MATH 7310 FALL 2010: INTRODUCTION TO GEOMETRIC REPRESENTATION THEORY

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CONTENTS

1. SPALTENSTEIN'S THEOREM AND HOTTA'S CONSTRUCTION

This section is a teaser, in three ways. First, we will state a few results but only prove them somewhat later. Second, we will be constructing representions of the Weyl group S_n, but certain key difficulties of the general case will keep this from generalizing to arbitrary Weyl groups. And third, representions of Weyl groups is not our really quarry – we want to construct representions of Lie groups.

Two permutations in S_n are conjugate iff they have the same cycle structure, so the conjugacy classes of are naturally indexed by partitions of n. If S_n were a general finite group, we could infer now only that the number of irreducible representations (hereafter **irreps**) is also $p(n)$, the number of partitions. But it is very special and there is a standard way to index its irreps by partitions.

There is another set naturally indexed by partitions – the conjugacy classes of nilpotent $n \times n$ matrices. (Proof: Jordan canonical form.) Their closures are affine varieties inside the space of all matrices, and this will be the source of the geometry. Our goal in this section is, for each partition, to construct the corresponding irrep directly from the corresponding conjugacy class.

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So let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ denote a partition of n, and let $\mathcal{O}_\lambda$ be the closure of the space of $n \times n$ matrices M with only nilpotent Jordan blocks, of size and multiplicity given by λ. Write $\lambda(M) = \lambda$. Let $(\mathfrak{b})$ denote the space of upper triangular matrices, and B the invertible ones.

1.1. **Orbital varieties.** The **orbital scheme** is the intersection $\mathcal{O}_\lambda \cap \mathfrak{b}$, and its irreducible components[1] are called **orbital varieties**.

Example. Let $n = 3$, and $\lambda = (2,1)$. Then $\mathcal{O}_\lambda = \{M : M^2 = 0\}$ (as a set), so

$$\mathcal{O}_\lambda \cap \mathfrak{b} = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : ac = 0 \right\} = \left\{ \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

The equation $ac = 0$ factors, giving the two components on the right.

Since $\mathcal{O}_\lambda$ carries an action of $\mathbb{G}_m \times GL(n)$ by dilation and conjugation (here $\mathbb{G}_m$ denotes the multiplicative group), and $(\mathfrak{b})$ carries one of $\mathbb{G}_m \times B$ by dilation and B-conjugation, the orbital scheme carries the $(\mathbb{G}_m \times B)$-action too. Since the group is irreducible ($\iff$ connected for groups), it acts on each orbital variety, too.

1.2. **Spaltenstein's map.** Spaltenstein defined a map from the orbital scheme to the set of standard Young tableaux. A **standard Young tableau** is an "English" partition diagram (meaning, in the 4th quadrant) with n boxes (a **Young diagram**), filled with the numbers $1 \ldots n$ increasing down and to the right. They correspond to increasing lists of partitions, from the empty partition to one of size n, adding one box at a time. Young invented them to index a basis of the irrep V_λ of S_n [Kn73].

If M is a nilpotent upper triangular matrix, let M_i denote the upper left $i \times i$ square in M, itself nilpotent upper triangular. Hence each M_i has an associated Jordan canonical form of nilpotent blocks, and associated partition of i, which we will denote $\lambda(M_i)$.

For example, in the $\lambda = (2,1)$ case from above, most elements in the $a = 0$ component go to $\begin{smallmatrix} 1 & 3 \\ 2 & \end{smallmatrix}$, and most elements in the second to $\begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix}$. The zero matrix (which lies in both components) goes to $\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}$.

Theorem 1 (Spaltenstein)**.**
1. *For any nilpotent upper triangular matrix M, the partitions $\lambda(M_0), \lambda(M_1), \ldots, \lambda(M_n)$ are an increasing chain. Hence there is a map from the* **nilpotent cone** *(of all nilpotent matrices of size n) to $SYT(n)$, the standard Young tableaux of size n.*
2. *Each orbital variety in $\mathcal{O}_\lambda \cap \mathfrak{b}$ has the same dimension, half that of $\mathcal{O}_\lambda$.*
3. *For each such orbital variety C, almost all $M \in C$ give the same chain $\lambda(M_0), \ldots, \lambda(M_n)$, hence the same SYT. Also $\lambda(M_n) = \lambda$.*
4. *Hence there is a map from the set of orbital varieties in $\mathcal{O}_\lambda \cap \mathfrak{b}$ to $SYT(\lambda)$, the standard Young tableaux of shape λ. This map is a bijection.*

Proof sketch.
1. An amusing linear algebra exercise.

[1] Don't worry about schemy issues here, e.g. embedded components; we only want the reduced scheme structure on the intersection. This intersection is definitely *not* reduced: upper triangular nilpotent matrices are automatically strictly upper triangular, but we haven't imposed those linear equations on the diagonal. Schemy issues rarely enter geometric representation theory.

(2) The "half" should ring alarm bells for any symplectic geometer, suggesting it should be improved to "Lagrangian", which will be how it is proven later.
(3) For the first conclusion, it is easy to see that Spaltenstein's map is **constructible**, meaning that each fiber is a finite union of locally Zariski-closed subsets. That forces it to have a well-defined generic value on each irreducible component.
(4) I don't know any cheap attack on this.

$\square$

So from $\mathcal{O}_\lambda$, we have constructed $\mathsf{SYT}(\lambda)$ geometrically. The goal will be to get S_n to act on the vector space with that basis.

1.3. **Hotta's construction.**

1.4. **Sweeping.** Given $Y \subseteq X$, and an action of G on X, we can define $G \cdot Y$ or even better the closure $\overline{G \cdot Y} \subseteq X$; call it the **sweep**. If Y is irreducible and G is connected, then $\overline{G \cdot Y}$ is again an irreducible subvariety of X. Proof: $G \cdot Y$ is the image of the composite $G \times Y \to G \times X \to X$, G connected makes it and $G \times Y$ irreducible, so the image has irreducible closure.

If Y is B-invariant for some $B \leq G$ (not, at this point, necessarily the upper triangular matrices), then we can say more. Let $G \times^B Y := (G \times Y)/B_\Delta$, where $B_\Delta = B$ acts on G on the right and also on Y. (This is *not* a fiber product, and hence shouldn't be written $G \times_B Y$, but one sees that sometimes.) Then the map $G \times Y \to X$ factors through $G \times^B Y$.

Proposition 1. *Let $G \geq B$ be such that G/B is compact. Let G act on X, with B preserving a closed subset Y. Then $G \cdot Y$ is already closed, of dimension $\leq \dim Y + \dim G/B$, and the map $G \times^B Y \to G \cdot Y$ is proper.*

Proof. The map $G \times^B Y \to G \times^B X$ is a closed inclusion. Then the map $G \times^B X \to X$ is a bundle with fibers G/B, hence proper. So the composite $G \cdot Y \to X$ has closed image. $\square$

1.5. **Hotta's construction.** To define an action of S_n on something, it suffices to define an action of each generator $(i \leftrightarrow i+1)$, $i = 1, \ldots, n-1$. We will be defining it on the free $\mathbb{Z}$-module with basis $\{[C_\tau]\}$, where C_τ denotes the orbital variety to which Spaltenstein associates the SYT τ and $[C_\tau]$ denotes the corresponding basis vector.

Given $i < n$, define $\mathfrak{p}_i$ to be the group of *almost*-upper triangular matrices – they are allowed to be nonzero in position $(i+1, i)$. Let P_i be the invertible such, and let $\mathrm{rad}(P_i)$ be the vector space of strictly upper triangular matrices with $M_{i,i+1} = 0$ (so named because it is the Lie algebra of the unipotent radical $\mathrm{Rad}(P_i)$ of the Lie group P_i). Note that P_i acts on $\mathfrak{p}_i$ by conjugation, preserving $\mathrm{rad}(P_i)$. Note also that $P_i/B \cong (P_i/\mathrm{Rad}(P_i))/(B/\mathrm{Rad}(P_i))$ $\cong PSL(2)/B_{PSL(2)} \cong \mathbb{P}^1$ is compact.

Given $i < n$ and C_τ an orbital variety, do three steps:

(1) *Cut.* Let $C'_\tau := C_\tau \cap \{M : M_{i,i+1} = 0\} = C_\tau \cap \mathrm{rad}(P_i)$. For this, we *do* want the scheme-theoretic intersection. Since B acts on C_τ and $B \leq P_i$ acts on $\mathrm{rad}(P_i)$, it acts on C'_τ and its components.
   If $C'_\tau = C_\tau$, let $(i \leftrightarrow i+1) \cdot [C_\tau] = [C_\tau]$. Otherwise...

(2) Let the components of C'_τ be $H_1, \ldots, H_f$, with scheme-theoretic multiplicities $m_1, \ldots, m_f$. (If C'_τ has lower-dimensional embedded components, toss them.) So each H_j is reduced by definition, and carries a B-action.

(3) *Sweep.* Let d_j be the degree of the map $P_i \times^B H_j \to P_i \times H_j$, or 0 if they are not the same dimension ($\iff$ H_j is P_i-invariant).

If $d_j \neq 0$, then $\dim P_i \times H_j > \dim H_j$, but we already know $\dim P_i \times H_j \leq \dim H_j + \dim \mathbb{P}^1$. Hence $\dim P_i \times H_j = \dim H_j + 1 = \dim C_\tau = \dim \mathcal{O}_\lambda \cap \mathfrak{b}$. So $P_i \times H_j$ must be an orbital variety.

Let $(i \leftrightarrow i+1) \cdot [C_\tau] = -\sum_j d_j m_j [P_i \times H_j] - [C_\tau]$.

Let's do this in the $lambda = (2,1)$ example above. The orbital variety $C_{\substack{1\,3\\2}}$ is P_1-invariant, and $C_{\substack{1\,2\\3}}$ is P_2-invariant, so $r_1 \cdot [C_{\substack{1\,3\\2}}] = [C_{\substack{1\,3\\2}}]$, $r_2 \cdot [C_{\substack{1\,2\\3}}] = [C_{\substack{1\,2\\3}}]$. (Here $r_i := (i \leftrightarrow i+1)$.)

The other C'_τs are both $H := C_{\substack{1\,3\\2}} = C_{\substack{1\,2\\3}}$, so reduced. Hence $m_j = 1$. Then $P_2 \cdot H = C_{\substack{1\,3\\2}}$ and $P_1 \cdot H = C_{\substack{1\,3\\2}}$, in each case with $d_j = 1$. In all, the matrices are

$$r_1 \mapsto \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \qquad r_2 \mapsto \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}.$$

This is isomorphic to the Coxeter representation (S_3 acting on sum-zero vectors in $\mathbb{Z}^3$), under $[C_{\substack{1\,3\\2}}] \mapsto (1,1,-2)^T$, $[C_{\substack{1\,2\\3}}] \mapsto (-2,1,1)^T$.

Theorem 2 (Hotta)**.** *These operators on the vector space spanned by the $\{[C_\tau]\}$ satisfy the defining relations of S_n, namely, faraway generators commute and nearby ones braid. So they define an action of S_n, and the corresponding representation is indeed the irrep V_λ.*

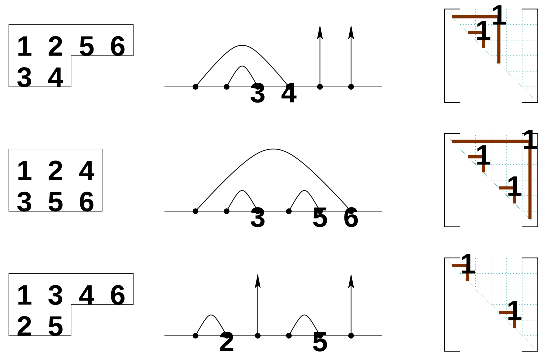
It is easy to prove that $GL_n(\mathbb{N})$ consists only of permutation matrices, so the only irrep (of any group) with all-natural entries is the trivial one. In this sense the positivity in the theorem above (namely, having predictable signs) is as much as we can expect. It would be nice to know what positivity holds for elements of S_n other than the generators; I do not know of any result in this direction.

One of the reasons for interest in geometric representation theory, of which the above is our first example, is that it produces representations *with canonical bases*, typically with integrality and positivity properties.

1.6. **Example: λ has only two rows.** This case is especially nice because $\mathcal{O}_\lambda$ is a "spherical variety", meaning that B acts on it with only finitely many orbits. In particular each orbital variety is a B-orbit closure. This makes the geometry much easier to analyze.

In this case the tableaux correspond to *partial matchings of* $1, \ldots, n$ *with no crossings.* The unmatched are "matched with ∞", and the numbers in the second row are the right sides of matches. That, plus no crossings, forces everything else.

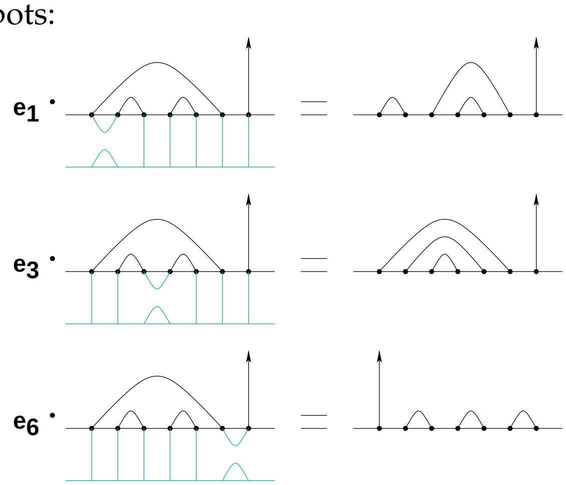
Turned 45°, each match gives a partial permutation matrix $M_\tau \in X_\tau$. In fact $X_\tau = \overline{B \cdot M_\tau}$, where B is the group of invertible upper triangular matrices, and $\cdot$ = conjugation.



If τ's matching doesn't connect i and $i+1$, then $X_\tau \subseteq \mathrm{rad}(P_i)$, so $r_i \cdot [X_\tau] = [X_\tau]$.

If it *does* connect i and $i+1$, things are more interesting...

Define the operator e_i on the set of crossingless partial matchings that glues an hour-glass under the $i, i+1$ spots:



Its image is $\{$ matchings τ with $i, i+1$ connected $\}$. Then the formula turns out to be

$$r_i \cdot [X_\tau] = -\sum_{\tau'} [X_{\tau'}] \qquad \text{sum taken over those } \tau' \text{ with } e_i \cdot \tau' = \tau.$$

Pleasantly, the coefficients m_γ are all $0, 1$. Alas,

Theorem [McLarnan–Warrington 2003]. *At* $N = 16$ *(and* λ *with more than two rows) one begins to encounter* $m_\gamma > 1$.

2. Equivariant cohomology and divided differences

Let $T_\mathbb{R} \cong U(1) \times \cdots \times U(1)$ be a **torus** group, and $T \cong \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$ be its complexification. Let $T^* := \mathrm{Hom}(T_\mathbb{R}, U(1))$ be its **weight lattice**, isomorphic to $\mathbb{Z}^{\dim T}$.

On the category of T-spaces and T-equivariant maps, there is a cohomology theory

$$H_T^* : T - \mathbf{Top} \to \text{graded-commutative rings}$$

with the following simple properties:

(1) $H_T^*(pt) = Sym(T^*) \cong \mathbb{Z}[x_1, \ldots, x_{\dim T}]$ with generators all of degree $2 = \dim_\mathbb{R} \mathbb{C}$ (so they commute, not anticommute).

(2) Pullback ring homomorphisms are actually $H^*_T(pt)$-algebra homomorphisms.
(3) There exists a pushforward for proper maps of oriented manifolds (or complex varieties) to oriented manifolds, that is $H^*_T(pt)$-linear, and changes degree by the change in dimension
(4) Complex vector bundles with compatible T-actions have "equivariant Euler classes". If V is a T-representation with weights $\mu_1, \ldots, \mu_{\dim V}$, thought of as an equivariant vector bundle over a point, then $e(V) = \prod_{i=1}^{\dim V} \mu_i$.
(5) If $S \to T$ is a torus homomorphism, making each T-space X also an S-space, then there is a natural map $H^*_S(pt) \otimes_{H^*_T(pt)} H^*_T(X) \to H^*_S(X)$, where the map $H^*_T(pt) \to H^*_S(pt)$ comes from the transpose map $T^* \to S^*$.

and the following much less obvious ones:

(1) If M is a smooth projective T-variety, the pullback map $H^*_T(M) \to H^*_T(M^T) \cong H^*_T(pt) \otimes H^*(M^T)$ is injective.
(2) The same holds if M is a vector space, or a vector bundle over a smooth projective variety.
(3) In these cases, the map $H^*_S(pt) \otimes_{H^*_T(pt)} H^*_T(X) \to H^*_S(X)$ is an isomorphism over $\mathbb{Q}$. For example, one can take $S = 1$ to compute $H^*(X; \mathbb{Q})$ from $H^*_T(X)$, which is often easier!

2.1. **Multidegrees.** For example, if $X \subseteq V$ is a T-invariant subvariety of a T-representation V, then we can push $1 \in H^0_T(X)$ into $H^*_T(V)$, and obtain a class $[X] \in H_T^{2\,\mathrm{codim}\,X}(V) \cong H_T^{2\,\mathrm{codim}\,X}(pt)$. Using an isomorphism of T with $(\mathbb{C}^\times)^{\dim T}$, this becomes a polynomial of degree $\dim T$, and is called a **multidegree** $\mathrm{mdeg}_V X$.

The theory of multidegrees needs less development than full equivariant cohomology (but is not as powerful). It can be characterized as follows:

(1) $\mathrm{mdeg}_{\{0\}}\{0\} = 1$.
(2) If X is a union of components X_i with multiplicities m_i, then

$$\mathrm{mdeg}_V X = \sum_{\dim X_i = \dim X} m_i \deg_V X_i.$$

(3) If H is a T-invariant hyperplane, and X prime,
   (a) and $X \not\subseteq H$, then $\mathrm{mdeg}_V X = \mathrm{mdeg}_H X \cap H$.
   (b) and $X \subseteq H$, then $\mathrm{mdeg}_V X = (\mathrm{mdeg}_H X)\, wt(V/H)$.
   where $wt(V/H) \in T^*$ is the weight of the torus action.

With this, we can make the Hotta result more precise, placing all the cut/sweep geometry into the multidegrees of the orbital varieties $\{C_\tau\}$.

Theorem 3 (Joseph 1984). *Let $\mathfrak{n}$ denote the strictly upper triangular matrices, and T the invertible diagonal matrices acting on $\mathfrak{n}$ by conjugation. Then S_N acts on T and T^* and $\mathrm{Sym}(T^*)$, and the multidegrees $\{\mathrm{mdeg}_{\mathfrak{n}} C_\tau\}$ span an S_N-irrep.*

In the $\lambda = (2,1)$ example, the weights of T on $\mathfrak{n}$ are

$$\begin{bmatrix} \cdot & y_1 - y_2 & y_1 - y_3 \\ \cdot & \cdot & y_2 - y_3 \\ \cdot & \cdot & \cdot \end{bmatrix}$$

So using the axioms above,

$$\operatorname{mdeg}_n[C_{\substack{1\ 3\\2}}] = y_1 - y_2, \quad \operatorname{mdeg}_n[C_{\substack{1\ 2\\3}}] = y_2 - y_3.$$

2.2. **Sweeping and divided differences.**

Lemma 1. *Let $P > B > T$ be a triple of Lie groups such that $P/B \cong \mathbb{P}^1$, and T is a torus acting with weight $\mu \neq 0$ on $\mathfrak{p}/\mathfrak{b}$. Let $\tilde{r} \in N_P(T)$ be an element of the normalizer, inducing an automorphism r of T^* such that $r \cdot \mu = -\mu$.*

Let V be a P-representation, and $X \subseteq V$ a B-invariant subvariety. Then

$$d[P \cdot X] = \frac{[X] - r \cdot [X]}{\mu} \qquad \in H_T^*(V)$$

where d is the degree of the map $P \times^B Y \to P \cdot Y$ (or 0 if Y is P-invariant).

Of course the $P > B > T$ one should think of are the groups from the Hotta section, with $\tilde{r}$ the permutation matrix of r_i, and $\mu = y_i - y_{i+1}$. In this case one writes ∂_i for the **divided difference operator** implemented on the right-hand side above. (Warning: this is very likely to be $-\partial_i$, depending on conventions!)

Proof. Just as X is a subvariety of V, the space $P \times^B X$ is a subvariety of the vector bundle $P \times^B V$ over P/B, and has a T-equivariant cohomology class.

On $P/B \cong \mathbb{P}^1$, we have an equation

$$[\vec{0}] - [\vec{\infty}] = \mu \qquad \in H_T^*(P/B)$$

where μ is from the base ring, $H_T^*(pt)$. (Proof: we can check by restricting to fixed points, since P/B is smooth and projective. But we can first restrict to the two patches on $\mathbb{P}^1$, and in each of those we are doing a multidegree calculation. The T-weights on the patches are $\mu, -\mu$.)

Pull that equation back to $P \times^B V$, to get

$$[B \times^B V] - [\tilde{r}B \times^B V] = \mu \qquad \in H_T^*(P \times^B V)$$

then multiply by $[P \times^B X]$:

$$[B \times^B V][P \times^B X] - [\tilde{r}B \times^B V][P \times^B X] = \mu[P \times^B X] \qquad \in H_T^*(P \times^B V)$$

These multiplications can be computed by transverse intersections:

$$[B \times^B X] - [\tilde{r}B \times^B X] = \mu[P \times^B X] \qquad \in H_T^*(P \times^B V)$$

Now push this forward to V:

$$[X] - r \cdot [X] = \mu d[P \cdot X] \qquad \in H_T^*(V).$$

$\square$

2.3. **Proof of part of Joseph's theorem.** We follow the $(T \times \mathbb{C}^\times)$-multidegrees inside $\mathfrak{p}_i$ in Hotta's construction, starting with C_τ. Here $\mu = y_i - y_{i+1}$.

(1) *Cut.* If $C'_\tau \neq C_\tau$, then $[C'_\tau] = \mu[C_\tau]$.
(2) $[C'_\tau] = \sum_j m_j[H_j]$.
(3) *Sweep.* Apply P_i to the $\{H_j\}$, and use the lemma. We get

$$\begin{aligned}\sum_j m_j d_j [P_i \cdot H_j] &= \sum_j m_j \frac{[H_j] - r_i \cdot [H_j]}{\mu} = \frac{1}{\mu}(\sum_j m_j[H_j] - r_i \cdot \sum_j m_j[H_j]) \\ &= \frac{1}{\mu}([C'_\tau] - r_i \cdot [C'_\tau]) = [C_\tau] + r_i \cdot [C_\tau]\end{aligned}$$

so

$$r_i \cdot [C_\tau] = \sum_j m_j d_j [P_i \cdot H_j] - [C_\tau] \qquad \text{in } H^*_{T \times \mathbb{C}^\times}(\mathfrak{p}_i).$$

Now, these are not quite the same polynomials as the ones Joseph studies, since he works in $\mathfrak{n}$, not $\mathfrak{p}_i$. So we have to divide by the product of the weights in $\mathfrak{p}_i/\mathfrak{n}$ (which is nonzero because we, unlike Joseph, included the dilation action). The only interesting one is the weight on $\mathfrak{p}_i/\mathfrak{b}$, as r_i negates it. But that doesn't change the result, which is that $r_i \cdot [C_\tau]$ is a $\mathbb{Z}$-linear combination of the multidegrees of other orbital varieties.

We haven't shown yet that the multidegrees are linearly independent, that their span is an irrep, or that the irreps for different λ are different.

2.4. **Joseph-Melnikov polynomials.** The T-multidegrees of orbital varieties are known as **Joseph polynomials** since A. Joseph invented multidegrees in order to define them.

We give a recipe to compute them, but only in the $\mathcal{O} = \{M^2 = 0\}$ case. The rule is inductive and uses more than just orbital varieties: it needs all the B-orbit closures $\mathcal{O}_\pi := \overline{B \cdot \pi_<}$, where $\pi_<$ denotes the strict upper triangle of the involution $\pi \in S_n$. To each π, we associate a chord diagram, as in figure 1, and a polynomial $\mathrm{mdeg}_{\mathbb{C}^\times \times T} \mathcal{O}_\pi \in \mathbb{Z}[A, y_1, \ldots, y_n]$ we will call the **extended Joseph-Melnikov** polynomial. ("Extended" for the extra dilation circle $\mathbb{C}^\times$.)

Proposition 2. [KnZJ, §2] *Let $\pi \in S_N$ be an involution, with an associated chord diagram also called π. Construct a new chord diagram ρ in one of three ways:*

(1) *If two arches in π border a common region, but don't cross, make them touch and turn that into a new crossing;*
(2) *if an arch and a half-line in π border a common region, but don't cross, make them touch and turn that into a new crossing;*
(3) *if an arch crosses all the half-lines, and borders the unbounded region, break it there into two half-lines.*

Then $\pi > \rho$ in the poset of orbit closures, i.e. $\overline{B \cdot \pi_<} \supset B \cdot \rho_<$. These are exactly the covering relations in the poset of B-orbit closures.

Theorem 4. [Ro] *Let π be an involution, and $a < b$ a minimal chord in π, i.e. $\pi(a) = b$ and $\nexists c, d$ with $a < c < d < b$, $\pi(c) = d$.*

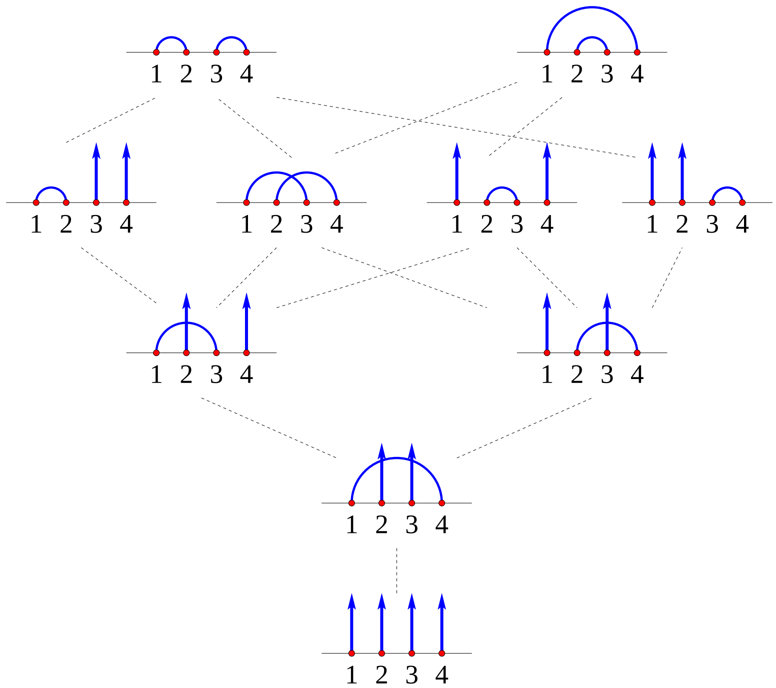


FIGURE 1. The poset of B-orbits for $N = 4$. The row gives the dimension, from 4 at the top down to 0 for the orbit $\{0\}$.

Let ρ vary over the set of involutions such that π covers ρ in the poset of B*-orbits, and there is no chord connecting a, b. Then for each such ρ we have $A + y_a - y_b \mid J_\rho$, and*

$$J_\pi = \sum_\rho \frac{J_\rho}{A + y_a - y_b}$$

Using this one can calculate inductively, where the base of the induction is $J_\mathbf{1} = \prod_{i<j}(A+y_i-y_j)$.

Proof sketch. Part of this is quite direct from the properties we used to define multidegrees. We slice $\overline{B \cdot \pi_<}$ with the hyperplane $\{M_{ab} = 0\}$, which does not contain it since $\pi(a) = b$. By the other condition on a, b, the intersection is again B-invariant.

Hence the intersection is supported on $\bigcup_\rho \overline{B \cdot \rho}$, and it remains to check that the multiplicities are all 1, a tangent space calculation done in [Ro].

(It is interesting to note that the same construction, applied to more general B-orbit closures in $\{M^2 = 0\}$, can produce multiplicities 1 or 2.) □

So in the $n = 3$ case, the extended Joseph-Melnikov polynomials are

$$J_{123} = (A + y_1 - y_2)(A + y_1 - y_3)(A + y_2 - y_3)$$
$$J_{321} = (A + y_1 - y_2)(A + y_2 - y_3)$$

so the extended Joseph polynomials are

$$J_{213} = (A + y_2 - y_3)$$
$$J_{132} = (A + y_1 - y_2)$$

with

$$r_1 \cdot J_{213} = J_{132} + J_{213}, \quad r_1 \cdot J_{132} = -J_{132} \qquad \mod A.$$

3. Review of: Borel subgroups, parabolic subgroups, the Bruhat decomposition

Let G be a connected complex reductive affine algebraic group, which forces it to be a product of factors of the following types:

- $\mathbb{C}^\times$
- $A_n = SL_{n+1}(\mathbb{C})$
- $B_n = SO_{2n+1}(\mathbb{C})$
- $C_n = Sp(\mathbb{C}^{2n})$
- $D_n = SO_{2n}(\mathbb{C})$
- five other cases called G_2, F_4, E_6, E_7, E_8

up to a finite group. For example, $GL_n(\mathbb{C}) \cong (\mathbb{C}^\times \times SL_n(\mathbb{C}))/Z_n$. Hereafter we will usually omit mention of the field $\mathbb{C}$.

A **Borel subgroup** B of G is a maximal solvable connected subgroup. They are motivated by Borel's theorem:

Theorem 5. *[Borel] Let a connected solvable group B act on a complete (e.g. projective) nonempty variety X. Then there is a fixed point.*

Proof sketch. If B is one-dimensional, we can take a point $x \in X$ and reduce to the case $X = \overline{B \cdot x}$, where it becomes easy. If $\dim B > 1$, then it turns out it has a normal subgroup N s.t. $\dim N, \dim B/N < \dim B$. So first take N-fixed points, and then in there, take B/N-fixed points. $\square$

A **parabolic subgroup** P of G is one that contains a Borel.

Corollary 1. *Let G act on a complete variety X, and $G \cdot x$ be an orbit of smallest dimension. Then $G \cdot x \cong G/P$ for some parabolic P.*

Standard stuff from algebraic groups books (e.g. Borel's):

- G/B is projective
- all Borels are conjugate
- $B = T \ltimes N$, where T is a maximal torus and $N = B'$ is the commutator subgroup and **unipotent radical**

Let $W := N_G(T)/T$ be the **Weyl group** of G. Note that the sequence $1 \to T \to N_G(T) \to W \to 1$ usually doesn't split, e.g. for $G = SL_2$. (It does for $G = GL_n$.)

Theorem 6 (Bruhat decomposition)**.** *Let T be a maximal torus inside B, a Borel subgroup of G. Let $\{\tilde{w} \in N(T)\}_{w \in W}$ be a system of representatives. Then*

$$G/B = \coprod_W N\tilde{w}B/B$$

where $N\tilde{w}B/B$ is isomorphic to affine space of dimension $\ell(w)$, the length of w as an element of the Coxeter group W.

Proof in the $G = GL_n$ case. Equivalently, $GL_n = \coprod_W N\tilde{w}B$, or every invertible matrix M can be reduced to a unique permutation matrix $\tilde{w}$ by upward row operations, rightward column operations, and scaling columns. Existence is easy to prove inductively by starting from the lower left. For uniqueness, observe that the ranks of the lower left rectangles of M are unchanged by these operations, and distinguish the permutation matrices.

The statement about $N\tilde{w}B/B$ is a sort of reduced row-echelon form statement. Using N, we can put stuff above the 1s in the permutation matrix $\tilde{w}$. Using B, we can cancel out stuff to the right of 1s. So a spot (i,j) is left over if $w(i) < i$ but $w^{-1}(j) < j$, i.e. if $(i, w(j))$ is an inversion of w. Hence the dimension is the number of inversions in w, which is its length. $\square$

Let G be reductive, and T a maximal torus within. For any subgroup $F \geq T$, let $\Delta_F \subseteq \Delta$ be the set of roots in its Lie algebra $\mathfrak{f}$, and $W_F := N_F(T)/T \leq W$. Then $B \geq T$ is a Borel iff Δ_B forms a system Δ_+ of positive roots. If $P \geq B$ is a parabolic subgroup, then Δ_P is characterized by its intersection with the set Δ_{-1} of negative simple roots, and every subset arises. Let $L \leq P$ be the subgroup containing T and all root spaces in $\Delta_P \cap -\Delta_P$; this reductive group is called a **Levi subgroup** and depends on the choice of T. Whereas the **unipotent radical** $Rad(P)$, which uses the roots $\Delta_P \setminus -\Delta_P$, is a normal subgroup.

Theorem 7 (Bruhat decomposition, parabolic case). *Let T be a maximal torus inside Q, P, two parabolic subgroups of G. Let $\{\tilde{w} \in N(T)\}_{w \in W}$ be a system of representatives of $W_Q \backslash W / W_P$. Then*

$$G/P = \coprod_W Q\tilde{w}P/P.$$

If $Q = B$ and each w is chosen minimal length in its coset in W/W_P (theorem: these exist uniquely, and the set of them is denoted W^P), then $Q\tilde{w}P/P$ is again an affine space of dimension $\ell(w)$.

Example: $G = GL_n$, G/P is projective space, so $W_P = S_1 \times S_{n-1}$. Then $W^P = \{k123\ldots k-1k+1\ldots n\}_{k=1,\ldots,n}$, giving the usual decomposition into n cells of dimensions $0, 1, \ldots, n-1$.

If $Q > B$, the orbits are not cells, e.g. if $Q = G$!

3.1. Grassmannians.

We will have particular need of the case of Grassmannians. Let $V = V_1 \oplus V_2 \oplus \ldots \oplus V_d$, and let $\mathbb{C}^\times$ act on V_i with weight n_i, increasing with i. This decomposition defines two parabolics $P_\pm \leq GL(V)$, where P_- preserves each $V_{\leq i} := V_1 \oplus \ldots \oplus V_i$ and P_+ preserves each $V_{\geq i} := V_i \oplus \ldots \oplus V_n$. Denote their intersection $L = \oplus_i GL(V_i)$.

A $\mathbb{C}^\times$-fixed point $W \in Gr_k(V)$ is exactly one of the form $\oplus_i (W \cap V_i)$. The fixed point components are indexed by their dimension vectors $(\dim(W \cap V_i))_i$, and are the closed L-orbits, of the form $\prod_i Gr_{\dim(W\cap V_i)}(V_i)$.

Each closed L-orbit should be in one P_+-orbit and one P_--orbit. How do we describe those larger orbits in terms of these intersections?

Given an arbitrary $W \in Gr_k(V)$, we can consider its limit $\lim_{z\to 0} z \cdot W$, necessarily a $\mathbb{C}^\times$-fixed point. To figure out which one, use the isomorphism of $gr_< V := \oplus_i V_{\leq i}/V_{\leq i-1}$ with V, taking $gr_< W$ to $\lim_{z\to 0} z \cdot W$. To figure out only which component will contain $\lim_{z\to 0} z\cdot W$, it is enough to know the dimensions $\dim(W \cap V_{\leq i})$. These pick out the P_--orbits.

Similarly, we get $gr_> W \mapsto \lim_{z\to\infty} z \cdot W$, with its component (and P_+-orbit) determined by the dimensions $\dim(W \cap V_{\geq i})$.

4. The Steinberg scheme

Fix a Lie algebra $\mathfrak{g}$, and let $\mathcal{N}$ denote the **nilpotent cone** in $\mathfrak{g}$. For $\mathfrak{g} = \mathfrak{gl}_n$ this is the usual nilpotent elements, defined by the vanishing of the characteristic polynomial. Let $\mathcal{B}$ be the space of Borel subgroups in G; if we fix a particular one B, then $\mathcal{B} \cong G/B$.

Let $\tilde{\mathcal{N}}$ be the set of pairs $(B \in \mathcal{B}, X \in \mathcal{N} \cap \mathfrak{B})$. It carries a G-action by conjugation on both factors, and G-equivariantly projects onto either factor.

Proposition 3. *$\tilde{\mathcal{N}} \cong T^*\mathcal{B}$. In particular it is smooth.*

Proof. The projection onto the first factor makes $\tilde{\mathcal{N}}$ into *some* vector bundle over $\mathcal{B}$. To identify it with the cotangent bundle, we need to put a perfect pairing on the fiber and the tangent space.

The tangent space to $\mathcal{B}$ at B is $\mathfrak{g}/\mathfrak{b}$, easily seen by using the transitive G-action. The fiber is $\mathfrak{n}$, for $N = B'$. To pair them, use the Killing form on $\mathfrak{g}$, which vanishes on $\mathfrak{b} \otimes \mathfrak{n}$ and so is well-defined.

(Maybe I should worry about why it's nondegenerate...) $\square$

The **Grothendieck-Springer resolution** of $\mathcal{N}$ is the projection map $T^*\mathcal{B} \twoheadrightarrow \mathcal{N}$. It is plainly proper; to see that it is a "resolution" we need to know it is birational, which is the statement that generic nilpotents are contained in unique Borel subalgebras.

The **Steinberg scheme** is the fiber product of two Grothendieck-Springer resolutions:

$$Z := \{(B_1, B_2 \in \mathcal{B}, X \in \mathcal{N}) : X \in \mathfrak{b}_1 \cap \mathfrak{b}_2)\}.$$

It is a G-invariant subset of $\mathcal{B} \times \mathcal{B} \times \mathcal{N}$. I have asked around, but nobody seems to know for sure what equations define it as a scheme.

Theorem 8. *As a set, Z is the union of the conormal bundles to the G-orbits on $\mathcal{B}^2$. In particular, its components are indexed by W, and all have the same dimension $(2 \dim \mathcal{B})$.*

This fact that the fiber square of $\tilde{\mathcal{N}} \twoheadrightarrow \mathcal{N}$ does not have surprisingly large components says that it is a "small" resolution, which basically means that its fibers are not too large too often. This implies that $\mathcal{N}$'s "intersection homology" (a replacement for ordinary cohomology when studying singular spaces; I hope to discuss it later) matches $\tilde{\mathcal{N}}$'s ordinary cohomology.

Proof.

$$\mathcal{B}^2/G \cong (G/B)^2/G \cong B\backslash G/B \cong W$$

This stratification of $\mathcal{B}^2$ induces a decomposition of Z; we will show that its pieces are all of the same dimension.

In each orbit we can find a representative (B, wBw^{-1}). The tangent space to the orbit at that point is isomorphic to

$$\mathfrak{g}/(\mathfrak{b} \cap w \cdot \mathfrak{b}) \cong \mathfrak{b}_- \oplus \mathfrak{n}/(\mathfrak{n} \cap w \cdot \mathfrak{b})$$

whereas the fiber of the projection $Z \twoheadrightarrow \mathcal{B}^2$ is

$$\mathfrak{b} \cap w \cdot \mathfrak{b} \cap \mathcal{N} = (\mathfrak{b} \cap \mathcal{N}) \cap w \cdot \mathfrak{b} = \mathfrak{n} \cap w \cdot \mathfrak{b}.$$

This makes it obvious that the pieces are all the same dimension. But it's also easy to check that these are orthogonal complements under the pairing we used to identify $\tilde{\mathcal{N}}$ with $T^*\mathcal{B}$. $\square$

Let $\mathcal{N}/G$ be the set of nilpotent orbits (which turns out to be finite, as we know directly for $G = GL_n$). The projection $Z \twoheadrightarrow \mathcal{N}$ descends to a map $W \twoheadrightarrow \mathcal{N}/G$ which we will write as $w \mapsto \mathcal{O}(w)$.

But one can do better. For each orbit $\mathcal{O} \in \mathcal{N}/G$, let $\mathcal{C}_\mathcal{O}$ be the set of orbital varieties. Then there is a natural map

$$W \quad \to \quad \coprod_{\mathcal{O} \in \mathcal{N}/G} \mathcal{C}_\mathcal{O} \times \mathcal{C}_\mathcal{O}$$

taking the component of Z through (B, wBw^{-1}) to the orbital varieties in $\mathcal{O}(w) \cap \mathfrak{b}$ containing $\mathfrak{n} \cap w \cdot \mathfrak{n}$ and $w^{-1} \cdot \mathfrak{n} \cap \mathfrak{n}$. One obvious property it has is that replacing w by w^{-1} switches the two results.

Theorem 9. [St88] *This map is onto. For $G = GL_n$, it is bijective, and was already studied combinatorially by Robinson and Schensted.*

In the case $G = GL_n$, this says that there is a correspondence between permutations and pairs of same-shape standard Young tableaux. This fits with our claim that SYT index bases of irreps of S_n, because for any finite group one has $|G| = \sum_{\text{irreps } V} (\dim V)^2$.

$\star$**but I don't yet have a real proof set out here**$\star$

What makes GL_n different (and easier) than other groups? The real issue turns out to be that the nilpotent orbits of GL_n are simply connected, but for other groups they often aren't (and their π_1s are calculated in [So98]).

5. Algebras of constructible correspondences

This section follows [Lu97].

5.1. **Motivation.** Let X be a set, to begin with, and $\mathcal{A}_X$ the vector space of functions on $X \times X$. Attempt to define an algebra structure by

$$f \star g := \int_{X_2} \pi_{12}^*(f)\pi_{23}^*(g)$$

where the integrand is a function on $X \times X \times X$, and $\pi_{ij} : X^3 \to X^2$ are the projections onto two of the factors.

What does $\int_{X_2}$ mean? If we make X a measure space, then we need to worry about whether f, g are in L^1 or L^2 or whatever. There are four solutions we will consider to this.

(1) Let X be finite, and $\int_X$ defined by counting measure. Then $\mathcal{A}_X$ is just a matrix algebra. This may sound silly, but it includes the case of counting $\mathbb{F}_q$ points on some variety X defined over $\mathbb{F}_q$.
(2) Let X be a compact manifold. Then instead of measures, we might use differential forms, or cohomology classes. This has the distinct benefits that we might consider classes that aren't top classes, and we might go beyond cohomology to K-theory.

(3) Let X be a noncompact manifold, but deal with clever f, g such that $\pi_{12}^*(f)\pi_{23}^*(g)$ is compactly supported on fibers of the π_{13} projection. This refinement of #2 is the one used in [CG, §2.7].

The fourth, that we will use, takes a little longer to describe.

5.2. **Constructible subsets and functions.** Let X be a scheme. A **constructible subset** of X is a finite union of locally closed subsets (closed subsets of open subsets). They are typically not subschemes. For example, let X be the plane, and $S \subset X$ be the plane minus an axis, union the origin. One motivation for the definition is that the image of a morphism of schemes is constructible; for the example just given, consider the self-map $(x, y) \mapsto (x, xy)$.

A **constructible function** $X \to R$ (where R is just considered as a set) is one with finite image, such that each fiber is a constructible subset.

Now we want to define $\int_X f$ where f is constructible and R is an abelian group. Since f takes finitely many values,

$$\int_X f = \sum_{r\in R} \int_{f^{-1}(r)} f = \sum_{r\in R} r \int_{f^{-1}(r)} 1$$

so we need the total measure of a constructible subset. *For this we use the topological Euler characteristic* χ, which turns out to have the remarkable property [Fu93, exercise, p95 & p141] that $\chi(A \coprod B) = \chi(A) + \chi(B)$. That is, $\int_X f := \sum_{r\in R} r\, \chi\left(f^{-1}(r)\right)$.

Define the **algebra $\mathcal{A}_X$ of constructible correspondences** as the constructible functions (taking values in R a commutative ring, probably $\mathbb{Z}$) on $X \times X$, where the Euler characteristic is used to compute the convolution as described above.

There is an anti-automorphism † of $\mathcal{A}_X$ given by $f^\dagger(x, y) = f(y, x)$.

To compute the $\chi(\bullet)$s we need, we will use the following (perhaps implicitly):

Lemma 2. *Let X be a separated scheme carrying a $S = \mathbb{C}^\times$ action, such that for all $x \in X$, the limit $\lim_{s\to 0} s \cdot x$ exists (uniquely, by the separatedness). (This existence is automatic if X is projective or more generally, complete.) Then $\chi(X) = \chi(X^S)$.*

In particular, if X^S is finite, then $\chi(X) = |X^S|$.

Proof. First, we replace X by its reduction, as the Euler characteristic is only defined from the topological space anyway.

Let C be the set of connected components of X^S. (If X is irreducible or smooth, then each component will be too.) For $c \in C$, define

$$X_c := \{x \in X \ : \ \lim_{s\to 0} s \cdot x \in c\}$$

so that $X = \coprod_C X_c$, a sort of algebraic version of a Morse decomposition. (Indeed, if X is smooth projective and the action of S is projective-linear, the moment map for its maximal compact subgroup $\cong U(1)$ is a Morse function giving this Morse decomposition [Fr59].)

Białynicki-Birula proved [BB] that the X_c are subschemes. Each X_c homotopy retracts to its c (indeed, if X is smooth, Białynicki-Birula proves that X_c is an affine bundle over c), so $\chi(X_c) = \chi(c)$. Hence

$$\chi(X) = \sum_c \chi(X_c) = \sum_c \chi(c) = \chi\left(\coprod_c c\right) = \chi(X^S).$$

□

5.3. **Realization of the group algebra.** These algebras are obviously godawful big. It is interesting that we can find some finite-dimensional subalgebras of them. One obvious way would be if we had a group G acting on X such that the action on $X \times X$ had finitely many orbits, and then only allow G-invariant constructible subsets and functions. [2] But in fact we will take $X = T^*G/B$, for which this finiteness never holds (unless G is abelian).

For s a simple reflection in W, let P_s be the corresponding minimal parabolic, and $\pi_s : G/B \to G/P_s$ the corresponding $\mathbb{P}^1$-bundle, so $\pi_s^{-1}(\pi_s(F))$ is the s**-line through** $F \in G/B$. Write $F_1 =_s F_2$ if $\pi_s(F_1) = \pi_s(F_2)$.

The corresponding component of Z is

$$\begin{aligned} Z_s := \{(F_1, F_2, X) \in G/B \times G/B \times \mathcal{N} \ : \\ \pi_s^{-1}(\pi_s(F_1)) \text{ contains } F_2 \text{ and is pointwise } \exp(X)\text{-invariant}\} \end{aligned}$$

This is obviously G-invariant, so a bundle over the left G/B. Fixing F_1, the fiber is

$$\{F_2 \in \pi_s^{-1}(\pi_s(F_1))\} \quad \times \quad \{X : \pi_s^{-1}(\pi_s(F_1)) \text{ is pointwise } \exp(X)\text{-invariant}\}$$

which is $\mathbb{P}^1$ times a hyperplane in $\mathfrak{n}$. Hence Z_s is irreducible. If we had only imposed "F_1 and F_2 are in the same s-line", we would get $Z_s \cup \{F_1 = F_2\}$.

Let f_s be the characteristic function of Z_s, so a very simple constructible function. Our goal for this section is to prove that the $\{f_s - 1\}$ satisfy the Coxeter relations, so generate a quotient ring of the group algebra $\mathbb{Z}[W]$. It will then be easy to prove (by counting components of Z) that they actually generate a copy of $\mathbb{Z}[W]$ inside the convolution algebra of constructible functions.

5.3.1. *Bott-Samelson manifolds.* To analyze $\star$-products of these $\{f_s\}$, we define the **Bott-Samelson manifold**

$$BS^Q := \{(G_0 = B, G_1, \ldots, G_{|Q|}) \in (G/B)^{|Q|} : \ (G_{i-1}, G_i) \in \overline{G \cdot (B, q_i B)}\}.$$

It carries a diagonal B-action, hence T-action. Since T acts on G/B with isolated fixed points, it does too on the submanifold BS^Q.

Lemma 3. *The T-fixed points on BS^Q can be indexed using subwords of Q, where absence of q_i means "inside the section $G_i = G_{i-1}$" (taking $G_0 = B$) and presence means "not inside that section". More specifically, the subword $R \subseteq Q$ corresponds to the tuple*

$$\left(\ldots,\ G_i = \left(\prod_{j \le i, q_j \in R} q_j\right) B,\ \ldots\right).$$

[2] At that point, it becomes more natural to think of f as a measure than a function, i.e. something that one can evaluate on subvarieties that aren't geometric points; then insist that they are G-invariant.

Proof. Plainly those are $2^{|Q|}$ many fixed points; we need to show there are no more.

Let Q' be Q minus its last letter. (If Q has no letters, then BS^Q is the singleton $\{(B)\}$ and the claims are trivial.) By forgetting the last $G_{|Q|}$, we can see that each Bott-Samelson manifold BS^Q is a B-equivariant $\mathbb{P}^1$-bundle over a smaller Bott-Samelson manifold $BS^{Q'}$, and it has a section $G_{|Q|} = G_{|Q|-1}$. For $S \leq B$, the S-fixed points $(BS^Q)^S$ must lie over those below $(BS^{Q'})^S$, and S acts on each $\mathbb{P}^1$ fiber over $(BS^{Q'})^S$.

If $S = T$, the action on each such fiber has isolated fixed points, hence 2 of them. We can distinguish them, as one lies in the $G_{|Q|} = G_{|Q|-1}$ section. $\square$

Associated to each $R \subseteq Q$ is not just a T-fixed point but a whole sub-Bott-Samelson, in which $G_i = G_{i-1}$ for each $i \notin R$. We can call this BS^R without confusion, since it is plainly isomorphic to the Bott-Samelson associated to the word R.

$\star$**this proposition is very likely wrong, for the reason mentioned just before the comment within**$\star$

Proposition 4. *Let* $Q = (q_1, \ldots, q_{|Q|})$ *be a word in the simple reflections (not necessarily reduced). Let* $f_Q := f_{q_1} \star f_{q_2} \star \cdots \star f_{q_{|Q|}}$. *Then*

$$f_Q(F_1, X_1, F_2, X_2) = \delta_{X_1,X_2} \#\{\textit{subwords of } Q \textit{ with product } w\}$$

where $(F_1, F_2) \in G \cdot (B, wB)$.

Sanity check: if $Q = (s)$, the only w are $1, s$, each occurring once.

Proof.

$$\begin{aligned} f_Q(F_1, X_1, F_2, X_2) &= \int_{G_1, Y_1, \ldots, G_{|Q|-1}, Y_{|Q|-1}} f_{q_1}(F_1, X_1, G_1, Y_1) f_{q_2}(G_1, Y_1, G_2, Y_2) \\ &\qquad \cdots f_{q_i}(G_{i-1}, Y_{i-1}, G_i, Y_i) \cdots f_{q_{|Q|}}(G_{|Q|-1}, Y_{|Q|-1}, F_2, X_2) \\ &= \delta_{X_1,X_2} \int_{G_1, \ldots, G_{|Q|-1}} f_{q_1}(F_1, X_1, G_1, X_1) f_{q_2}(G_1, Y_1, G_2, X_2) \\ &\qquad \cdots f_{q_i}(G_{i-1}, X_1, G_i, X_1) \cdots f_{q_{|Q|}}(G_{|Q|-1}, X_1, F_2, X_1) \end{aligned}$$

This function is G-invariant, and we can use the G-action to move F_1 to B; on that slice it is B-invariant. Since each factor f_{q_i} is a characteristic function, the product is the characteristic function of the intersection. The condition placed on the tuple (G_i), independent of X_1, is that $(G_1, \ldots, G_{|Q|-1}, F_2) \in BS^Q$.

On that Bott-Samelson manifold, we still have the action of $\exp(X_1)$, and we must consider the subscheme fixed by it. $\star$**still having trouble doing this in any way better than Lusztig's detailed, gross calculation**$\star$ $\square$

Let $\mathbf{1} = \delta_{X_1,X_2}\delta_{F_1,F_2}$ denote the identity of convolution, and $f'_s = f_s - \mathbf{1}$. Let $f'_Q = f'_{q_1} \star \cdots \star f'_{q_{|Q|}}$.

5.4. The geometric basis.

Lemma 4. *If* Q *is a reduced word for* w, *then* f_Q *is supported on the preimage in* T^*G/B *of* $\overline{G \cdot (B, wB)}$, *and* $f_Q(B, wB, X = 0) = 1$.

Proof. If we think about the convolution formula set-theoretically (rather than keeping track of actual Euler characteristics), we get upper bounds on the support of $f \star g$. If we project from $(T^*G/B)^2$ to $(G/B)^2$, we get a weaker upper bound still. In this case it turns into the claimed upper bound.

The latter statement boils down to the statement that the fiber of the map $BS^Q \to X^w$ over the point wB/B is a point, hence has $\chi = 1$. $\square$

Let $\mathcal{F}$ denote the algebra generated by the (f_s) under convolution.

Theorem 10. $\mathcal{F} \cong \mathbb{Z}[W]$.

Proof. Having checked the defining relations $\star$**which we haven't**$\star$, we have a map $\phi : \mathbb{Z}[W] \twoheadrightarrow \mathcal{F}$. If $\sum_w c_w w$ is a linear combination and w is chosen maximal with $c_w \neq 0$, then by lemma 4 $\phi(\sum_w c_w w)\big(B, wB, 0\big) = c_w$. So if $\phi(\sum_w c_w w) = 0$, each $c_w = 0$, hence ϕ is also injective. $\square$

Proposition 5. *cite[proposition 4.3]Lusztig97 There exists uniquely a basis $(f_w)_{w\in W}$ of $\mathcal{F}$ such that*

- $f_1 = \mathbf{1}$
- f_s *is as defined already, for $s \in W$ a simple reflection*
- $f_w = 1$ *on* Z_w
- $\phi^{-1}(f_w)$ *is a $\mathbb{Z}$-combination of* $\{w' \leq w\}$
- $f_w = 0$ *on an open set in each* $Z_{w'}, w' < w$.

Proof. Without the last condition, $f_w = \phi(w)$ would work. With it, we essentially need to row-reduce the unipotent matrix $\phi(w)\big(\text{general point of } Z_{w'}\big)$, which can be done uniquely. $\square$

Alas, noone can really calculate this unipotent matrix.

And actually, this is not even the most important basis of $\mathbb{Z}[W]$! The wrong one is the group elements, this one is more interesting, but the really interesting one is related to representation theory.

6. Hall algebras

This section follows [Sc09].

Let $\mathcal{C}$ be a finitary linear category over $\mathbb{F}$, meaning that

- it's an abelian category
- each $\mathrm{Hom}(A, B)$ is a finite-dimensional $\mathbb{F}$-vector space
- each $\mathrm{Ext}^{i>0}(A, B)$ is a finite-dimensional $\mathbb{F}$-vector space.

Main example: the category of finite-dimensional modules of an $\mathbb{F}$-algebra. (If you're not familiar with Exts, just pretend for the moment.)

Let $K(\mathcal{C})$ denote the Grothendieck group, the free abelian group on isomorphism classes of objects modulo short exact sequences. It has a $\mathbb{Z}$-basis given by irreducible objects.

One could try to define an inner product on the set of isomorphism classes of objects,

$$\langle M, N\rangle_a \sim \dim \mathrm{Hom}(M, N),$$

but this has two problems; it is asymmetric, and doesn't descend to $K(\mathcal{C})$. We can fix the second one:

$$\langle M, N\rangle_a := \sum_i (-1)^i \dim \mathrm{Ext}^i(M, N)$$

(I claim that if you were now locked in a cell and told to define Ext, the above clues would lead you to it uniquely.) Then fix the first in a silly way:

$$(M, N)_a := \langle M, N\rangle_a + \langle N, M\rangle_a.$$

The $()_a$ is for "additive"; there is a more general definition when $\mathcal{C}$ is only abelian but not linear, but it requires that the Exts are actually *finite* not just finite-dimensional:

$$\langle M, N\rangle_m := \left(\prod_i \#\mathrm{Ext}^i(M, N)^{(-1)^i}\right)^{1/2}, \qquad (M, N)_m = \langle M, N\rangle_m \langle N, M\rangle_m.$$

The two finitenesses intersect when $\mathbb{F} = \mathbb{F}_q$, giving

$$\langle M, N\rangle_m := \sqrt{q}^{\langle M,N\rangle_a}, \qquad (M, N)_m := \sqrt{q}^{(M,N)_a}.$$

These four are called the (additive vs. multiplicative) (asymmetric vs. asymmetric) **Euler forms** on $K(\mathcal{C})$.

Hereafter we assume the stronger finiteness, that every Ext^i is actually a finite set.

I'm not sure about the motivation for the $\sqrt{\ }$. Maybe it's because we want to define $()_m$ as a product of two $\langle\rangle_m$s. In Roger Howe's 230-page opus "A century of Lie theory" he recommends that people who want to know should ask Lusztig.

The **Hall algebra $\mathbf{H}_\mathcal{C}$** of $\mathcal{C}$ is the free $\mathbb{Z}[\sqrt{q}^{\pm}]$-module on the set of isomorphism classes of objects (not just simple objects, like $K(\mathcal{C})$ is). Alternately, one can (and we will) think of it as the set of constructible functions on the moduli space of objects. This is the sense in which it is a basic construction in geometric representation theory.

The product is defined by

$$(f \cdot g)(R) = \sum_{Q \subseteq R} \langle R/Q, Q\rangle_m f(R/Q) g(Q).$$

It is associative (though not usually commutative); either calculation of $f \cdot g \cdot h$ gets you[3]

$$(f \cdot g \cdot h)(M) = \sum_{R \subseteq S' \subseteq M} \langle M/S, R\rangle_m \langle S'/R, R\rangle_m \langle M/S', S'/R\rangle_m f(M/S') g(S'/R) h(R).$$

This algebra is naturally graded by $K(\mathcal{C})$, or even just the cone inside $\mathbb{N}$-spanned by simple objects. If $\mathcal{C}$ has the **Krull-Schmidt** property that each object is isomorphic to a unique finite direct sum of indecomposables, then as a vector space, $\mathbf{H}_\mathcal{C}$ is isomorphic to a polynomial ring in the indecomposables.

Much more restrictively, if $\mathcal{C}$ is **semisimple**, meaning that every object is a direct sum of simple objects, then $\mathbf{H}_\mathcal{C}$ is just a polynomial ring in the simple objects (and in particular, commutative). For example, if $\mathcal{C}$ were the category of representations of a group algebra $\mathbb{F}_q[\Gamma]$ with $(|\Gamma|, q) = 1$, this boring case would hold.

[3]I have S' instead of S only to follow [Sc09, §1.3].

There is another way to write the product, on the basis:

$$[M][N] = \langle M, N\rangle_m \sum_R \frac{\#\{\text{exact sequences } 0 \to N \to R \to M \to 0\}}{|\mathrm{Aut}(M)| \quad |\mathrm{Aut}(N)|}$$

Proof sketch: the number of orbits of the automorphism group on the space of exact sequences is the number of subobjects N' of R with $N' \cong N$ and $R/N' \cong M$.

Call this number of exact sequences the **Hall number** $P^R_{M,N}$.

6.1. **The Steinitz-Hall-Macdonald example.** Let $\mathcal{C}$ be the category of finite abelian p-groups. (This isn't an $\mathbb{F}_p$-linear category; most honestly, it is $\mathbb{Z}_p$-linear, where $\mathbb{Z}_p$ is the p-adics.) There is only one simple object, $\mathbb{Z}/p$, so the Hall algebra $\mathbf{H}_{\mathcal{C}}$ is only singly graded.

By the classification of finite abelian (p-)groups, the set of objects is indexed by partitions. Let Γ_λ denote $\prod_i \mathbb{Z}/p^{\lambda_i}$, where $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$.

To analyze the product, we need to count short exact sequences. For example,

Proposition 6. *If there exists a short exact sequence $0 \to \Gamma_\lambda \to \Gamma_\nu \to \Gamma_\mu \to 0$, then $\lambda_1 + \mu_1 \geq \nu_1$.*

Proof. The **exponent** of a finite group Γ is the least m such that $g^m = 1$ for all $g \in \Gamma$. Now combine the two easy statements

- The exponent of Γ_λ is p^{λ_1}
- In a short exact sequence $0 \to N \to R \to M \to 0$ of finite abelian groups, the exponent of R is at most the product of those of M and N.

□

Amazingly, this will turn out to be an analogue of the statement "The largest eigenvalue ν_1 of the sum $H_\nu := H_\lambda + H_\mu$ of two Hermitian matrices is at most the sum of the two individual largest eigenvalues, λ_1 and μ_1."

A bunch of these Hall numbers are calculated in [Sc09, §2.2].

6.2. **Green's coproduct on a Hall algebra.** A **coproduct** on a vector space V is a map $\Delta : V \to V \otimes V$. Up to dualization issues, it is the same as a product on V^*, and so one can talk about it being coassociative or cocommutative, and having a counit.

So why bother? Because we will want to define both an algebra and coalgebra structure on the same vector space. The most familiar case is a group algebra $\mathbb{F}[\Gamma]$, which we consider to be dual to the commutative ring $\mathbb{F}^\Gamma$ of pointwise multiplication of functions on Γ.

Of course, we're trying to do this on a Hall algebra. The definition is

$$\Delta([R]) := \sum_{M,N} \langle M, N\rangle_m \frac{1}{|\mathrm{Aut}(R)|} P^R_{M,N}([M] \otimes [N])$$

and the counit is

$$\epsilon : H_A \to \mathbb{Z}, \qquad \epsilon([M]) := \delta_{M,0}.$$

But there's already a problem: why is this sum finite? Actually, in general, it is not; it is only graded-finite (so taking values in a completed tensor product). Such things are called **topological coproducts**. I'm going to completely ignore this issue, as the sum *is* finite for quivers.

The coassociativity can eventually be blamed on the associativity [Sc09, §1.4]. This suggests that the coalgebra isn't really more interesting than the algebra. We have an obvious basis for the algebra, hence a dual basis for the coalgebra; if we identify them we could just infer some coalgebra structure from the algebra.

That almost gives the above rule, but not quite; the right inner product to use is

$$([M],[N]) := \frac{\delta_{M,N}}{|\mathrm{Aut}(M)|}.$$

Then the Hall algebra/coalgebra becomes self-dual.

6.3. **The bialgebra structure.** Presumably one wants the algebra and coalgebra structures to be compatible in some sense. (Otherwise any algebra + basis could be made self-dual in the above sense, which wouldn't be especially interesting.)

A **bialgebra** is one for which the coproduct is an algebra homomorphism. This has some extra subtleties when the coproduct is only "topological", as the completed tensor product isn't an algebra! So already one has to worry that the multiplications one is attempting to compute are "convergent" (in the algebraic sense, of course; they should have only finitely many summands in each degree).

Even granting that, the coproduct above is *not* an algebra homomorphism for general Hall algebras. Green's rather difficult theorem [Sc09, §1.5] is that it is enough to ask that every Ext^i vanishes for $i \geq 2$. The category $\mathcal{C}$ is then said to have **global dimension** ≤ 1 or, for reasons I don't understand, to be **hereditary**.

7. Geometric construction of $U_q(\mathfrak{n}_+)$ for simply-laced Lie algebras

Idea: apply the Hall algebra construction to the right abelian category. First we figure out what properties it might have.

7.1. **Universal enveloping algebras.** Fix a commutative base ring $\mathbf{k}$. Given an associative $\mathbf{k}$-algebra A, we can define a Lie bracket on it by $[a,b] = ab - ba$. This is a functor from $\mathbf{Assoc}/\mathbf{k}$ to $\mathbf{Lie}/\mathbf{k}$, and it has a left adjoint U called **universal enveloping algebra**:

$$U\mathfrak{g} := T\mathfrak{g}/\langle a\otimes b - b\otimes a - [a,b]\rangle_{a,b\in\mathfrak{g}}$$

where $T\mathfrak{g} := \oplus_n \mathfrak{g}^{\otimes_{\mathbf{k}} n}$ is the tensor algebra (itself a left adjoint, of $\mathbf{Assoc}/\mathbf{k} \to \mathbf{Vec}/\mathbf{k}$).

Since the generators $\{a\otimes b - b\otimes a - [a,b]\}$ are inhomogeneous, this is only a filtered algebra, not graded. The associated graded obviously satisfies the relations $a\otimes b - b\otimes a \cong 0$, but a priori may satisfy more. (This is a noncommutative Gröbner basis statement.)

Theorem 11 (Poincaré-Birkhoff-Witt). *The natural map* $\mathrm{Sym}\ \mathfrak{g} \twoheadrightarrow \mathrm{gr}\, U\mathfrak{g}$ *is an isomorphism. Hence given an ordered basis of* $\mathfrak{g}$*, one can construct a basis of* $U\mathfrak{g}$ *from ordered monomials in the basis.*

More generally, if $\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{d}$ *as vector spaces, the multiplication map* $U\mathfrak{c}\otimes U\mathfrak{d} \to U\mathfrak{g}$ *is a vector space isomorphism.*

A universal enveloping algebra is not only an algebra, but a coalgebra: it has a comultiplication $\Delta : U\mathfrak{g} \to U\mathfrak{g}\otimes U\mathfrak{g}$ defined uniquely by the Leibniz rule $\Delta(pq) = \Delta(p)\otimes q + \Delta\otimes c(q)$ for $p, q \in \mathfrak{g}$. Even better, $U\mathfrak{g}$ is a "Hopf algebra", possessing an "antipode" map

$S : U\mathfrak{g} \to U\mathfrak{g}$ defined by $S(p) = -p$ for $p \in \mathfrak{g}$, satisfying several relations with multiplication and comultiplication.

Our interest is in the "triangular decomposition" $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{t} \oplus \mathfrak{n}_+$ of a semisimple Lie algebra, where $\mathfrak{t}$ is a toral subalgebra and $\mathfrak{n}_\pm$ are the sums of the positive or negative root spaces. For example, if $G = SL_n$, we could have $\mathfrak{t} =$ diagonal matrices and $\mathfrak{n}_\pm =$ strictly upper (or lower) triangular matrices.

Let V be a representation of G. By Borel's theorem, there is a B-fixed point on $\mathbb{P}V$; a vector $\vec{v} \neq 0$ in it is called a **high weight vector**. So $\mathbf{k}\vec{v}$ is a $U\mathfrak{b}$-submodule. Under the adjunction

$$\mathrm{Hom}_{U\mathfrak{b}}(\mathbf{k}\vec{v}, \mathtt{Forget}_{U\mathfrak{b}}^{U\mathfrak{g}} V) \cong \mathrm{Hom}_{U\mathfrak{g}}(U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbf{k}\vec{v}, V)$$

we get a natural nonzero map

$$U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbf{k}\vec{v} \to V.$$

If V is irreducible, this map is automatically onto. This left guy is called a **Verma module**. Using the PBW theorem and the triangular decomposition, we can describe it very well as a T-representation:

$$U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbf{k}\vec{v} \cong U\mathfrak{n}_- \otimes_{\mathbf{k}} \mathbf{k}\vec{v} \cong \mathrm{Sym}(\mathfrak{n}_-) \otimes_{\mathbf{k}} \mathbf{k}\vec{v}$$

For this reason, we are very interested in $U\mathfrak{n}_-$, the **algebra of lowering operators**. But there's no real difference in working with $U\mathfrak{n}_+$, so to match [Sc09] we'll take that one.

7.1.1. *Which category to take the Hall algebra of?* Recall that $\mathbf{H}_{\mathcal{C}}$ is graded by the cone spanned by the simple objects of $\mathcal{C}$, and if each object is isomorphic to a unique sum of indecomposables, then it is isomorphic as a vector space to a polynomial ring in these indecomposables.

This sounds a great deal like $U_q(\mathfrak{n}_+)$, suggesting that we want to find an abelian category $\mathcal{C}$ whose indecomposables correspond to the positive roots.

7.2. **Gabriel's theorem.** Let $Q = (V, E)$ be a directed graph, where V is the vertex set and $E \subseteq V \times V$ the edge set. In the following context this will be called a **quiver**, which is a place where you keep a bunch of arrows. A **representation** τ **of the quiver** Q is an assignment of a vector space W_v to each vertex v, and a linear transformation $\tau_e : W_{t(e)} \to W_{h(e)}$ to each edge $e = (t(e), h(e))$. If you like, you can think of it as a functor from the free category on Q to **Vec**. The **dimension vector** of τ is the function $V \to \mathbb{N}$, $v \mapsto \dim W_v$.

There is an obvious notion of direct sum of representations, hence of indecomposable representations. For example, if Q is a chain of n vertices, called the **equioriented** A_n **quiver**, then the indecomposables are of the form

$$0 \to 0 \to \cdots \to 0 \to \mathbb{A}^1 \cong \mathbb{A}^1 \cong \cdots \cong \mathbb{A}^1 \to 0 \to \cdots \to 0$$

which, for $n = 2$, is the best version of the nullity plus rank theorem.

Call a quiver Q **of finite type** if it has only finitely many indecomposables, up to isomorphism (e.g. the equioriented A_n quiver above).

Theorem 12. [Ga73] *A quiver is of finite type iff its underlying undirected graph is a disjoint union of* ADE *Dynkin diagrams. In this case, there is exactly one indecomposable for each positive root of the corresponding Lie algebra. An indecomposable* τ *corresponds to a positive root* β *if for each simple root* $\alpha \in V$, $\dim W_\alpha$ *equals the coefficient of* α *in the expansion of* β.

In the A_n root system, the simple roots are $x_i - x_{i+1}$ for $i = 1 \ldots n$, and the positive roots are $x_i - x_j$, $i < j$. As $x_i - x_j = (x_i - x_{i+1}) + (x_{i+1} - x_{i+2}) + \ldots + (x_{j-1} - x_j)$, this matches the indecomposables above.

Corollary 2. *Fix a dimension vector (d_α) of a finite type quiver, and let $\mu = \sum_\alpha d_\alpha \alpha$ be the corresponding element of the root lattice.*

Then the number of isomorphism classes of quiver representation with that dimension vector equals the number of ways of writing μ as an $\mathbb{N}$-combination of positive roots.

(These isomorphism classes are the same as the $\prod_V GL(W_v)$-orbits on the space $\prod_E \mathrm{Hom}(W_{t(e)}, W_{h(e)})$.)

For the rest of this section, Q will be a quiver of finite type and $\mathfrak{g}$ will be the corresponding semisimple Lie algebra. Our goal is to use the geometry of representation spaces of quivers to produce the Lie algebra.

7.3. **The construction.** Let T act on V with some weights (λ_i). Then the dimension of the μ weight space of the polynomial ring $\mathrm{Sym}(V)$ is the number of ways of writing μ as $\sum_i n_i \lambda_i$.

7.4. **The algebra.** Since $U\mathfrak{n}_- \cong \mathrm{Sym}(\mathfrak{n}_-)$ as a T-vector space, the dimension of its μ weight space is the number of ways of writing μ as a combination of the (negative) roots of $\mathfrak{g}$. By the corollary above, that is the number of representations of Q with dimension vector corresponding to μ. So we have a natural T^*-graded vector space

$$\mathcal{A} = \bigoplus_\mu \mathcal{A}_\mu, \quad \mathcal{A}_\mu := \left(\left(\prod_v GL(W_v) \right) \text{-invariant functions on } \prod_e \mathrm{Hom}(W_{te}, W_{h(e)}) \right)$$

where $\mathcal{A}_\mu$ has a basis given by characteristic functions on the orbits.

Now we want a way to muliply two such functions, $f \in \mathcal{A}_\mu$, $g \in \mathcal{A}_\nu$, so $f * g \in \mathcal{A}_{\mu+\nu}$. The first version of the formula is this:

$$(f * g)(X) = \sum_{W \leq V} f(X|_W) g(X|_{V/W})$$

where the sum is over all subspaces of V which are preserved by X of fixed graded dimension (so X induces a map on W and then quotient V/W). Now, this sum will generally be infinite, which we fix by taking $\mathbf{k} = \mathbb{F}_q$. (Later we will fix it a different way.)

7.5. sl_3, $\bullet \to \bullet$. We study this example in detail. Let f, g be the characteristic functions

$$f = \chi\left(k_1(1 \to 0) \oplus c_1(0 \to 1) \oplus r_1(1 \cong 1)\right)$$

$$g = \chi\left(k_2(1 \to 0) \oplus c_2(0 \to 1) \oplus r_2(1 \cong 1)\right)$$

so X is a representation $(V_{\mathrm{left}}, V_{\mathrm{right}})$ with dimension vector $(k_1+k_2+r_1+r_2, c_1+c_2+r_1+r_2)$. Hence $X \cong X_r$ for some $r \in [0, \min(k_1 + k_2 + r_1 + r_2, c_1 + c_2 + r_1 + r_2)]$, where

$$X_r := (k_1 + k_2 + r_1 + r_2 - r)(1 \to 0) \oplus (c_1 + c_2 + r_1 + r_2)(0 \to 1) \oplus r \cdot (1 \cong 1).$$

Call these three summands K_r, C_r, R_r, so the vector spaces are $K_r \oplus R_r^{\mathrm{left}}$, $C_r \oplus R_r^{\mathrm{right}}$.

For $(W_{\mathrm{left}}, W_{\mathrm{right}})$ to be a subrepresentation of X_r, we need $W_{\mathrm{right}} \geq \tau(W_{\mathrm{left}})$. For f(this subrepresentation) to be 1 not 0, we need

$$\dim \tau(W_{\mathrm{left}}) = \dim W_{\mathrm{left}}/(W_{\mathrm{left}} \cap K_r) = r_1$$

or

$$\dim(W_{\text{left}} \cap K_r) = k_1.$$

The rank of τ on $V_{\text{left}}/W_{\text{left}} \to V_{\text{right}}/W_{\text{right}}$ is $\dim R_{\text{right}}/(R_{\text{right}} \cap W_{\text{right}})$, so we need

$$\dim(R_{\text{right}} \cap W_{\text{right}}) = r - r_2.$$

...

8. Convolution in Borel-Moore homology

8.1. **Borel-Moore homology.** To begin with, let M be a (probably noncompact) real manifold, and $\mathbf{k}$ some base ring. On M we can define homology $H_*(M)$, cohomology $H^*(M)$, and compactly supported cohomology $H^*_c(M)$. There are always graded maps

$$H_*(M) \to H_*(pt) \cong \mathbf{k}, \qquad H^*(M) \otimes H^*_c(M) \to H^*_c(M)$$

and with trickier grading,

$$H^i(M) \otimes H_j(M) \to H_{j-i}(M)$$

hence

$$H^i(M) \otimes H_i(M) \to H_0(M) \to H_0(pt) \cong \mathbf{k}.$$

If M is oriented, we also have an integration map of degree $-\dim M$,

$$H^*_c(M) \to H^*_c(pt) \cong \mathbf{k},$$

hence a pairing

$$H^*(M) \otimes H^*_c(M) \to H^*_c(M) \to \mathbf{k}.$$

The usual statement of Poincaré duality is for M oriented *and compact*. But that is really just to make $H^*(M) = H^*_c(M)$. More generally,

$$H_i(M) \cong H_c^{\dim M - i}(M).$$

`http://en.wikipedia.org/wiki/Cohomology_with_compact_support`

Intuitively, these *should* be related as both of these can be paired against $H^i(M)$. So what pairs with $H^i_c(M)$?

Define the **Borel-Moore homology** $H^{lf}_*(M)$ of a manifold M using infinite chains that are locally finite (each point has an open neighborhood meeting only finitely many). For example, on $M = \mathbb{R}$ we can add up all the intervals $[i, i+1]_{i\in\mathbb{Z}}$ to get a locally finite chain with no boundary, a sort of fundamental class of $\mathbb{R}$.

What if M isn't a manifold? This should be a covariant functor w.r.t. proper maps, such as the inclusion $X \hookrightarrow M$ of a closed subset. Then the desired dualities (on M) lead to the alternate description

$$H^{lf}_i(X) := H^{\dim M - i}(M, M \setminus X)$$

which one can prove is independent of M. (In particular, if X is smooth so $M = X$ works, then $H^{lf}_i(X) = H^{\dim X - i}(X)$, and ordinary homology becomes a module over Borel-Moore homology.) This is often taken as the definition, which is good for people unlike myself who are fluent with relative cohomology. One such reference is [Na, §8.2].

One nice consequence is a Mayer-Vietoris sequence for any F closed in X admitting an embedding in an oriented manifold:

$$\ldots \to H^{lf}_i(F) \to H^{lf}_i(X) \to H^{lf}_i(X \setminus F) \to H^{lf}_{i-1}(F) \to \ldots$$

but mainly we will use the fact that for X a quasiprojective variety, $H^{lf}_{2\dim_{\mathbb{C}} X}(X)$ has a basis given by the top-dimensional components.

8.2. **Convolution.** The product structure on cohomology extends to relative cohomology:

$$H^i(X, Y_1) \otimes H^j(X, Y_2) \to H^{i+j}(X, Y_1 \cup Y_2)$$

We only need this for Y_1 empty, giving a module structure

$$H^i(X) \otimes H^j(X, Y_2) \to H^{i+j}(X, Y_2)$$

which we can interpret as an **intersection pairing** on Borel-Moore homology, if X is a manifold:

$$H^{lf}_i(M) \otimes H^{lf}_j(Z) \to H^{lf}_{j-(\dim_{\mathbb{R}} M - i)}(Z) = H^{lf}_{i+j-\dim_{\mathbb{R}} M}(Z)$$

Now let M_1, M_2 be oriented (noncompact) manifolds, with $Z \subseteq M_1 \times M_2$, such that

$$\pi_2 : Z \to M_2 \text{ is proper.}$$

(In particular Z should be closed). We will use it to define an operator

$$H^{lf}_j(M_1) \to H^{lf}_{j+\dim_{\mathbb{R}} Z - \dim_{\mathbb{R}} M_1}(M_2), \qquad c \mapsto (p_2)_*(p_1^* c \cap [Z]).$$

Let's unpack:

$$\begin{array}{rcl} c \in H^{lf}_j(M_1) & \cong & H^{\dim_{\mathbb{R}} M_1 - j}(M_1) \\ & \xrightarrow{p_1^*} & H^{\dim_{\mathbb{R}} M_1 - j}(M_1 \times M_2) \\ & \cong & H^{lf}_{j+\dim_{\mathbb{R}} M_2}(M_1 \times M_2) \\ & \xrightarrow{\cap [Z]} & H^{lf}_{j+\dim_{\mathbb{R}} Z - \dim_{\mathbb{R}} M_1}(Z) \end{array}$$

Now we use the properness assumption so that the map

$$H^{lf}_{j+\dim_{\mathbb{R}} Z - \dim_{\mathbb{R}} M_1}(Z) \xrightarrow{(\pi_2)_*} H^{lf}_{j+\dim_{\mathbb{R}} Z - \dim_{\mathbb{R}} M_1}(M_2)$$

exists.

In analysis, the analogue of Z is a "kernel" and this map, called a "transform", is written

$$f(a \in M_1) \qquad \mapsto \qquad \left(b \in M_2 \mapsto \int_{a \in M_1} f(a) z(a, b)\, da \right)$$

The natural holding place for these homological transforms turns out to be

$$H^{lf}_*(M_1) \otimes H_*(M_2),$$

as we can define a convolution product

$$(H^{lf}_*(M_1) \otimes H_*(M_2)) \otimes (H^{lf}_*(M_2) \otimes H_*(M_3)) \xrightarrow{\star} (H^{lf}_*(M_1) \otimes H_*(M_3))$$

by

$$(c_1 \otimes c_2) \star (c_2' \otimes c_3) := (\int c_2 \cap c_2') c_1 \otimes c_3$$

where $c_2 \cap c_2'$ comes the H^{lf}_*-module structure of H_*, and $\int$ is the pushforward to a point. This should be awfully reminiscent of §5.

9. Grojnowski-Nakajima quiver varieties

We largely follow Ginzburg's lectures [Gi08]. The goal is to give a geometric derivation of the whole $U\mathfrak{g}$ (not just $U\mathfrak{n}_+$), and also of its finite-dimensional (or more generally, "integrable") representations. More specifically, here is what is known, and by whom, to have a geometric interpretation:

algebra	geometric construction?	by
$\mathbb{Z}$[Weyl group]	yes (convolution algebra)	Kazhdan-Lusztig?
Hecke algebra	no	
affine Hecke algebra	yes	Kazhdan-Lusztig
$U_q\mathfrak{n}_+$	yes (Hall algebra)	Ringel
$U\mathfrak{g}$	yes	Nakajima
$U_q\mathfrak{g}$	no	
$U_q\widehat{L\mathfrak{g}}$	yes	Nakajima

Of course, one may consider the construction of $U\mathfrak{g}$ without the q-deformation to be a feature, rather than a bug, of the Nakajima construction. I will not attempt to tease out the history, and the extent to which it is properly Grojnowski's construction, as Grojnowski is one of the most non-publishing mathematicians I know [Gr2].

For $Q = (V, E)$ a quiver and $(A_v)_{v\in V}$ an assortment of vector spaces, let $GL := \prod_V GL(A_v)$ and $\mathrm{Hom} := \bigoplus_E \mathrm{Hom}(A_{t(e)}, A_{h(e)})$. So GL acts on Hom. Gabriel's theorem is that that action has finitely many orbits (in particular, a dense one) exactly for ADE quivers. So taking the quotient Hom/GL is not so interesting geometrically. We can soup it up in three ways:

(1) Go beyond ADE quivers, obviously. But since we're eventually trying to do representation theory that's not so desirable.
(2) Go beyond ordinary quotients to symplectic quotients.
(3) Introduce phantom vector spaces.

Ideas 2 and 3 require some explanation. But let's first check out the simplest case of 1, the **Jordan quiver** with one vertex and one loop. Then $\mathrm{Hom} = \mathrm{End}(A)$, and GL's orbits on it are indexed by Jordan canonical forms. The quotient Hom/GL is non-Hausdorff, since Hom has many orbits that are not closed; the standard fix is to take the *geometric invariant theory quotient*, $\mathrm{Hom}//GL$, which identifies smaller orbits with larger ones in whose closure they lie, or throws them away entirely (this only in the projective situation). In this case one keeps only those matrices M such that M has one Jordan block for each eigenvalue, or equivalently, M's centralizer is generated by M as an algebra. Then $\mathrm{Hom}//GL \cong \mathbb{C}^n/S_n \cong \mathbb{C}^n$ by taking a matrix to its eigenvalues (unordered) or to its characteristic polynomial.

For later, let $\mathrm{Hom}^{\mathrm{nilp}}$ denote the subscheme of Hom in which the product transformation around any oriented cycle is nilpotent. (So if Q has no cycles, $\mathrm{Hom}^{\mathrm{nilp}} = \mathrm{Hom}$.)

9.1. GIT quotients of cones.

For a rough-and-ready description of more general GIT quotients, one place to look on-line is [Kn00, §5-6].

Proposition 7. *Let X be an irreducible affine G-variety. Let $X//_\mu G$ be a nonempty GIT quotient, and $X//_0 G$ the affine quotient. Then there is a natural projective map $\pi : X//_\mu G \twoheadrightarrow X//_0 G$.*

Now assume X is conical (meaning, invariant under rescaling), defined over $\mathbb{C}$, and the G-action commutes with but probably doesn't contain the scaling action. Then $X//_\mu G$ homotopy retracts to the projective variety $\pi^{-1}(0)$ (where 0 denotes the image of $0 \in X$ in $X//_0 G$).

Proof. In general, the inclusion $R_0 \hookrightarrow R$ of the zero part of a graded ring R, finitely generated over R_0, induces a projective map $\operatorname{Proj} R \to \operatorname{Spec} R_0$. Here that map is π.

One can compute the image of the map, or the ideal defining it, as the annihilator of $R_{>>0}$ as an R_0-module. By assumption, $\operatorname{Fun}(X)$ is a domain, so $\operatorname{Fun}(X)[\ell]$ is too. Since $X//_\mu G$ is nonempty, we learn $\operatorname{Fun}(X)[\ell]^G_+$ is nonzero, so is torsion-free over $\operatorname{Fun}(X)^G$, giving the surjectivity.

The scaling action retracts X to 0. Since G commutes with scaling, each $X//_\mu G$ has a scaling action, and $X//_0 G$ retracts to its 0. We can use this to analyze the retraction of a general $X//_\mu G$ to its $\pi^{-1}(0)$. $\square$

The subscheme $\pi^{-1}(0)$ is often called the **core** of $X//_\mu G$. Its top-dimensional components provide a basis for the top homology of $X//_\mu G$. In the cases we will consider it is equidimensional of dimension $1/2 \dim(X//_\mu G)$. Warning: as a scheme-theoretic fiber, it is often not reduced!

Already the case of $X = \mathbb{A}^n$, $G = T^k$ a torus acting linearly is interesting. By change of basis, we can assume G is a subtorus of the diagonal matrices.

$$G \to T^n$$

The T^n-action on $\operatorname{Fun}(X)$ has a weight basis given by monomials, and the corresponding weights are $\mathbb{N}^n$.

$$\mathbb{N}^n \hookrightarrow \mathbb{Z}^n \cong (T^n)^* \to G^*$$

Taking G-invariants corresponds to taking the fiber over 0 of the above map $\mathbb{N}^n \to G^*$ of monoids. The result is a cone, and $X//G$ is the semigroup algebra of this cone. This is pretty boring if all the weights of G's action on $\mathbb{A}^n$ lie in a half-space, where the cone is a point.

If we instead look at $X//_\theta G$, for $\theta \in G^*$, the natural replacement for the cone is the fiber over θ. This is a polyhedron that looks like the previous cone, in the large; that cone is called the **recession cone** of this polyhedron.

If all the weights of G's action on $\mathbb{A}^n$ lie in a half-space, then this fiber will be a convex polytope. The space $X//_\theta G$ is then a **toric variety**, as it carries a residual action of T^n/G.

It is easy to see that any rational polytope arises from some G and θ. Picture the desired k-dimensional polytope P inside $\mathbb{R}^n_{\geq 0}$. Start with $G = 1$ acting on $\mathbb{A}^k$. For each desired facet F, enlarge the space by an $\mathbb{A}^1$, and G by $\mathbb{G}_m \to T^n$, where the (hyperplane) kernel of the dual map $(T^n)^* \to \mathbb{G}_m{}^*$ is parallel to F, and θ is determined by the distance of the parallel translate F to that hyperplane. At each step, the resulting polytope P_i has more and more cut away, until at last only P is left.

Two remarks:

(1) Oddly, one can introduce gratuitous extra inequalities defining P, which seems pointless but will become relevant in a moment.
(2) The construction is perhaps more natural if we start with $G = 1$ acting on $(\mathbb{G}_m)^k$, so that we're cutting P out of $\mathbb{R}^n$ instead of $\mathbb{R}^n_{\geq 0}$.

(3) Not every "toric variety" arises this way, e.g. $\mathbb{P}^2$ minus a coordinate point. Not even every compact one, though a counterexample is more annoying to give. `http://mathoverflow.net/questions/28551`

9.2. **Doubling a quiver.**

9.2.1. *Symplectic quotients of cotangent bundles.* The most complete reference for this concept is [Pr].

Let G act on X, and assume for the moment that X is smooth and G acts nicely enough that the map $X \to X/G$ is a G-bundle. (Freeness isn't enough; consider $\mathbb{C}^\times$ acting on $\mathbb{C}^2 \setminus \vec{0}$ by $z \cdot (a, b) := (za, z^{-1}b)$.)

Then G acts on the cotangent bundle T^*X as well, and we could consider the two spaces $T^*(X/G)$ and $(T^*X)/G$, of dimensions $2(\dim X - \dim G)$ and $(2\dim X) - \dim G$. The dualization can make it confusing to see what the relation between them should be. To simplify, consider the extremely simple case that G is a linear subspace of X, a vector space, acting by translation. Then

$$T^*(X/G) \cong (X/G) \oplus (X/G)^* \cong (X/G) \oplus (G^\perp \leq X^*), \qquad (T^*X)/G \cong (X \oplus X^*)/G \cong (X/G) \oplus X^*$$

so $T^*(X/G)$ should be a subset (not further quotient) of $(T^*X)/G$.

Really, we should hope to cut a G-invariant subvariety out of T^*X, of codimension $\dim G$, and divide it by G to get something isomorphic to $T^*(X/G)$. In the extremely simple case, it is the kernel of the composite

$$X \oplus X^* \to X^* \to G^*.$$

In general, this is what the G-equivariant **moment map** $\Phi : T^*X \to \mathfrak{g}^*$ does, taking

$$(x \in x, \vec{v} \in T^*_x X) \quad \mapsto \quad \left(A \mapsto \langle A|_x, \vec{v}\rangle\right)$$

where the $\langle , \rangle$ pairing is on $T_xX \otimes T^*_xX$. Then $T^*(X/G) \cong \Phi^{-1}(0)/G$.

I will not spell out the requirements of a moment map Φ in general – see e.g. [Kn00] – but will say that $\Phi^{-1}(0)/G$ is called the **symplectic quotient**.

In fact we will not want to assume that X/G is a G-bundle, and relatedly our quotient $\Phi^{-1}(0)/G$ will have to be replaced by a GIT quotient $\Phi^{-1}(0)//G$.[4] The cool idea, though, is that while X/G (and even its replacement $X//G$) may be a bad space its "cotangent bundle" may be nice!

(It may be that the "correct" thing to do is study equivariant $\mathcal{D}$-modules on X. When X is smooth, these have "characteristic varieties" that live inside T^*X.)

When we take a GIT quotient of a level set of a moment map, we will denote it by

$$X////_\theta G := \Phi^{-1}(0)//_\theta G.$$

If G is the complexification of a group preserving a "hyperKähler" structure on X (slightly more special than quaternionic), the reduction is again hyperKähler. See e.g. [Pr, ch. 2].

[4] You may have heard, e.g. in [Kn00], that complex GIT quotients can be reinterpreted as real symplectic quotients. This is a different statement from the one being made here, which has a complex moment map! Rather, in the situation here something quaternionic or "hyperKähler" is going on.

9.2.2. *The cotangent bundle to* Hom. Let the **doubled** Hom **space** be the space $\bigoplus_E(\mathrm{Hom}(A_{t(e)}, A_{h(e)}) \oplus \mathrm{Hom}(A_{h(e)}, A_{t(e)}))$. We can identify this with the cotangent bundle to the usual Hom space by putting on an antisymmetric inner product using traces. Generally one thinks of this as doubling all the arrows, which I will perhaps foolishly call $2E$, as in

$$\overline{Q} := (V, 2E);$$

there is a slight annoyance that the signs in the symplectic form (the antisymmetric inner product) require the orientation.

The moment map turns out to be

$$\Phi : T^*\mathrm{Hom} \to \prod_V \mathrm{End}(A_v)$$

$$(\varphi_e)_{e\in 2E} \quad \mapsto \quad \left(\sum_{e:h(e)=v} \varphi_e \circ \varphi_{rev\cdot e} - \sum_{e:t(e)=v} \varphi_{rev\cdot e} \circ \varphi_e \right)_{v\in V}$$

where we have identified $\mathfrak{gl}_n^* = \mathrm{End}(\mathbb{C}^n)^*$ with $\mathrm{End}(\mathbb{C}^n)$ using the trace form.

(Here rev denotes the involution on $2E$ taking an edge to its chosen reversal. This notation only becomes important if we have multiple edges or self-loops, which we may.)

This map is easily seen to be GL-equivariant. It is a standard construction in symplectic geometry to take a fiber of Φ lying over a GL-invariant point (namely, a list of scalar matrices) and divide by the action of the group.

In the world of compact groups, we would have our space: $\Phi^{-1}(p)/K$. But here we again have the problem that $\Phi^{-1}((\lambda_v \mathbf{1})_{v\in V})/GL$ is not Hausdorff, so we will have to impose a "stability condition".

This doubling construction does not, in fact, make the ADE case any more interesting; while GL does not have a dense orbit on $T^*\mathrm{Hom}$, it does on each fiber of the moment map. (Where the doubling has given, the Φ^{-1} has taken away.)

Still, let us apply it to the Jordan case. Now $T^*\mathrm{Hom} = \{(X, Y) \in \mathrm{End}(A)\}$, and the moment map is

$$(X, Y) \mapsto XY - YX.$$

Since this has trace 0, the only scalar we can take the fiber over is $0 \cdot \mathbf{1}$, and the fiber is then the commuting scheme. We leave the discussion of the GIT stability condition for later, but observe the following. There is an open set in the commuting scheme consisting of pairs of *simultaneously diagonalizable* matrices. When we divide that by $GL(n)$, we get $\mathbb{C}^{2n}/S_n$, or the **Chow variety** of unordered n-tuples of points in the plane.

9.2.3. *The McKay correspondence: doubling an affine ADE quiver.* Let $\Gamma \leq SL(2) = Sp(1)$ be a finite group, and V the set of isomorphism classes of its representations. So we have a finite set and a vector space attached to each one, but we still need edges. Draw an edge from $A_1 \to A_2$ for each time[5] A_2 occurs in $A_1 \otimes \mathbb{C}^2$, where the latter is the God-given Γ-representation. This turns out to be symmetric, so our quiver is $\overline{Q}$ of some graph, which turns out to be an affine ADE graph (ordinary ADE + the lowest root), and the $\dim(A_v)$ are the coefficients of the unique linear dependence! This is called the "McKay correspondence".

[5]"Each time" will always be at most one time, unless $|\Gamma| = 1, 2$.

One can check here that without the doubling, $\dim \mathrm{Hom}/GL = 1$, so can't be that interesting. With the doubling, $\dim \Phi^{-1}(0)/GL = 2$, so we can hope to get some interesting surface associated to the ADE diagram.

Theorem 13. [Kr] *The affine GIT quotient $\Phi^{-1}(0)//GL$ is isomorphic to $\mathbb{C}^2/\Gamma$. The other quotients are resolutions of it.*

This resolution of a "simple surface singularity" is well known; the core is a union of $\mathbb{P}^1$s, one for each vertex of the (non-affine) Dynkin diagram, and the multiplicity of each $\mathbb{P}^1$ is the corresponding coefficient on the simple root.

Let us do the case of $\Gamma = \mathbb{Z}/2$ in detail. The quiver is $\widehat{A_1}$, and both vector spaces are 1-dimensional. There are two arrows each way, which we can think of as four numbers (a, b, x, y), and the action is $(s, t) \cdot (a, b, x, y) = (sat^{-1}, sbt^{-1}, txs^{-1}, tys^{-1})$. The moment map is $(a, b, x, y) \mapsto (ax - by, xa - yb)$. Reduction at 0 gives the affine GIT quotient

$$\{(a, b, x, y) : ax = by\}//(a, b, x, y) \sim (sa, sb, s^{-1}x, s^{-1}y)$$

The invariant algebra has three generators, $ay, ax = by, bx$, satisfying the relation $ay \cdot bx = (ax)^2$. This is isomorphic to the algebra $\mathbb{C}[c^2, cd, d^2]$ of Γ-invariant functions on $\mathbb{C}^2$.

For a more interesting GIT quotient, introduce a projective parameter ℓ with $(s, t) \cdot \ell := st^{-1}\ell$. Then inside $\mathbb{C}[a, b, x, y, \ell]/\langle ax - by\rangle$ we have the additional invariants $x\ell, y\ell$, with the relations $ax \cdot y\ell = ay \cdot x\ell, bx \cdot y\ell = by \cdot y\ell$. This is isomorphic to the algebra

$$\mathbb{C}[c^2, cd, d^2, p, q]/\langle c^2p = cdq, cdp = d^2q\rangle$$

whose Proj is the blowup of the previous one at the origin $(0, 0, 0)$, adding the $\mathbb{P}^1$ there (with coordinates $[p, q]$).

In general, $\widetilde{\mathbb{C}^2/\Gamma}$ retracts to its core $\pi^{-1}(0)$, which is a union of $\mathbb{P}^1$s glued according to the ordinary (non-affine) Dynkin diagram. The multiplicity of a $\mathbb{P}^1$ is the coefficient of the corresponding simple root. In particular, $\pi^{-1}(0)$ is reduced exactly in the A_n case, when Γ is abelian.

9.2.4. *Various choices of quotient.* The affine GIT quotient $\Phi^{-1}(0)//GL$ requires no choices. The Proj quotients require a choice of action of GL on the trivial line bundle over Hom, which is to say a 1-D rep of GL, necessarily of the form $(g_v)_{v\in V} \mapsto \prod_V (\det g_v)^{n_v}$, $n_v \in \mathbb{Z}$. There are no invariants in the algebra unless all these n_v are ≥ 0.

It turns out, using the symplectic/GIT correspondence (described in [Kn00]), that one can extend these choices to $n_v \in \mathbb{R}$.

There is actually an additional choice: we could look at $\Phi^{-1}((c_v \cdot \mathbf{1}))$, where $c_v \in \mathbb{C}$. All together, the choice of (n_v, c_v) naturally lives in the imaginary quaternions, and the diffeomorphism type of the space $\Phi^{-1}((c_v))//_{(n_v)}GL$ is constant away from a family of codimension 3 subspaces of $\{(n_v, c_v)\}$. (As we saw before, $n_v \equiv 0$ lies outside this good open set.)

In particular, this open subset in this base space is connected, so the diffeomorphism type of the quotient does not really depend on these choices, and better yet simply connected, which allows one to canonically identify the homology groups of two different regular quotients.

In the case that we have a scaling action, this shows that the cores of two different generic reductions are homotopic. They are often not homeomorphic! One very interesting situation, related to hyperplane arrangements and their Orlik-Solomon algebras, is treated in [HS02]. We describe it briefly here.

Consider again the construction that cut a polytope out of affine space, constructing an n-dimensional toric variety as a reduction of ${\mathbb{G}_m}^n \times \mathbb{A}^k$ by ${\mathbb{G}_m}^k$. Now it starts with $T^*{\mathbb{G}_m}^n$, and each step involves multiplying by $T^*\mathbb{A}^1$. If the eventual torus acts on $\mathbb{A}^1$ with weight λ, then it acts on $T^*\mathbb{A}^1$ with weights $\pm\lambda$. The right polyhedral interpretation turns out to be the following: instead of *cutting* with a hyperplane, which we can only see the λ side of, we only *introduce* the hyperplane, which we can see both the λ and $-\lambda$ sides of.

In particular, even if our goal was to cut out a particular polytope P, there may be many compact regions in the hyperplane arrangement. (Especially if used some gratuitous hyperplanes that don't define facets of P.)

Theorem 14. [HS02] *Consider an arrangement $\mathcal{A}$ of k rational hyperplanes in $\mathbb{R}^n$, and let $\mathcal{A}_0$ be the corresponding central arrangement (every hyperplane translated, by $-\theta$, to be through 0). Then under the corresponding map*

$$T^*({\mathbb{G}_m}^n \times \mathbb{A}^k)////_\theta T^k \to T^*({\mathbb{G}_m}^n \times \mathbb{A}^k)////T^k$$

the core is a union of toric varieties, one for each compact region in $\mathcal{A}$.

9.3. **The stable set.** The space $T^*\mathrm{Hom}////_0 GL$ will be singular, so we need to pick an action $\theta : V \to \mathbb{N}$ of GL on the trivial line bundle over $T^*\mathrm{Hom}$, i.e. $(g_v) \mapsto \prod_V (\det g)^{-\theta(v)}$. Then $T^*\mathrm{Hom}////_{\alpha,\theta} GL$ will be the quotient of some open set inside $\Phi^{-1}(\alpha)$.

Lemma 5. *Let G be a reductive group acting on a pair $X \subseteq Y$ of affine varieties X. Pick an action θ of G on the trivial line bundle over X. Then with respect to this choice, the semistable sets satisfy $Y^{ss} = Y \cap X^{ss}$.*

Proof. Let $X = \mathrm{Spec}\, R$, $Y = \mathrm{Spec}\, R/I$. The unstable sets in $\mathrm{Proj}\,(R[\ell]^G), \mathrm{Proj}\,(R/I[\ell]^G)$ are given by by $\mathrm{Spec}\, R^G$ and $\mathrm{Spec}\,(R/I)^G$; by the reductivity, $(R/I)^G = R^G/I^G$.

In terms of ideals, the preimage of $(R/I)_+$ is $I + R_+$.

(For a nonreductive action, it can happen that $(R/I)^G \supsetneq R^G/I^G$.) ∎

Corollary 3. *To figure out the stable set inside $\Phi^{-1}(\alpha)$, it's enough to just consider the stable set inside $\mathtt{Hom}_{\bar{Q}}$.*

Recall that a **subrepresentation** of a quiver representation is a collection of subspaces $(S_v \le A_v)$, preserved under the maps. Two obvious ways for this to happen are that each S_v is contained in all kernels of maps out of A_v, or S_v contains all images of maps into A_v.

It will turn out that the open set will consist of those representations that don't contain subrepresentations that are "too big". Define the **slope** of a subrepresentation as $\mathtt{slope}_\theta(S) := (\sum_v \theta(v) \dim S_v)/\sum_v \dim S_v$.

Proposition 8. [Ki94] *Fix $Q, (A_v), \theta$ and a representation ϕ. Then ϕ is semistable iff for any subrep (S_v), $\mathtt{slope}_\theta(S) \le \mathtt{slope}_\theta(A)$.*

All this stability technology and slope terminology is stolen from Mumford's original work on moduli spaces of vector bundles, which are stable iff they contain no subbundles of larger slope. Unstable vector bundles come with a canonical "Harder-Narasimhan" filtration by subbundles with large slope, such that each quotient is itself stable. I don't know if this technology has been brought to bear on the quiver side.

9.4. **Adding phantoms.** Let $Q = (V, E)$ be a quiver, and rather than just fixing a family (A_v) of vector spaces at the vertices, we attach a second family (B_v) which we think of as connected only to their corresponding A_v. That is, let

$$Q^\heartsuit := (V \times \{\text{real,phantom}\}, E \cup \{((v, \text{real}) \to (v, \text{phantom})\}).$$

Then for $\overline{Q^\heartsuit}$,

$$\mathrm{Hom} := \bigoplus_E \big(\mathrm{Hom}(A_{t(e)}, A_{h(e)}) \oplus \mathrm{Hom}(A_{h(e)}, A_{t(e)})\big) \oplus \bigoplus_V (\mathrm{Hom}(A_v, B_v) \oplus \mathrm{Hom}(B_v, A_v))$$

On this space, we have an action of $GL := \prod_V GL(A_v)$ and a separate, commuting action of $\prod_V GL(B_v)$. *But we do not divide by this latter group, keeping it around to act on the quotient.* In particular, we only choose θ on the original, non-phantom vertices. The moment map is much the same as above.

Unless otherwise stated, each $\dim B_v = 1$.

Example. $Q = A_1$. Here E is empty, so we have $GL(A)$ acting on

$$(\phi_1 : A \to \mathbb{A}^1, \quad \phi_2 : \mathbb{A}^1 \to A)$$

and the moment map is $[\phi_1, \phi_2]$. Again, the only possible scalar fiber is over 0.

⋆**fix this**⋆

9.4.1. *Crawley-Boevey's take on phantoms.* Let $(w_v)_{v \in V}$ denote the dimension vector of the phantoms. Make a new quiver Q^w with one new vertex ∞, and w_v many edges from $v \to \infty$. Put a one-dimensional space $\mathbb{C}$ on it. Then we can correspond

$$\mathrm{Hom}_{Q^\heartsuit} = \bigoplus_E \mathrm{Hom}(A_{t(e)}, A_{h(e)}) \oplus \bigoplus_V \mathrm{Hom}(A_v, B_v)$$

with

$$\mathrm{Hom}_{Q^w} = \bigoplus_E \mathrm{Hom}(A_{t(e)}, A_{h(e)}) \oplus \bigoplus_V \mathrm{Hom}(A_v, \mathbb{C})^{w_v}$$

in a GL-equivariant way (but not $GL \times \prod_v GL(B_v)$, since we need to pick bases of those).

This actually identifies the $\mathrm{Hom}_{Q^\heartsuit}//GL$ phantom quotients with $\mathrm{Hom}_{Q^w}//GL$ non-phantom quotients, which is mildly nonobvious because one would expect the group acting on Hom_{Q^w} to include a $GL(1)$ acting on the ∞ vertex. But the action of the smaller group, leaving that $GL(1)$ out, has exactly the same orbits, since we can rescale at either end. In this way we see that the construction with phantom vertices is *less* general than without.

9.5. **The stable set for $\bar{Q}^\heartsuit$.** The following is a translation of King's stability criterion (with no phantoms) through Crawley-Boevey's isomorphism.

Proposition 9. [Gi08, proposition 5.15] *Fix the particular case $\bar{Q}^\heartsuit$, a stability condition $\theta : V \to \mathbb{N}$, and a representation ϕ.*

If (S_v) is a subrep contained in all kernels of maps to phantoms, and ϕ is θ-stable, then

$$\sum_v \theta(v) \dim S_v \leq 0.$$

If (S_ν) is a subrep containing all images of maps from phantoms, and ϕ is θ-stable, then

$$\sum_\nu \theta(\nu) \dim S_\nu \leq \sum_\nu \theta(\nu) \dim A_\nu.$$

If these inequalities hold for all (S_ν) with either property, then ϕ is θ-stable.

Consider the Jordan case [Gi08, §5.6], with a 1-dimensional phantom.

$$T^*\mathrm{Hom} = \{(X, Y, \vec{v}, f) : X, Y \in \mathtt{End}(A), \vec{v} \in A, f \in A^*\}$$

Let $S \leq A$ be the subspace $S \leq A$ generated by $\vec{v}$ and the endomorphisms X, Y. (Possibly 0, if $\vec{v} = 0$.)

If $\theta < 0$, the kernel condition is vacuous, but the subspace condition is $\dim S \geq \dim A$. This says that considering A as a module over $k\langle X, Y\rangle$, $\vec{v}$ must be a cyclic vector, i.e. $A \cong k\langle X, Y\rangle / I$ for some I of codimension $\dim A$.

The moment map condition $(\Phi^{-1}(\alpha \mathbf{1}))$ says that $XY - YX + f \otimes \vec{v} = \alpha \mathbf{1}$. Taking traces, we get $f \cdot \vec{v} = \alpha \dim A$. Unsurprisingly, there are really only two cases, $\alpha = 0$ and $\alpha \neq 0$.

If $\alpha = 1$, this space is called the **Calogero-Moser** moduli space, and the moment map *almost* gives the Heisenberg relation $[X, Y] = \mathbf{1}$. Of course, that is impossible in finite dimensions (because of the trace), so it gives the next best thing, $[X, Y] = \mathbf{1} +$ a rank 1 operator.

If $\alpha = 0$, then $f \cdot \vec{v} = 0$, and $f \otimes \vec{v}$ is a rank 1 nilpotent. There is some tricky linear algebra to prove [Gi08, proposition 5.6.5] that actually $f = 0$ if $\theta = -1$. So the moduli space consists of $A \cong k[X, Y]/I$ (commuting variables!) for some I of codimension $\dim A$; dividing by $GL(A)$ forgets the isomorphism, giving the **Hilbert scheme** of n points in the plane.

Interestingly, the Calogero-Moser space and Hilbert scheme are diffeomorphic, although C-M is affine and the Hilbert scheme is not. The best statement in this direction is that there is a hyperKähler structure in which these form two of the complex structures.

9.6. **The Hilbert scheme of n points in the plane.** Our references for this section are [Na] and [MS].

The T^2 action on $\mathbb{C}^2$, by dilating the axes, gives an action on $\mathtt{Hilb}_n(\mathbb{C}^2)$. The Hilbert-Chow morphism $\mathtt{Hilb}_n(\mathbb{C}^2) \to \mathtt{Chow}_n(\mathbb{C}^2)$, taking a subscheme to its cycle, is T^2-equivariant. This map is an example of proposition 7, where the core is the **punctual Hilbert scheme** $\mathtt{Hilb}_n(\mathbb{C}^2)_{\vec{0}}$ consisting of ideals I of colength n with $\sqrt{I} = \langle x, y\rangle$, or as turns out to be equivalent, $I \geq \langle x, y\rangle^n$.

Since the only T^2-invariant cycle has all n points at the origin, all the T^2-fixed points on $\mathtt{Hilb}_n(\mathbb{C}^2)$ lie in the punctual Hilbert scheme.

Lemma 6. *The fixed points $\mathtt{Hilb}_n(\mathbb{C}^2)^{T^2}$ correspond $1:1$ to partitions of n, by $\lambda \mapsto \langle x^i y^j : i, j \notin \lambda\rangle$. (Here λ is considered as an order ideal in $\mathbb{N}^2$.)*

There is a decomposition of the core of $\mathtt{Hilb}_n(\mathbb{C}^2)$ into affine spaces, one for each partition. Hence $\dim H_*(\mathtt{Hilb}_n(\mathbb{C}^2)) = p(n)$.

Proof. If a polynomial ideal I is homogeneous under some weighting of the variables, one can split any generator up into its homogeneous components, each of which must be in the ideal. So one may assume that the generators are homogeneous. In this case, they are required to be homogeneous in each of the variables, hence monomial.

Identifying $\mathbb{N}^2$ with the set of monomials in $\mathbf{k}[x, y]$, the monomials in a monomial ideal correspond to the complement of a partition.

This description generalizes to $\mathrm{Hilb}_n(\mathbb{P}^2)^{T^2}$; fixed points on there correspond to triples (λ, μ, ν) describing the fat points at $[0, 0, 1]$, $[0, 1, 0]$, $[1, 0, 0]$ of total length n.

For the second statement, consider the circle action $t \cdot (x, y) = (t^a x, t^b y)$, where $a, b > 0$ are chosen large enough such that this circle action has no new fixed points. (I.e. (a, b) is not a multiple of any T^2-weight on any of the tangent spaces at the fixed points.) This gives a Białynicki-Birula decomposition of the projective manifold $\mathrm{Hilb}_n(\mathbb{P}^2)$:

$$\mathrm{Hilb}_n(\mathbb{P}^2) = \coprod_{\lambda,\mu,\nu:|\lambda|+|\mu|+|\nu|=n} \mathrm{Hilb}_n(\mathbb{P}^2)^{(\lambda,\mu,\nu)}$$

where each

$$\mathrm{Hilb}_n(\mathbb{P}^2)^{(\lambda,\mu,\nu)} := \{Y : \lim_{t\to\infty} t \cdot Y = (\lambda, \mu, \nu)\}$$

is an affine space I like to call a "Gröbner basin".

We now claim that the punctual Hilbert scheme (which is not a manifold) is a union of certain of these Gröbner basins. If we dilate a point-scheme Y out from the origin $[0, 0, 1]$ (which is where we use $a, b > 0$), its limit $\lim_{t\to\infty} t \cdot Y$ will meet the line at infinity iff Y was not in the punctual Hilbert scheme. Hence

$$\mathrm{Hilb}_n(\mathbb{C}^2)_{\vec{0}} = \coprod_{\lambda:|\lambda|=n} \mathrm{Hilb}_n(\mathbb{P}^2)^{(\lambda,\emptyset,\emptyset)},$$

so is also a union of affine spaces. □

Proposition 10. *The punctual Hilbert scheme has dimension* $n - 1$.

More generally, consider the stratum $S^n_\lambda(\mathbb{C}^2)$ *in the Chow variety* $S^n\mathbb{C}^2 := (\mathbb{C}^2)^n/S_n$ *where the* n *points sit in* k *places, with multiplicities* $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$. *Then* $\dim S^n_\lambda(\mathbb{C}^2) = 2k$, *and the dimension of the preimage of* $S^n_\lambda(\mathbb{C}^2)$ *under the Hilbert-Chow morphism is* $2k+\sum_i(\lambda_i-1) = n+k$.

One way to prove the first statement is to compute the T-weights on each tangent space $T_\lambda \mathrm{Hilb}_n(\mathbb{C}^2) \cong \mathrm{Hom}_{\mathbb{C}[x,y]}(I_\lambda, \mathbb{C}[x, y]/I_\lambda)$. The second follows easily from the first.

This dimension $n - 1$ is not quite half that of the Hilbert scheme. The Hilbert scheme factors as $\mathbb{C}^2 \times \mathrm{Hilb}_n(\mathbb{C}^2)^0$, where the latter space is the subfamily of n points whose center of mass is at the origin. This obviously contains the punctual Hilbert scheme, and in fact possesses a symplectic structure with respect to which the punctual Hilbert scheme is Lagrangian.[6]

$n = 2$. The punctual Hilbert scheme, of 2 points at the origin, is $\mathbb{P}^1$, which is the right dimension.

$n = 3$. The punctual Hilbert scheme is 2-dimensional, and since we have a T^2 action it turns out to be a toric variety. To figure out which one, we need to know what stabilizers can occur on it. Three points in the plane lie on a unique circle (or straight line), which has

[6]The general situation is this. If S is a smooth symplectic surface with a Lagrangian L, then the Hilbert scheme of n points on S is smooth, and possesses a Lagrangian in which the n points have collided, and the collision point is on L.

a well-defined center. If we consider the subscheme of $\mathrm{Hilb}_3(\mathbb{C}^2)_{\vec{0}}$ in which that center lies on an axis of $\mathbb{P}^2$ – either the x-axis, y-axis, or line at infinity – we get the three T^2-invariant $\mathbb{P}^1$s on $\mathrm{Hilb}_3(\mathbb{C}^2)_{\vec{0}}$.

The case where the circle is $x^2 + y^2 = 0$ turns out to be the unique singularity of that toric surface, locally isomorphic to $\mathbb{C}^2/Z_3$. If we looked at a wrong B-B decomposition of $\mathrm{Hilb}_3(\mathbb{C}^2)_{\vec{0}}$, e.g. with respect to $(a, b) = (-4, -1)$, one of the strata would be that $\mathbb{C}^2/Z_3$. So the result that we can pave the punctual Hilbert scheme by affine spaces is rather delicate.

One can get some ideas about the geometry of the Hilbert scheme and its subschemes by using a moment map. Fix a Hermitian inner product on $\mathbb{C}^2$, so $\mathbb{C}[x, y]$ has an orthogonal projection to $I^\perp$. Then the moment map is

$$\Phi_{\mathbb{R}} : I \mapsto \sum_{i,j\in\mathbb{N}} d_{ij} \cdot (i, j) \in \mathbb{R}^2 \cong (\mathfrak{t}^2)^*$$

where $x^i y^j \mapsto d_{ij} x^i y^j + \ldots$ (This d_{ij} is a diagonal entry of a Hermitian operator, so real.) In any case, if $I = I_\lambda$, then $\Phi_{\mathbb{R}}(I) = \sum_{(i,j)\in\lambda}(i, j)$. The utility of such a moment map is that each T^2-invariant subvariety Y will map properly to a (possibly unbounded) polygon, and if Y is *pointwise* invariant under a circle $S \le T^2$, then $\Phi_{\mathbb{R}}(Y)$ will lie in a line parallel to $\mathfrak{s}^\perp \le (\mathfrak{t}^2)^*$.

On the punctual scheme, the sum defining the moment map can be restricted to $\sum_{i,j<n} d_{ij} \cdot (i, j)$ since $I \ge \langle x, y\rangle^n$.

Let $H_*\mathrm{Hilb}_\bullet(\mathbb{C}^2) := \oplus_n H_*\mathrm{Hilb}_n(\mathbb{C}^2)$, for right now a graded vector space whose nth graded piece has dimension $p(n)$.

9.6.1. *The ring of symmetric functions.* Consider the inverse system $\ldots \xleftarrow{f_n} \mathbb{C}[x_1, \ldots, x_n]^{S_n} \ldots$ of graded rings, where f_n sets x_n to 0. The degree d part of $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$ has a basis consisting of symmetrizations e_λ of monomials $\prod x_i^{\lambda_i}$, with $\lambda_1 \ge \ldots \ge \lambda_n \ge 0$, $\sum_i \lambda_i = d$. This stabilizes for $n \ge d$, so

$$\dim \mathrm{Symm}_n = p(n)$$

where Symm is the inverse limit, of **symmetric functions**.

(Minor annoyance: the actual inverse limit is not the direct sum of its graded pieces. We really want to take the inverse limit in each degree, then add those up, so that $\mathrm{Symm}_\bullet$ will be $\oplus_n \mathrm{Symm}_n$ and not something like $\prod_n \mathrm{Symm}_n$. One way to reconcile the notation is to let $\mathrm{Symm}_\bullet$ be smaller than the true, unused, inverse limit Symm.)

Obviously this ring is commutative, and contains a subring generated by the $(e_{(n)})$ whose partitions have only one part. It is not hard to show that subring is free, i.e. a polynomial ring. Then by comparing graded dimensions, we see $\mathrm{Symm}_\bullet$ is that polynomial ring. (This is where we use $\mathrm{Symm}_\bullet = \oplus_n \mathrm{Symm}_n$.)

Another natural source of symmetric functions comes from functors $\tau : \mathbf{Vec} \to \mathbf{Vec}$. (To get elements of $\oplus_n \mathrm{Symm}_n$, we need to assume something like $\dim \tau(\mathbb{C}^d) = O(d^n)$ for some n.) There is an obvious one for each n, $V \mapsto V^{\otimes n}$, and S_n acts on this functor by natural automorphisms. If we decompose under S_n, whose irreps are indexed by partitions of n, we get

$$V^{\otimes n} \cong \bigoplus_{\lambda\vdash n} W_\lambda \otimes (S_\lambda V)$$

where $S_\lambda V := \mathrm{Hom}_{S_n}(W_\lambda, V^{\otimes n})$ is the **Schur functor** associated to λ.

Example: $n = 2, 3$.

Then the **Schur polynomial** associated to λ is the function $(x_1, \ldots, x_m) \mapsto \mathrm{Tr}(S_\lambda \cdot \mathrm{diag}(x_1, \ldots, x_m))$. It is easy to check that these polynomials fit into the inverse system, so have a well-defined associated symmetric function s_λ of degree $|\lambda|$. One famous way to compute it is

$$s_\lambda = \sum_{\tau \text{ of shape } \lambda} \prod x_{\tau_i}$$

where the sum is over semistandard Young tableaux τ of shape λ.

Example: $n = 2, 3$.

These Schur functors are related to $GL(n)$ representation theory as follows. Consider the three categories $\mathbf{Vec} \supset End(\mathbb{C}^n) \supset GL(n)$, considering $End(\mathbb{C}^n)$ as the full subcategory on the single object $\mathbb{C}^n$, and $GL(n)$ the further subcategory with only invertible endomorphisms. Then any functor $\mathbf{Vec} \to \mathbf{Vec}$ restricts to a linear representation of the monoid $\mathrm{End}(\mathbb{C}^n)$ and the group $GL(n)$.

Theorem 15. (1) *If λ has more than n rows, then the representation $S_\lambda : \mathrm{End}(n) \to \mathbf{Vec}$ is an action on a 0-dimensional space.*
(2) *If λ has at most n rows, then the representations $S_\lambda : GL(n) \to GL(S_\lambda\mathbb{C}^n), \mathrm{End}(\mathbb{C}^n) \to \mathrm{End}(S_\lambda\mathbb{C}^n)$ are irreducible.*
(3) *Every algebraic irrep of $\mathrm{End}(\mathbb{C}^n)$ arises from a unique λ.*
(4) *Every algebraic irrep of $GL(n)$ is of the form $S_\lambda \otimes \det^{-k}$, where λ is uniquely determined except for the number of n-columns.*

The $GL(n)$ vs. $\mathrm{End}(\mathbb{C}^n)$ distinction here often confuses people, as many references do not emphasize that there is a difference. The ring of characters on $GL(n)$ is symmetric *Laurent* polynomials, and one cannot construct an inverse limit of those, as there is no $f_n : x_n \mapsto 0$ homomorphism. One way in which the $GL(n)$ theory is nicer is that instead of studying the S_2-symmetric coefficients $c^\nu_{\lambda\mu}$ in $V_\lambda \otimes V_\mu \cong \oplus_\nu c^\nu_{\lambda\mu} V_\nu$, one can equivalently study the S_3-symmetric coefficients $c_{\lambda\mu\nu} := (V_\lambda \otimes V_\mu \otimes V_\nu)^{GL(n)}$, related by $c_{\lambda\mu\nu} = c_{\lambda\mu}^{-w_0\cdot\nu}$.

By dimension count, $H_* Hilb_\bullet(\mathbb{C}^2) \cong \mathrm{Symm}_\bullet$ as graded vector spaces. So we should look for a specific isomorphism, and look for geometric interpretations of the (many) interesting structures on $\mathrm{Symm}_\bullet$.

We already know that $\mathrm{Sym}_\bullet$ is a polynomial ring in the infinitely many generators $(e_i)_{i\in\mathbb{N}}$. So to establish the isomorphism, we want to make $H_* Hilb_\bullet(\mathbb{C}^2)$ into a rank 1 free module over this polynomial ring. It is almost sufficient to define the $\widehat{e}_i$ multiplication operators, but that would only make it a module, not necessarily free.

9.6.2. *Correspondences for a Heisenberg algebra action.* The trick is to notice that $\mathbb{Z}[(e_i)]$ supports not only multiplication operators, but differentiation operators, making it it a representation of the infinite-dimensional **Heisenberg Lie algebra**. This is defined by

$$\mathfrak{s} = \langle (p_i, p_{-i}, c)_{i\in\mathbb{N}_+} : [p_i, p_j] = \delta_{i,-j} c \text{ for } i > 0 \rangle$$

and becomes commutative once one mods out the central element c. (In the more classical Heisenberg description, p_{-i} is written as q_i, a "position operator".)

(We will actually want some constants in $[p_i, p_{-i}]$, which can obviously be scaled away if we pass to rational coefficients.)

For each $a \in A^\times$, the Heisenberg Lie algebra acts on the **bosonic Fock space** $A[(e_i)]$ *irreducibly* by $p_i \mapsto \widehat{e}_i$, $p_{-i} \mapsto ai\frac{d}{de_i}$, $c \mapsto \widehat{a}$. (That these are the only irreps is one of the many versions of the "Stone-von Neumann theorem".)

For $i > 0, n \geq 0$, define

$$P[i]_n := \left\{ (J_1, J_2) \in \mathrm{Hilb}_{n-i}(\mathbb{C}) \times \mathrm{Hilb}_n(\mathbb{C}) \ : \ J_1 \supseteq J_2, \ \mathrm{supp}(J_1/J_2) = \{\vec{0}\} \right\}$$

of dimension $2(n-i) + (i-1) = 2n - i - 1$ (via proposition 10). Here J_1, J_2 are the ideals of codimension $n-i, n$. Geometrically, the n-point scheme must contain the $(n-i)$-point scheme, with the i extra points at the origin.

For $i < 0$, it is slightly different:

$$P[i]_n := \{ (J_1, J_2) \in \mathrm{Hilb}_{n-i}(\mathbb{C}) \times \mathrm{Hilb}_n(\mathbb{C}) \ : \ J_1 \subseteq J_2, \ |\mathrm{supp}(J_1/J_2)| = 1 \}$$

of dimension $2n + (-i-1) + 2 = 2n - i + 1$.

Theorem 16. [Na, Gr] *Under the convolution algebra construction from §8, these $\coprod_{\mathbb{N}} P[i]_n$ give the $a = 1$ representation of the Heisenberg algebra. (At least over $\mathbb{Q}$, because of the $(-1)^{i-1}i$ coefficient below.)*

More specifically, let $\widetilde{P}[i]_n \in H_*^{lf}(\mathrm{Hilb}_{n-i}(\mathbb{C})) \otimes H_*(\mathrm{Hilb}_n(\mathbb{C}))$ be associated to the cycle $P[i]_n$ defined above. Then

$$\widetilde{P}[i]_{n-i} \star \widetilde{P}[j]_n = \widetilde{P}[j]_{n-j} \star \widetilde{P}[i]_n, \qquad ij > 0$$

$$\widetilde{P}[i]_{n-i} \star \widetilde{P}[j]_n = \widetilde{P}[j]_{n-j} \star \widetilde{P}[i]_n + (-1)^{i-1} i \delta_{i,-j} [\mathrm{Hilb}_{n-i}(\mathbb{C}^2)_\Delta], \qquad i > 0 > j$$

as elements of $H_*^{lf}(\mathrm{Hilb}_{n-i}(\mathbb{C})) \otimes H_*(\mathrm{Hilb}_{n+j}(\mathbb{C}))$.

Proof. $\underline{i, j > 0.}$ Then we are considering

$$\left\{ (J_1, J_2) \in \mathrm{Hilb}_{n-i}(\mathbb{C}) \times \mathrm{Hilb}_n(\mathbb{C}) \ : \ J_1 \subseteq J_2, \mathrm{supp}(J_1/J_2) = \{\vec{0}\} \right\}$$
$$\star \left\{ (J_2', J_3) \in \mathrm{Hilb}_n(\mathbb{C}) \times \mathrm{Hilb}_{n+j}(\mathbb{C}) \ : \ J_2' \subseteq J_3, \mathrm{supp}(J_1/J_2) = \{\vec{0}\} \right\}$$

but we can replace the latter with the homologous cycle

$$\{ (J_2', J_3) \in \mathrm{Hilb}_n(\mathbb{C}) \times \mathrm{Hilb}_{n+j}(\mathbb{C}) \ : \ J_2' \subseteq J_3, \mathrm{supp}(J_1/J_2) = \{\vec{q}\} \}$$

where q is some other point in the plane.

We want to intersect this with $\{J_2 = J_2'\}$, and project out the J_2 factor. The intersection is

$$\{ (J_1, J_2, J_3) \in \mathrm{Hilb}_{n-i}(\mathbb{C}) \times \mathrm{Hilb}_n(\mathbb{C}) \times \mathrm{Hilb}_{n+j}(\mathbb{C})$$
$$: \ J_1 \subseteq J_2 \subseteq J_3, \ \mathrm{supp}(J_2/J_1) = \{\vec{0}\}, \ \mathrm{supp}(J_3/J_2) = \{q\} \}$$

First we show that the cases $\vec{0} \in \mathrm{supp}(J_1)$, $q \in \mathrm{supp}(J_1)$ can be neglected. Stratify according to the number k resp. ℓ of points that J_1 has at $\vec{0}$ resp. q. Then to specify a pair (J_1, J_3) in the projection, we specify

- the $n - i - k - \ell$ points away from the $\vec{0}, q$ picking up dimension $2(n - i - k - \ell)$,
- the $i + k$ at $\vec{0}$, picking up dimension $i + k - 1$, and
- the $j + \ell$ at q, picking up dimension $j + \ell - 1$,

for a total of $2n-i-k+j-\ell-2$. This only achieves its top dimension $2n-i+j-2$ for $k=\ell=0$. As our precise goal is to compute the homology class of the image, this says that we can compute the image of the $k=\ell=0$ open locus and take its closure.

In that locus, J_2's scheme is J_1's scheme with i points added at $\vec{0}$ and j added at q. We can now add them in the opposite order, corresponding to the other multiplication, which shows the desired commutativity.

$\underline{i, j < 0.}$ This is very similar; we have to show the $p=q$ case is negligible.

$\underline{i > 0 > j.}$ We are considering

$$\left\{(J_1, J_2) \in \mathrm{Hilb}_{n-i}(\mathbb{C}) \times \mathrm{Hilb}_n(\mathbb{C}) \;:\; J_1 \supseteq J_2, \mathrm{supp}(J_1/J_2) = \{\vec{0}\}\right\}$$

$$\star\{(J_2', J_3) \in \mathrm{Hilb}_n(\mathbb{C}) \times \mathrm{Hilb}_{n+j}(\mathbb{C}) \;:\; J_2' \subseteq J_3, \exists q,\ \mathrm{supp}(J_3/J_2) = \{q\}\}$$

which geometrically corresponds to adding i points at $\vec{0}$ and then removing $-j$ points from q.

Again, there is an open set on which $\vec{0} \neq q$. Then $J_2 = J_1 \cap J_3$ (the scheme-theoretic union), and we can let $J_2' = J_1 + J_3$ to think about removing the points at q before adding the points at $\vec{0}$. Since each of J_2, J_2' is determined by J_1, J_3, it is easy to compute the dimension of the projection: $2(n-i)+2(n+j)$.

Now consider $\vec{0} = q$. If $i > -j$, then J_1 must have $\geq i+j$ points there, and the dimension of the projection is $2(n-i-(i+j)) + (i+j-1) + 2(n+j)$. Similarly, if $i < -j$, then J_3 must have $-(i+j)$ points there. Either way we meet a negligible set.

This shows that if $i \neq -j$, the operators commute. It remains to consider $i=-j$.

...

□

9.7. **The modified universal enveloping algebra.** Given a finite-dim rep V of $\mathfrak{t}$ (say of $\mathfrak{g}$), and λ a weight in V, one can cook up a projection operator in $U\mathfrak{t}$ that picks out the λ weight space, essentially the same way that given finitely many integers, one can find a polynomial that vanishes on all but one of them. However, there is no polynomial that vanishes on *all* integers but one.

Let $\tilde{U}\mathfrak{g}$ denote the larger algebra in which we include these projection operators (the $q=1$ specialization of a definition due to Lusztig in "Introduction to quantum groups"). Any representation of $U\mathfrak{g}$ on which $\mathfrak{t}$ acts diagonalizably (e.g. the finite-dimensional ones) will extend to a representation of $\tilde{U}\mathfrak{g}$, so it is essentially harmless, and is a more convenient algebra to present via generators and relations, see e.g. [Na98, §2]. (While this larger algebra is still T^*-graded, I think it may not have a reasonable filtration extending the usual one on $U\mathfrak{g}$, since the new elements added to $\mathfrak{t}$ should be like polynomials of infinite order.)

Let $\mathcal{M}_\theta(v, w)$ be the moduli space of θ-stable representations of $Q^\heartsuit$ with dimension vector v on the real vertices and w on the phantom vertices. (We will only be reducing at $\alpha = 0$, like the Hilbert scheme not the Calogero-Moser space, so we drop it from the notation.) Mostly we assume $\theta > 0$ at each vertex, and drop it from the notation too.

Think of w as a dominant weight (specifically, $\sum_i w_i\omega_i$). The corresponding irrep of $\mathfrak{g}_Q$ will soon be

$$L_w := \bigoplus_v H_{top}(\mathrm{core}(\mathcal{M}(v,w))).$$

Each $H_{top}(\mathrm{core}(\mathcal{M}(v,w)))$ will be a weight space, specifically, the $\sum_i w_i\omega_i - \sum_i v_i\alpha_i$ weight space. Note that this homology group has a basis indexed by the components of the core.

Start with the case Q is ADE (not affine). Let v, v' be a pair of dimension vectors. Then there are natural embeddings $\mathcal{M}_0(v,w), \mathcal{M}_0(v',w) \hookrightarrow \mathcal{M}_0(v+v',w)$, hence natural morphisms $\mathcal{M}_\theta(v,w), \mathcal{M}_\theta(v',w) \hookrightarrow \mathcal{M}_0(v+v',w)$. Define

$$Z_{v,v',w} := \mathcal{M}_\theta(v,w) \times_{\mathcal{M}_0(v+v',w)} \mathcal{M}_\theta(v',w) \qquad \subseteq \mathcal{M}_\theta(v,w) \times \mathcal{M}_\theta(v',w)$$

and call it, by analogy, the **associated Steinberg variety**.

If we now assume θ is positive on all vertices, then the ambient space $\mathcal{M}_\theta(v,w) \times \mathcal{M}_\theta(v',w)$ is smooth.

Theorem 17. *(Nakajima)* [Gi08, 7.1.4 + 7.2.4 + 7.2.5] *In the ADE case, each component of $Z_{v,v',w}$ has half the total dimension. (More specifically, each $\mathcal{M}_\theta(v,w)$ is holomorphic symplectic, and $Z_{v,v',w}$ is Lagrangian with respect to the symplectic structure $\pi^*(\omega) - \pi'^*(\omega')$.)*

More generally, one must define a "good" open set in $\mathcal{M}_0(v+v',w)$, only take the components of $Z_{v,v',w}$ lying over that, and obtain something with some junk lower-dimensional components. (Already in the affine ADE case, one does need to define the "good" set, but in this case there are at least no junk components.)

The $Z_{v,v',w}$ above are supposed to tell $\tilde{U}(\mathfrak{g}_Q)$ how to act on L_w, i.e. to define the corresponding quotient of $\tilde{U}(\mathfrak{g}_Q)$. In the ADE case, where the irreps are finite-dimensional, the definition is

$$H_w := \bigoplus_{v,v'} H^{lf}_{top}(Z_{v,v',w})$$

and only finitely of these $Z_{v,v',w}$ are nonempty. $\star$**I should prove this, at least**$\star$ Outside ADE one must take a certain completion of this direct sum [Gi08, §7.3]. Using the convolution algebra technology, H_w is naturally an algebra that acts on L_w.

Theorem 18. *(Nakajima) There is a natural homomorphism $\tilde{U}(\mathfrak{g}_Q) \to H_w$, making L_w into an irrep with high weight $\sum_i w_i\omega_i$.*

10. The affine Grassmannian

We spend a while looking at the modifications of the Bruhat decomposition and the isomorphism $K/T \cong G/B$ necessary for the infinite-dimensional case.

10.1. **The differential-geometric viewpoint.**

Let K be a compact connected Lie group, e.g. $U(n)$, and let

$$LK = Map(S^1, K), \qquad \Omega K = Map_\bullet(S^1, K)$$

be the spaces of smooth maps, and smooth pointed maps, respectively. Then LK is a group by pointwise multiplication of values, a sort of limit of K^n for $n \to \infty$. One can obviously think of ΩK as a (normal) subgroup, but it is actually more useful to think of it as a quotient

$$\Omega K \cong LK/K$$

where $K \hookrightarrow LK$ is embedded as the constant loops. In particular, this gives a transitive action of LK on ΩK, unlike the conjugation action regarding it as a normal subgroup.

There is an action of S^1 on LK by rotating the loop

$$(t \cdot \phi)(s) = \phi(ts)$$

fixing only K. This descends to a more interesting action on ΩK (when considered as a quotient):

$$(t \cdot \phi)(s) = \phi(ts)\phi(t)^{-1}$$

where the fixed points are exactly the one-parameter subgroups.

We run into them in another way: let $H(\phi) = \int_{S^1} |\phi'|^2 \, dt$ be the "action" of the loop. (This is the function for which Bott invented Morse-Bott theory.) It uses the Riemannian metric on K, and its critical points are geodesics. Any geodesic on a Lie group is a left-right translate of a one-parameter subgroup, but here since $\phi \in \Omega K$ it's an actual subgroup.

We have run into this "fixed points = critical points" phenomenon before: H should be the Hamiltonian for a symplectic action of S^1. It is not hard to guess what the symplectic structure on ΩK should be. (First figure out how to describe the tangent space $T_\phi \Omega K$, then how to build a number antisymmetrically out of two tangent vectors.) Oddly, though the action of LK on ΩK preserves the symplectic structure, it is not Hamiltonian; one must enlarge LK by a central extension to get the "affine Lie group".[7]

Regardless, let us study the one-parameter subgroups $\gamma : S^1 \to K$ of a group, which are obviously determined by their derivatives $\gamma' : \mathbb{R} \to \mathfrak{k}$ at the identity. We can conjugate by K to get γ' to land inside $\mathfrak{t}$; this changes the loop but not the connected component considered of the critical set. Now γ' is an element of the **coweight lattice** of T. Using $N(T)$, we can get it into the positive Weyl chamber.

Proposition 11. *The critical points of* H *on* ΩK *coincide with the* S^1*-fixed points. There is a component for each dominant coweight* $\lambda \in \ker(\exp : \mathfrak{t} \to T)$ *on which* K *acts transitively. That component* $K/\mathrm{Stab}(\lambda)$ *is an adjoint orbit, hence partial flag manifold for* K.

Exercise (I wasted a long time on, fruitlessly): compute the Hessian of H at λ, or equivalently, the S^1-weights on the complexified tangent space. The number of negative weights should be the height of the coweight λ, hence finite.

10.2. **Morse-Bott theory and the algebraic picture.** Let $H : M \to \mathbb{R}$ be a proper Morse-Bott function, bounded below, with finitely many critical manifolds $\{F\}$. Let $M^F := \{m \in M :$ upward gradient flow of m limits into $F\}$, which by the Morse-Bott assumption is a vector bundle over F. The usual statement is

$$M = \coprod_F M^F, \qquad \text{for } M \text{ compact.}$$

In the standard example of $M = T^2$ and a Morse function, this is $M = pt \cup \mathbb{R}^1 \cup \mathbb{R}^1 \cup \mathbb{R}^2$.

[7] An analogous situtation holds for the action of $\mathbb{R}^{2n}$ on $T^*\mathbb{R}^n$ by translation, which only admits a moment map when we enlarge $\mathbb{R}^{2n}$ to the Heisenberg group.

If M is not compact, we only get

$$M \supseteq \coprod_F M^F, \qquad \text{to which it homotopy retracts.}$$

If we puncture the top point of T^2 (and stretch it up to ∞, to keep H proper), this says the punctured surface retracts to a figure eight, $pt \cup \mathbb{R}^1 \cup \mathbb{R}^1$.

If M is infinite-dimensional, it may still happen that each M^F is finite-dimensional, as happens with $M = \Omega K$. (We would have seen this had I been able to do the Hessian computation.) What will turn out in this example is that each $Gr_\lambda := \overline{\Omega K^\lambda}$ is actually a complex projective variety, and any finite union thereof is projective. The whole union is called Gr.

Algebraic geometers study these "ind-schemes" mostly working on larger and larger finite unions. There is a very weird subtlety: while reduced schemes are generically smooth, it can happen (and does for Gr) that any given point in an ind-scheme is a singular point for all large such unions, making the ind-scheme "singular everywhere".

10.3. **Lattices, and their ind-scheme** Gr**.** Let $\mathcal{O}$ be a principal ideal domain and $\mathcal{K}$ its fraction field. A **lattice** L is a free $\mathcal{O}$-submodule of the vector space $\mathcal{K}^n$ such that $\mathcal{K} \otimes_{\mathcal{O}} L \to \mathcal{K}^n$ is an isomorphism. The name is taken from the case $\mathcal{O} = \mathbb{Z}$, $\mathcal{K} = \mathbb{Q}$.

The space of lattices, Gr, has a transitive action of $GL_n(\mathcal{K})$, identifying it with the coset space $GL_n(\mathcal{K})/GL_n(\mathcal{O})$. The determinant map from this to $\mathcal{K}^\times/\mathcal{O}^\times$ takes a lattice to its **volume**. In the $\mathcal{O} = \mathbb{Z}$ case this group of volumes is $\mathbb{Q}_+$.

The case we care about is $\mathcal{O} = \mathbb{C}[[t]]$, power series, so $\mathcal{K} = \mathbb{C}((t))$, Laurent series. Then the space of lattices is the **affine Grassmannian for** GL_n. Every nonzero Laurent series is uniquely of the form $t^n f(t)$ where $n \in \mathbb{Z}$ and $f(0) \neq 0$, so $f \in \mathcal{O}^\times$. Hence the group of volumes is $\mathbb{Z}$, and in this case the volume of a lattice is also called its **index** $ind(L)$. It can be computed as the difference

$$ind(L) = \dim L/(L \cap \mathcal{O}^{\oplus n}) - \dim \mathcal{O}^{\oplus n}/(L \cap \mathcal{O}^{\oplus n})$$

where both numbers are finite.

While Gr may seem scarily infinite-dimensional, it turns out to be quite comprehensible; secretly it is supposed to be the ind-scheme made of the Morse-Bott strata in $\Omega U(n)$. That suggests that we should look at the analogue of the Morse-Bott decomposition, namely the Białynicki-Birula decomposition. But to avoid circularity (since the B-B decomposition is only defined for schemes or suchlike, and we don't have such a structure yet) we will take a different approach to seeing that this is an ind-scheme.

Given a lattice L, let $V_a(L) = (t^{-a}L \cap \mathcal{O}^{\oplus n})/(t\mathcal{O})^{\oplus n}$, considered as a subspace of the constant space $\mathcal{O}^{\oplus n}/(t\mathcal{O})^{\oplus n} \cong \mathbb{C}^n$.

Lemma 7. *For any* L*,* $V_a(L)$ *is increasing in* a*, and goes from* 0 *to all of* $\mathcal{O}^{\oplus n}/(t\mathcal{O})^{\oplus n}$*.*

Proof. The increasingness is just the fact that $L \geq tL$. Hence $V_{-\infty}(L), V_\infty(L)$ are well-defined.

For any $a < 0$, $\dim_{\mathbb{C}} L/(L \cap \mathcal{O}^{\oplus n}) \geq (-a) \dim V_a(L)$. For this to be bounded in a (as it must be to compute L's index), we need $V_a(L) = 0$ for $a \ll 0$.

To extend scalars to $\mathcal{K}$, we only need to introduce t^{-1}, so $\mathcal{K} \otimes_{\mathcal{O}} L = V_\infty(L) \otimes \mathcal{K}$. By assumption, this is $\mathcal{K}^n$, so $V_\infty(L) = \mathbb{C}^n$. ☐

Corollary 4. Gr *is a projective ind-scheme.*

More specifically, for each $a < b$, *let* $\mathrm{Gr}[a,b]$ *be the set of lattices* L *with* $t^a\mathcal{O}^{\oplus n} \geq L \geq t^b\mathcal{O}^{\oplus n}$. *For each fixed index, this can be identified with a closed subscheme (more specifically, a Springer fiber) inside a Grassmannian of* $t^a\mathcal{O}^{\oplus n}/t^b\mathcal{O}^{\oplus n} \cong \mathbb{C}^{(b-a)n}$. *Then* $\mathrm{Gr} = \cup_{a\leq b}\mathrm{Gr}[a,b]$.

On a Grassmannian of k-planes, there is a natural line bundle $\Lambda^k\mathcal{E}$, the kth exterior power of the tautological bundle. Its dual is ample, and generates the Picard group, which is isomorphic to $\mathbb{Z}$. (Perhaps you know these facts for $k = 1$.) If we restrict this bundle to each $\mathrm{Gr}[a,b]$, we get one whose fibers can be naturally identified with $\Lambda^{top}(L/(L \cap \mathcal{O}^{\oplus n}))\otimes\Lambda^{top}(\mathcal{O}^{\oplus n}/(L \cap \mathcal{O}^{\oplus n}))^*$, which doesn't depend on a, b. With this one can define a natural line bundle on the ind-scheme Gr.

10.4. **The Bruhat decomposition of** Gr**.** Let D denote the action of $\mathbb{C}^\times$ on the space of lattices by scaling the variable t. This restricts to $\mathrm{Gr}[a,b]$, and extends to the Grassmannian containing that. We already studied the B-B decomposition of Grassmannians, so can use this to determine that of Gr.

Since a lattice is free, it has a basis, whose elements g have some least power of t. When we scale t by z and let $z \to 0$, this picks out that least power; call this $\mathrm{init}\, g$. So the D-fixed points are lattices with a basis of the form $(\vec{v}_i\otimes t^{e_i})$. If we put these in order $e_1 \geq e_2 \geq \ldots e_n$, we get an element of $\mathbb{Z}^n_{dec}$, a dominant coweight of $U(n)$.

One can do better – ask invariance not only under D, but also under the diagonal matrices $T \leq U(n)$. Then the $(\vec{v}_i)$ must run over the basis vectors. If we order by those, the corresponding vector $(e_1, e_2, \ldots, e_n)$ is now an arbitrary element of $\mathbb{Z}^n$, or a general coweight of $U(n)$. More specifically, each component of the D-fixed points has an S_n-orbit of $(D \times T)$-fixed points, just as the dominant coweights index the S_n-orbits of general coweights.

Now, what are the B-B strata associated to a dominant coweight λ? Let $\mathcal{O}_- := \mathbb{C}[t^{-1}]$, and pick a D-fixed lattice L_λ as in the last paragraph. By applying the technology from §3.1 to the subvarieties $\mathrm{Gr}[a,b]$, then taking the union, we get

$$\mathrm{Gr}^\circ_\lambda = \{L \in \mathrm{Gr} \,:\, \dim(L \cap t^i\mathcal{O}_-^{\oplus n}) = \dim(L_\lambda \cap t^i\mathcal{O}_-^{\oplus n})\}$$

$$\mathrm{Gr}_\circ^\lambda = \{L \in \mathrm{Gr} \,:\, \dim(L/L \cap t^i\mathcal{O}^{\oplus n}) = \dim(L_\lambda/L_\lambda \cap t^i\mathcal{O}^{\oplus n})\}$$

These are exactly the $G(\mathcal{O}_-)$, $G(\mathcal{O})$ orbits, where the first is finite-codimensional and the second finite-dimensional. Put "$\geq$" and "$<$" for the closures Gr_λ, Gr^λ respectively.

The closure of a group orbit is a union, so we can ask which occur in Gr^λ :

$$\mathrm{Gr}^\lambda = \bigcup_{\mu \text{ dominant}, \mu\in\lambda-\mathbb{N}\check{\Delta}_+} \mathrm{Gr}^\mu_\circ$$

i.e. μ is less than λ in **dominance order**. Usually, when Gr is thought of as an ind-scheme it is through $\mathrm{Gr} = \bigcup_\lambda \mathrm{Gr}^\lambda$.

It is interesting to note the T-fixed points on Gr^λ.

$$(\mathrm{Gr}^\lambda_\circ)^T = S_n \cdot \lambda$$

$$(\mathrm{Gr}^\lambda)^T = (\lambda + \{\text{coweights}\}) \cap \mathrm{hull}(S_n \cdot \lambda)$$

where the latter is the lattice point inside the convex hull of the S_n orbit. If we misinterpret $\lambda \in \mathbb{Z}^n$ as a *weight*, rather than a coweight, then this is exactly the set of weights that occur in the irrep V_λ!

10.5. **Thanksgiving break: juggling on the affine flag manifold.** Since some people are away, we're going to discuss a side story not directly related to geometric representation theory, but using what we've developed about the affine Grassmannian of GL_n.

Recall that we interpreted LK/K as a partial flag manifold for the loop group, containing the homotopically equivalent ind-scheme $G(\mathcal{K})/G(\mathcal{O})$. The corresponding full flag manifold should be LK/T, containing an ind-scheme $G(\mathcal{K})/\hat{B}$, where $\hat{B} \leq G(\mathcal{O})$ specializes at $t \mapsto 0$ to $B \leq G$.

In the $G = GL_n$ case, where $G(\mathcal{K})$ had Dynkin diagram a cycle with n vertices, and $G(\mathcal{O})$ was the parabolic that omitted one vertex, we can think of $\hat{B}$ as the intersection of n subgroups related to $G(\mathcal{O})$ by outer automorphism. In lattice terms,

$$\mathtt{AffFlag} = \{(\ldots \leq L_1 \leq \ldots \leq L_n \leq \ldots) : L_i \in Gr, \dim(L_{i+1}/L_i) = 1, L_{i+n} = tL_i\}$$

so instead of one subspace of infinite dimension and codimension, we have a periodic flag of them, each automatically of the same index.

If we introduce a formal nth root z of t^{-1}, we can identify $\mathbb{C}[[t]][t^{-1}]^{\oplus n}$ with $\mathbb{C}[[z^{-1}]][z]$, where the ith summand is multiplied by z^i. Because of this latter trick, the circle rescaling z is diagonally embedded in $T \times \mathbb{C}^\times$ in such a way that it has isolated fixed points.

In these coördinates, and shifting L_i by z^{-i}, we can rewrite

$$\mathtt{AffFlag} \cong \{(\ldots, L_1, \ldots, L_n, \ldots) : L_i \leq \mathbb{C}[[z^{-1}]][z], L_{i+1} \geq z^{-1}L_i, \dim(L_{i+1}/z^{-1}L_i) = 1, L_{i+n} = L_i\}$$

(Note the condition $L_i \geq tL_i = z^n L_i$ is automatic.)

The fixed points (under scaling z) correspond to periodic chains of subspaces $L_i \leq \mathbb{C}[[z^{-1}]][z]$, each generated by some $z^a\mathbb{C}[[z^{-1}]][z]$ and finitely many monomials. Think of dots along a $\mathbb{Z}$-line, where all spots far to the left have dots, none far to the right do, and near the origin all bets are off. To get from L_i to L_{i+1}, push everything one to the left, and add one dot at position $g(i)$. Then $f(i) = g(i) + i$ is an **affine permutation**, a bijection $\mathbb{Z} \to \mathbb{Z}$ with the periodicity property $f(i + n) = f(i) + n \; \forall i$.

The group of affine permutations induces permutations of $\mathbb{Z}/n$, giving a split exact sequence

$$1 \to \mathbb{Z}^n \hookrightarrow W(LGL_n) \twoheadrightarrow S_n \to 1$$

where the image of the splitting $S_n \to W(LGL_n)$ is affine permutations f taking $1, \ldots, n$ back into $1, \ldots, n$. Though we are calling it the Weyl group of the loop group of GL_n, it is not a Coxeter group; rather it fits into another exact sequence

$$1 \to W(LSL_n) \hookrightarrow W(LGL_n) \twoheadrightarrow \mathbb{Z} \to 1$$

where the second map is $f \mapsto avg(f(i) - i)$. The group $W(LSL_n)$ *is* a Coxeter group, generated by

$$r_i(m) = \begin{cases} m+1 & \text{if } m \equiv i \bmod n \\ m-1 & \text{if } m+1 \equiv i \bmod n \\ m & \text{otherwise} \end{cases}$$

The image of the splitting S_n is the Coxeter subgroup generated by $r_1, \dots, r_{n-1}$, leaving out $r_0 = r_n$.

The Coxeter length $\ell(\pi)$ of an element of S_n is the number of inversions, $\{(i < j) : \pi(i) > \pi(j)\}$. There is a similar, slightly trickier, formula for the Coxeter length of an element of $W(LSL_n)$:

$$\sum_{i=1}^{n} \#\{j > i : \pi(j) < \pi(i)\}$$

Note that the same $j \bmod n$ can be counted several times, if $\pi(i) \gg i$.

We can apply the same formula to $W(LGL_n)$, obtaining the dimension of opposite Bruhat cells on $\mathtt{AffFlag}$, or the codimension of Bruhat cells.

10.5.1. *The juggling interpretation.* Consider a one-handed juggler, throwing one ball every second, who makes no collisions. Let the ball thrown at time i be next time thrown at time $f(i)$. "No collisions" means f is injective. If we ask the pattern be periodic, i.e. $f(i+n) = f(i) + n\ \forall i$, then f is also surjective. Under usual circumstances one would assume $f(i) \geq i$, that balls land after they are thrown.

This is an unpleasant assumption for a mathematician (it cuts the group down to a monoid), but we can interpret throws with $f(i) < i$ as antimatter throws.

The lattices L_i themselves have a juggling interpretation; each one is a history of when in the past ($\mathbb{C}[[z^{-1}]]$) balls were caught and when in the future ($\mathbb{C}[z]$) balls will be caught, and the index of L_i is the net ball number (matter balls minus antimatter balls).

In particular, the lattices of (vague!) interest to jugglers are the ones that contain $\mathbb{C}[[z^{-1}]]$ (no antimatter). The T-fixed ones of index k correspond to k-element subsets of $\mathbb{N}_+$.

10.5.2. *Snider's opposite Bruhat cells.* Given $\sigma \in \binom{[n]}{k}$, associate a k-ball juggling pattern $f_\sigma \in \ker(W(LGL_N) \twoheadrightarrow S_n)$:

$$f_\lambda(i) := i + \begin{cases} 0 & i \in \lambda \bmod n \\ n & i \in \lambda \bmod n. \end{cases}$$

The corresponding list of lattices (L_i) has $L_i = \mathbb{C}[[z^{-1}]] \oplus \bigoplus_{s \in \sigma} \mathbb{C}t^s$ where $\chi \in S_n$, $\chi(j) = j + 1 \bmod n$.

11. The geometric Satake correspondence

This is a correspondence between tensor categories, the familiar one defined representation-theoretically, the new one defined geometrically. One tricky bit is that they are associated to different groups.

11.1. Rep(**the Langlands dual group**)**.** Let's start with the representation-theoretic one. Given a connected reductive Lie group G, we standardly construct the weight lattice and root system (which may not span the weight lattice; the quotient is the Pontrjagin dual of the center). Define the **coweight lattice** as the $\mathbb{Z}$-dual of the weight lattice, or equivalently the kernel of $\exp : \mathfrak{t} \twoheadrightarrow T$. Then define the **coroot** $\check{\beta}$ associated to a root β as the unique element of $[\mathfrak{g}_\beta, \mathfrak{g}_{-beta}]$ satisfying $\langle \check{\beta}, \beta \rangle = 2$.

(Note that the weight lattice has an inner product derived from the Killing form, so it's not such a surprise that we could correspond some elements of the two lattices.)

A **root datum** is the quadruple (weight lattice, root system, coweight lattice, coroot system) with the dual pairing and the bijection between roots and coroots. This serves two functions; one is to classify not just semisimple Lie algebras (as root systems already do), but reductive Lie groups (i.e. keeping track of the center).

The other is that we can switch the lattices and the finite subsets therein, and obtain a new root datum, of the **Langlands dual group** G^L. Already on the root system level this does something nontrivial; it exchanges short and long roots. If you are *extremely* careless and only want to see what happens up to isomorphism, you might say that Langlands duality exchanges B_n with C_n and takes all other diagrams to themselves. But really, it flips F_4 and G_2. On the group level it is more interesting; *if* G *is semisimple,* so $\pi_1(G)$ and $Z(G)$ are finite abelian groups measuring the distance of G from the largest and smallest groups with its Lie algebra, these will be switched (and dualized, not so's you'd notice) for G^L. For example, the Langlands dual of a simply-connected group is a centerless group.

If $G \to G'$ is a finite map, then there is a natural map $G'^L \to G^L$ of the same degree. Hence the covers

$$\mathbb{G}_m \times SL_n \to GL_n \to \mathbb{G}_m \times PGL_n$$

dualize to

$$\mathbb{G}_m \times PGL_n \leftarrow GL_n \leftarrow \mathbb{G}_m \times SL_n$$

so GL_n is its own Langlands dual!

Anyway the point of this construction (for us) is that it provides a way to interpret the dominant coweights of G as something more familiar: dominant coweights, but of G^L.

Now we can define one of the tensor categories: $Rep(G^L)$. This has simple objects indexed by dominant weights for G^L.

11.2. **A whirlwind view of the other side.** If X is a stratified space, we can consider the constructible sheaves on X that are constant[8] on each stratum. The K-group of this category is free abelian on the strata. In our case, $X = Gr$ and the stratification is by $G(\mathcal{O})$-orbits, so the strata correspond to dominant coweights of G. So far so good!

That's not the category we want, though. First we pass to the category of bounded complexes of sheaves. Then we take a quotient of that, calling two complexes isomorphic if there exists a morphism between them inducing an isomorphism on homology. (This is not as forgetful as saying "Call them isomorphic if they have the same cohomology"; the analogy I like is that one can have two different spaces with isomorphic homotopy groups, but there may not be a single map that induces all those isomorphisms.) This is the **derived category** of constructible sheaves for this stratification.

Why does one want to do this? The usual tensor product on sheaves is not exact, and so to have a place to put the higher derived functors of it, we need something like a complex. In order to think of a sheaf and a projective resolution of it as "the same", we go

[8]Or locally constant, i.e. constant on some covering space. In our example the strata are simply connected.

down to the derived category.[9] If we wanted to compute cohomology of this stratified space, we might ask that the supports of these constructible sheaves be transverse to the singularities. (More specifically, one would ask this of the homology groups of the complex, themselves sheaves.) If we wanted to compute homology of this stratified space, we wouldn't ask anything. Instead we ask something halfway in between ("middle perversity") that I will not define.

The very nonobvious part is that for the derived category of $G(\mathcal{O})$-orbit-constructible sheaves on Gr, there is a tensor product $\star$. In the formulation in [Gi], this is derived from the multiplication $m : \mathrm{Gr} \times \mathrm{Gr} \to \mathrm{Gr}$ based on the identification $\mathrm{Gr} \cong \omega K$ with a group: $L \star M := m_*(L \otimes M)$, where $\otimes$ is the derived tensor product.

11.3. **Tannaka-Krein reconstruction.** The theorem will be

Theorem 19. *(Geometric Satake Correspondence) There is an equivalence of tensor categories between the derived category of $G(\mathcal{O})$-orbit-constructible sheaves on Gr, and the representation category of G^L.*

Note that the latter category has a **fiber functor** to **Vec**, forgetting the group action. So we should understand what that is on the sheaves, particularly in light of the amazing fact that this structure characterizes representation categories:

Theorem 20. *(Tannaka-Krein) Let $\mathcal{C}$ be a semisimple tensor category with a tensor identity ("rigid"), and a fully faithful tensor functor to* **Vec**. *Then $\mathcal{C}$ is equivalent to the category of representations of some unique reductive group.*

Put another way, once one defines the tensor product and fiber functor on the derived category of $G(\mathcal{O})$-orbit-constructible sheaves on Gr, a group pops out automatically, which one checks without much difficulty to be the Langlands dual group.

The simple objects on the rep theory side are of course the irreps. The simple objects on the other side are the "intersection homology sheaves" associated with the $\{\mathrm{Gr}^\lambda\}$, and the fiber functor takes an intersection homology sheaf to the intersection homology.[10] Of course we haven't defined any of that, but it gets the flavor across: each irrep V_λ of G^L arises as $IH^*(\mathrm{Gr}^\lambda)$. (The latter is graded, corresponding to the weight decomposition of V_λ paired with the smallest regular dominant coweight $\check{\rho}$.)

12. Mirković-Vilonen cycles and polytopes

Consider the T-action on the singular variety Gr^λ, and pick a regular dominant coweight $S : \mathbb{C}^\times \to T$. (There is a canonical choice called $\check{\rho}$, but it won't matter which one.) The regularity will say that the S-fixed points = the T-fixed points.

With this, we can look at the Białynicki-Birula decomposition of each Gr^λ. Recall that the T-fixed points μ on Gr^λ correspond 1:1 to the weights that occur in the G^L-irrep V_λ.

[9] As I understand it, one of the points of this approach is to allow one to think about the derived functors of composites of more than two functors. Already for two, the standard approach to study them was "pages" of spectral sequences; for three one would imagine three-dimensional versions.

[10] This is *not* another name for the Chow ring. It is very unfortunate that Goresky and MacPherson, who were all set to name this important theory "perverse homology", were persuaded not to by Sullivan.

Theorem 21. [MV] *Each* $(\mathrm{Gr}^\lambda)_\mu$ *is equidimensional, of dimension* $\langle \lambda - \mu, \check\rho\rangle$*, and the closures of its components define elements in* $\mathrm{IH}^*(\mathrm{Gr}^\lambda)$ *giving a basis of the* μ *weight space in* V_λ*.*

Example: the adjoint representation of SL_3.

These closures are called the **Mirković-Vilonen cycles** in Gr^λ. Let $MV^\lambda(\mu)$ denote the set of these varieties, but for the fixed point $\lambda - \mu$, so every variety in $MV^\lambda(\mu)$ has dimension $\langle \mu, \check\rho\rangle$. If $\lambda' \geq \lambda$ in dominance order, there is a natural inclusion

$$MV^\lambda(\mu) \hookrightarrow MV^{\lambda'}(\mu)$$

so we can take the union over all λ, obtaining a set $MV(\mu)$. This is finite, of size the Kostant partition function of μ, i.e. the dimension of the μ weight space in $U\mathfrak{n}$. This suggests there should be a way to index MV cycles by $\mathbb{N}^{\dim \mathfrak{n}}$.

J. Anderson [A] had the great idea of studying the moment polytopes of MV cycles, w.r.t. the T-action on the line bundle on Gr described before, and christened them **MV polytopes**. Note that the inclusion $MV^\lambda(\mu) \hookrightarrow MV^{\lambda'}(\mu)$ doesn't quite preserve the moment polytope – it translates it by $\lambda' - \lambda$. So the actual definition of the MV polytope $P(X)$ associated to $X \in MV^\lambda(\mu)$ will be the $-\lambda$ translate of the moment polytope $\Phi(X)$. Hence for each $X \in MV(\mu)$, the $\check\rho$-highest point of $P(X)$ will be 0, and the lowest will be $-\mu$.

Theorem 22. [A]

- *If* $X \in MV(\mu)$ *arises inside* Gr^λ*, then* $\lambda + P(X)$ *is contained in the convex hull of* $W \cdot \lambda$*. (This is very easy, as the latter is* $\Phi(\mathrm{Gr}^\lambda)$*.)*
- *The converse is true!*
- *Therefore, the dimension of the* $\lambda - \mu$ *weight space of* V_λ *is the number of* X *such that* $\lambda + P(X)$ *is contained in the convex hull of* $W \cdot \lambda$*.*
- *The multiplicity of* V_ν *inside* $V_\lambda \otimes V_\mu$ *is the number of* $X \in MV(\lambda + \mu - \nu)$ *such that* $P(X) \subseteq \mathrm{conv}(W \cdot \lambda) \cap (\nu + \mathrm{conv}(W \cdot (-\mu)))$.

Hence the tensor product multiplicity is bounded above by a weight multiplicity, and in a certain limit, will be equal. In turn, the weight multiplicity is bounded above by a Kostant partition function, and in a certain limit will be equal. These two facts are well known and provide a sanity check on the above result.

Example [A]: the MV polygons for A_2.

Anderson's result is a sort of existence result, saying that there is a magic set of polytopes with which to calculate these multiplicities, but he doesn't really say what they are in a useful fashion.

Theorem 23. [K]

- *The map* $X \mapsto P(X)$ *is injective.*
- *The fan of an MV polytope is a coarsening of the fan of* $\mathrm{conv}(W \cdot \check\rho)$*. Loosely speaking, any MV polytope is a degenerate permutahedron, where one cannot turn faces but one can let the edges shrink to zero length. Call these* **pseudo-Weyl polytopes***.*

Each face of pseudo-Weyl polytope has an associated Weyl subgroup of W. The geodesics on $\mathrm{conv}(W \cdot \check\rho)$ from 1 to w_0 are in 1:1 correspondence with reduced words for w_0, and each gives a geodesic on every pseudo-Weyl polytope (though a number of steps may have shrunk to 0 length).

Theorem 24. [K]

- *An integral pseudo-Weyl polytope is an MV polytope iff each 2-face is an MV polygon for the corresponding rank 2 group.*
- *Fix a geodesic from* 1 *to* w_0*. Then the map*

$$P(X) \mapsto \textit{the vector of lengths of the edges along the geodesic}$$

is a bijection.

With this, and the MV polygons for A_2, one obtains a uniform combinatorial rule for tensor product multiplicities in ADE types. (Kamnitzer also includes a description of the B_2 polygons, with which one can get all the remaining groups except G_2.)

It was already well understood how to index Lusztig's canonical basis of $U\mathfrak{n}_-$ by $\mathbb{N}^{\dim \mathfrak{n}_-}$, with a different indexing for each reduced word for w_0. If one is willing to give up $\mathbb{N}^{\dim \mathfrak{n}_-}$ for Anderson's collection of MV polytopes, then one need not choose reduced words.

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