver

$$\begin{array}{ccccccccl} \mathbb{C}^{h_1} & & \mathbb{C}^{h_2} & & \cdots & & \mathbb{C}^{h_d} & \text{"framed vertices"} \\ \uparrow\downarrow & & \uparrow\downarrow & & & & \uparrow\downarrow & \\ \mathbb{C}^{w_1} & \begin{smallmatrix}\rightarrow \\ \leftarrow\end{smallmatrix} & \mathbb{C}^{w_2} & \begin{smallmatrix}\rightarrow \\ \leftarrow\end{smallmatrix} & \cdots & \begin{smallmatrix}\rightarrow \\ \leftarrow\end{smallmatrix} & \mathbb{C}^{w_d} & \text{"gauged vertices"} \end{array}$$

such that at each gauged vertex, $\sum(\text{go out then in}) = 0$, plus some open "stability" condition. We mod out by $\prod_i GL(\mathbb{C}^{w_d})$. Let $\mathcal{M}(\vec{h}) := \coprod_{\vec{w}} \mathcal{M}(\vec{h}, \vec{w})$.

Theorems. (Nakajima) $U\mathfrak{sl}_{d+1}$ acts on $H_{top}(\mathcal{M}(\vec{h}))$, making it $V_{\sum_i h_i \omega_i}$, and $H_{top}(\mathcal{M}(\vec{h}, \vec{w}))$ is the $\sum_i h_i \omega_i - \sum_i w_i \alpha_i$ weight space.
(Varagnolo) $U_q(\mathfrak{gl}_{d+1}[y])$ acts on $H_*(\mathcal{M}(\vec{h}))$.
(Nakajima) $U_q(\mathfrak{gl}_{d+1}[e^{\pm y}])$ acts on $K(\mathcal{M}(\vec{h}))$.
As modules, $K(\mathcal{M}(\lambda + \mu)) \cong K(\mathcal{M}(\lambda)) \otimes K(\mathcal{M}(\mu))$.
If $\vec{h} = (n, 0, \ldots, 0)$, then $\mathcal{M}(\vec{h}, \vec{w}) \cong T^* \coprod(\{\text{partial flags in } \mathbb{C}^n \text{ with dims } \vec{w})$.
This last is fun to check; consider powers of $\mathbb{C}^n \to \mathbb{C}^{w_1} \to \mathbb{C}^n$ vs. the images $\mathbb{C}^{w_i} \to \mathbb{C}^n$, and one recognizes the Springer resolution.

Maulik–Okounkov's geometric $\check{R}$-matrices, and $d = 1$ puzzles.

[MO] dress up the natural map $\prod_i \mathcal{M}(\lambda_i) \xrightarrow{\oplus} \mathcal{M}(\sum_i \lambda_i)$ to a "stable envelope" Lagrangian relation, giving a convolution in homology. (If all these spaces are cotangent bundles, we can equivalently map the CSM classes on the base.)

In particular, if the λ_i are minuscule, then the LHS is points indexing the stable basis of $H^*_{T\times\mathbb{C}^\times}(\mathcal{M}(\sum_i \lambda_i))[\hbar^\pm]$, *depending crucially on the order of summands.* If we change this order (say by a simple transposition), then the basis changes, and this change of basis is the generic rational $\check{R}$-matrix!

The boundary labels of $d = 1$ puzzles are restricted to 0 or 1 not 10; correspondingly the A_2 quiver varieties involved reduce to A_1 quiver varieties. (*N.B.* The subspaces $\mathbb{C}^2 \leq \mathbb{C}^3$ on the three sides are Z_3-related, *not* the same!)

Theorem. Consider these two Lagrangian relations relating quiver varieties, the first a stable envelope and the second a symplectic reduction:

$$\mathcal{M}\begin{pmatrix} n & \\ k & 0 \end{pmatrix} \times \mathcal{M}\begin{pmatrix} n & \\ n & k \end{pmatrix} \to \mathcal{M}\begin{pmatrix} 2n & \\ n+k & k \end{pmatrix} \xrightarrow[\mathrm{Id}]{//\,\mathrm{Rad}(P_n)} \mathcal{M}\begin{pmatrix} & n \\ k & k \end{pmatrix}$$

Then the puzzle scattering amplitudes using the generic $\check{R}$-matrix compute the induced map on stable classes. In cohomology, they compute the product in the basis $\{MO_\lambda/[\text{zero section}]\}$ of $K_{T\times\mathbb{C}^\times}(T^*Gr(k,n)) \otimes \mathrm{frac}\, K_{T\times\mathbb{C}^\times}(pt)$.