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Combinatorics and Geometry of K-orbits on the Flag Manifold

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Introduction

Let G be a reductive algebraic group over an algebraically closed field F of characteristic $\neq 2$ and let θ be an automorphism of G of period two. Let K be a subgroup of the fixed point subgroup G^θ which contains $(G^\theta)^0$, the identity component of G^θ. Let $\mathcal{B} = \mathcal{B}(G)$ denote the variety of Borel subgroups of G; $\mathcal{B}$ is the flag manifold of G. If $B \in \mathcal{B}$, we may identify $\mathcal{B}$ with the coset space $B\backslash G$. We consider the K-orbits on the flag manifold $\mathcal{B}$ or, equivalently, the set $V = B\backslash G/K$ of $(B \times K)$-orbits of G. It is known that V is finite and there is a natural partial order on V given by inclusion of the orbit closures.

The classification and properties of these orbits play an important role in the representation theory of real semisimple groups (see [**V1**] and [**HMSW**]) and in a number of geometric problems. If $G = H \times H$, where H is a reductive group, and if θ is given by $\theta(x, y) = (y, x)$, $\quad (x, y) \in H \times H$, then $K = \{ (h, h) \mid h \in H \}$ and the K-orbits on $\mathcal{B}(G) = \mathcal{B}(H) \times \mathcal{B}(H)$ can be naturally parametrized by $W(H)$, the Weyl group of H. In this case the partial order on the set V of orbits corresponds to the usual Bruhat order on the Weyl group $W(H)$. One would like a similar description of the poset $V = B\backslash G/K$ in the general case.

In this paper we will give an informal discussion, mostly without proofs, of some recent results of the authors on the set V of orbits. In the joint paper [**RS**], we developed some techniques for the analysis of the orbits. Using this machinery, we were able to give a purely combinatorial description of the partial order on V and to generalize to the poset V a number of standard properties of the usual Bruhat order on the Weyl group $W(G)$. Although the results of [**RS**] are based on simple geometric ideas, the discussion there sometimes gets bogged down in technical detail and parts of the paper are difficult to read (even for the authors). In Sections 1-4 of this paper, we discuss the main ideas and theorems

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of [RS]. We have tried to omit most of the technical detail and to emphasize the underlying geometric ideas.

Let G be simple (over its center) and simply connected. Then we say that the pair (G,θ) is of "Hermitian symmetric type" if the center of $K = G^\theta$ has positive dimension. Roughly speaking, pairs (G,θ) of Hermitian symmetric type correspond to Hermitian symmetric spaces. In Section 5, we discuss some recent (unpublished) results on the parametrization of the set V of orbits for pairs (G,θ) of Hermitian symmetric type. In this case, there is an elementary combinatorial model for the poset V in terms of combinatorial data involving only the Weyl group.

In Section 6, we consider the transcendental case $F = \mathbb{C}$. In this case, there is a real form $G_\mathbb{R}$ of G such that $K_\mathbb{R} = G_\mathbb{R} \cap K$ is a maximal compact subgroup of both $G_\mathbb{R}$ and K. Matsuki has shown that there is a natural duality between the K-orbits and the $G_\mathbb{R}$-orbits on $\mathcal{B}$ which reverses the natural partial order on these orbits and he has proved a number of results concerning both the K-orbits and the $G_\mathbb{R}$-orbits [M1-M5, MO]. It has been shown recently by one of us (RWR) that the machinery of [RS] can equally well be used for the analysis of the $G_\mathbb{R}$-orbits on $\mathcal{B}$. We discuss these results and indicate some extensions of Matsuki's work.

In the applications to the representation theory of real semisimple Lie groups, an important role is played by certain representations of the Hecke algebra $\mathcal{H}$ of the Weyl group W of G (see [LV]). In Section 7, we give a description of these representations of $\mathcal{H}$ in terms of our analysis of the set V of orbits.

§1 Some basic constructions of [RS]

1.1. Preliminaries. Our main reference for algebraic groups and algebraic geometry will be Borel's book [B] and we will usually follow the terminology and notation there. All algebraic varieties will be taken over an algebraically closed field F with $\mathrm{char}(F) \neq 2$.

Since we consider left, right and two sided actions of groups, we need to be careful about the notation for the sets of orbits. Let the groups H and L act on the set E, with H acting on the left and L acting on the right. Then we let $H\backslash E$ denote the set of H-orbits of E and let E/L denote the set of L-orbits. If the actions of H and L commute, so that we get an action of $H \times L$ on E, we let $H\backslash E/L$ denote the set of $(H \times L)$-orbits of E.

Throughout the paper, G, θ and K will be as in the Introduction. In order to conform with the notation of [RS], we will always assume that $K = G^\theta$.

REMARKS 1.1.1. (a) If G is semisimple and simply connected, then it follows from a theorem of Steinberg [St, Thm. 8.1] that $K = G^\theta$ is connected. For a general reductive G however, the fixed point subgroup $K = G^\theta$ is not necessarily connected, so that the K-orbits on $\mathcal{B}$ are not necessarily connected. However, the B-orbits on G/K are always connected (and irreducible), so that one can apply standard irreducibility arguments to these orbits.

(b) The assumption that $G^\theta = K$ is not an essential restriction. The argument goes as follows. Let $\theta_1 : G_1 \to G_1$ be an involutive automorphism of a reductive group G_1 and let K_1 be a subgroup of the fixed point subgroup $G_1^{\theta_1}$ containing the identity component $(G_1^{\theta_1})^0$. We note that the center $Z(G_1)$ acts trivially on $\mathcal{B}(G_1)$, so that we get an induced action of $\mathrm{Ad}(G_1)$ on $\mathcal{B}(G_1) \cong \mathcal{B}(\mathrm{Ad}(G_1))$. A straightforward argument, using the result of Steinberg mentioned in (a) above, shows that there exists a covering group G of $\mathrm{Ad}(G_1)$ and an involutive automorphism θ of G, such that, setting $K = G^\theta$, we have $\mathrm{Ad}(K) = \mathrm{Ad}(K_1)$ (as subgroups of $\mathrm{Ad}(G_1) = \mathrm{Ad}(G)$). Thus the K-orbits of $\mathcal{B}(G) \cong \mathcal{B}(\mathrm{Ad}(G_1))$ are the same as the K_1-orbits, and we are in the case with $K = G^\theta$.

The group G acts on $\mathcal{B}$ (on the left) by conjugation. In order to avoid confusing notation, such as $K \cdot B$ for the K-orbit on $\mathcal{B}$ of a Borel subgroup B, we will often let X denote the projective G-variety corresponding to $\mathcal{B}$. If $x \in X$, we let B_x denote the corresponding Borel subgroup; thus $K \cdot x$, the K-orbit of x on X, is equal to $\{\, {}^kB_x \mid k \in K \,\}$.

1.2. In order to bring the Weyl group into the picture, we need to consider maximal tori. Let $\mathcal{T} = \mathcal{T}(G)$ denote the variety of maximal tori of G. Then θ acts on $\mathcal{T}$ and we let $\mathcal{T}^\theta$ be the subvariety of θ-stable maximal tori. If $T \in \mathcal{T}$, we let $W(T) = N_G(T)/T$ denote the corresponding Weyl group. Let

$$\mathcal{C} = \{\, (B,T) \in \mathcal{B} \times \mathcal{T} \mid T \subset B \,\} \quad \text{and let} \quad \mathcal{C}_\theta = \mathcal{C} \cap (\mathcal{B} \times \mathcal{T}^\theta).$$

The group G acts on $\mathcal{C}$ (on the left) by conjugation and it is clear that $\mathcal{C}_\theta$ is K-stable. Define $\pi : \mathcal{C} \to \mathcal{B}$ by $\pi(B,T) = B$. Then the restriction of π to $\mathcal{C}_\theta$ determines a map $\gamma : K\backslash\mathcal{C}_\theta \to K\backslash\mathcal{B}$ of the sets of K-orbits.

PROPOSITION 1.2.1. *$\gamma : K\backslash\mathcal{C}_\theta \to K\backslash\mathcal{B}$ is a bijection.*

PROOF. We will give the proof, since it is quite easy. Let $B \in \mathcal{B}$. Since $B \cap \theta(B)$ is of maximal rank, it follows from another theorem of Steinberg [St, Thm. 7.5] that $B \cap \theta(B)$ contains a θ-stable maximal torus. Thus γ is surjective. To prove that γ is injective, we need to prove that if T and T' are θ-stable maximal tori of B, then they are conjugate by an element of $B \cap K$. Now T and T' are maximal tori of $B \cap \theta(B)$, hence they are conjugate by an element u of $R_u(B \cap \theta(B))$, the unipotent radical of $B \cap \theta(B)$. It follows from this that $u^{-1}\theta(u) \in N_G(T) \cap R_u(B \cap \theta(B))$. But $R_u(B \cap \theta(B)) \subset R_u(B)$, so that $u^{-1}\theta(u) \in N_G(T) \cap R_u(B) = \{1\}$; thus $\theta(u) = u$ and hence $u \in K$

In the transcendental case $F = \mathbb{C}$, Proposition 1.2.1 is due to Matsuki [M1] and Rossman [Ros]. The basic idea is due to Wolf [W], who proved a similar theorem for the $G_\mathbb{R}$-orbits on $\mathcal{B}$. (Here $G_\mathbb{R}$ is as in the Introduction.)

We say that a pair $(B,T) \in \mathcal{C}$ is a *standard pair* if both B and T are θ-stable. It follows from [St, Thm. 7.5] that standard pairs exist. We choose a standard pair (B_0, T_0), which will remain fixed throughout the paper. Let $W = W(T_0)$ and $N = N_G(T_0)$. Let $\Phi = \Phi(T_0, G)$ be the set of roots of G

relative to T_0, let $\Phi^+ = \Phi(T_0, B_0)$ be the set of positive roots determined byB_0 and let $\Delta = \Delta(T_0, B_0)$ be the set of simple roots corresponding to Φ^+. If $\alpha \in \Phi$, let $s_\alpha \in W$ be the corresponding reflection. Let $S = \{ s_\alpha \mid \alpha \in \Delta \}$. Then $W = (W, S)$ is a Coxeter group. Let l denote the length function on W and let $\leq$ be the Bruhat order on W.

For each subset I of Δ, let W_I be the subgroup of W generated by $\{ s_\alpha \mid \alpha \in I \}$ and let $P_I = B_0 W_I B_0$ be the "standard" parabolic subgroup of G corresponding to I. We let w_I denote the longest element of W_I and let $w_0 = w_\Delta$ be the longest element of W.

Define $\zeta_0 : G \to \mathcal{C}$ by $\zeta_0(g) = ({}^{g^{-1}}B_0, {}^{g^{-1}}T_0)$. Then ζ_0 is constant on right T_0 cosets of G and induces a isomorphism of varieties $\zeta : T_0\backslash G \to \mathcal{C}$. The group G acts on $T_0\backslash G$ on the right and we have $\zeta(x \cdot g) = g^{-1} \cdot \zeta(x)$, $g \in G$, $x \in T_0\backslash G$. Set

$$\mathcal{V} = \{ g \in G \mid g\theta(g)^{-1} \in N \} = \{ g \in G \mid {}^{g^{-1}}T_0 \in \mathcal{T}^\theta \} = \zeta_0^{-1}(\mathcal{C}_\theta).$$

Since $N_G(B_0) = B_0$ and $B_0 \cap N = T_0$, we see that the restriction of ζ_0 to $\mathcal{V}$ induces a bijection of $T_0\backslash\mathcal{V}$ onto $\mathcal{C}_\theta$. Thus there is an induced bijection of orbit sets $T_0\backslash\mathcal{V}/K \to K\backslash\mathcal{C}_\theta$.

The map $g \mapsto {}^{g^{-1}}B_0$ of G to $\mathcal{B}$ is constant on left B_0-cosets and induces an isomorphism of K-varieties $B_0\backslash G \to \mathcal{B}$. Thus we obtain a bijection $B_0\backslash G/K \cong K\backslash\mathcal{B}$.

Combining all of these results with Proposition 1.2.1, we obtain:

PROPOSITION 1.2.2. *There exist canonical bijections between the following four sets of orbits*: (a) $T_0\backslash\mathcal{V}/K$; (b) $K\backslash\mathcal{C}_\theta$; (c) $K\backslash\mathcal{B}$; *and* (d) $B_0\backslash G/K$.

We observe that the bijection $T_0\backslash\mathcal{V}/K \cong B_0\backslash G/K$ is induced by the inclusion map $\mathcal{V} \to G$.

We let V denote the set $T_0\backslash\mathcal{V}/K$ of $(T_0 \times K)$–orbits on $\mathcal{V}$. By Proposition 1.2.2, we may identify V with each of the orbit sets given in (b), (c) and (d) of the Proposition. Occasionally we will make make such identifications without being very explicit about it.

1.3. Notation and remarks. (a) We let $x_0 \in X$ correspond to B_0, so that $B_{x_0} = B_0$. If $v \in V$, then v is a (T_0, K) double coset, $v = T_0gK$, with $g \in \mathcal{V}$. In this case we let B_0vK denote the set-theoretic product; thus $B_0vK = B_0(T_0gK)K = B_0gK$. It $v \in V$, we will let $\mathcal{O}(v)$ denote B_0vK, and let $\mathcal{K}(v)$ denote the corresponding K–orbit on X. Thus, if $v = T_0gK$ and if $x = g^{-1} \cdot x_0$ (so that $B_x = {}^{g^{-1}}B_0$), then $\mathcal{K}(v)$ is equal to $K \cdot x$, the K–orbit of x on X.

(b) The varieties $\mathcal{T}^\theta$, $\mathcal{C}_\theta$ and $\mathcal{V}$ are not connected. In fact, it follows from [**R1**, Thm. A] that K^0, the identity component of K, acts transitively on each irreducible component of $\mathcal{T}^\theta$. Using this, one can show that K^0 acts transitively on each irreducible component of $\mathcal{C}^\theta$ and that $T_0 \times K^0$ acts transitively on each irreducible component of $\mathcal{V}$. In particular, all K-orbits on $\mathcal{C}_\theta$ are closed and all $(T_0 \times K)$–orbits on $\mathcal{V}$ are closed. Thus, one cannot obtain any information about closures of K–orbits in X or closures of $(B_0 \times K)$-orbits in G from the closures of the corresponding orbits on $\mathcal{C}_\theta$ or $\mathcal{V}$.

1.4. The map ϕ and the W–action on V. Since T_0 and N are θ–stable, we get an induced action of θ on the Weyl group W. Let $\mathcal{I} = \{ w \in W \mid \theta(w) = w^{-1} \}$. We say that the elements of $\mathcal{I}$ are *twisted involutions*. If θ is an inner automorphism of G, then θ acts trivially on W and $\mathcal{I}$ is just the set of ordinary involutions of W. We define the "twisted action" of the group W on the set W by the following rule : if w', $w \in W$, then $w' * w$, the twisted action of w' on w, is equal to $w'w\theta(w'^{-1})$. It is clear that the set $\mathcal{I}$ of twisted involutions is stable under the twisted action of W. We have the following elementary result [**S1**, §3] concerning $\mathcal{I}$:

LEMMA 1.4.1. *Let $s \in S$ and $a \in \mathcal{I}$ and assume that $s*a \neq a$. If $l(sa) > l(a)$, then $l(s*a) = l(a) + 2$ and if $l(sa) < l(a)$, then $l(s*a) = l(a) - 2$.*

Define $\kappa : G \to G$ by $\kappa(g) = g\theta(g)^{-1}$. Then κ is constant on left K-cosets, and induces an isomorphism of G/K onto the closed subvariety $\kappa(G)$ of G. We note that $\mathcal{V} = \kappa^{-1}(N)$. Let $\pi : N \to W$ be the canonical projection. Then the map $g \to \pi(\kappa(g))$ from $\mathcal{V}$ to W is constant on $(T_0 \times K)$–orbits and hence induces a map $\phi : V \to W$. If $g \in G$, then it is easy to check that $\theta(\kappa(g)) = \kappa(g)^{-1}$, which shows that $\phi(V) \subset \mathcal{I}$. Let $v = T_0gK \in V$ and let $\phi(v) = a$. Then $\kappa(\mathcal{O}(v)) \subset B_0aB_0$; one can use this to give an alternate definition of the map ϕ.

The map ϕ was introduced in [**S1**]. A geometric interpretation of ϕ in terms of the *canonical Weyl group $\mathcal{W}$* is given in 1.7. The map ϕ plays an important role in the study of the set V of orbits. For example, we have:

PROPOSITION 1.4.2. *Let $v \in V$. Then the orbit $\mathcal{O}(v)$ is closed if and only if $\phi(v) = 1$.*

See [**S1**, §6.6].

We also have the following characterization of closed orbits:

PROPOSITION 1.4.3. *Let $x \in X$. Then the orbit $K \cdot x$ is closed in X if and only if the Borel subgroup B_x is θ–stable.*

Let $V_0 = \{ v \in V \mid \phi(v) = 1 \}$ denote the set of closed orbits. An easy argument shows that all closed K-orbits on X have dimension equal to the dimension of $\mathcal{B}(K^0)$, the flag manifold of K^0; this is equivalent to the statement that if $B \in \mathcal{B}^\theta$, then $K^0 \cap B$ is a Borel subgroup of K^0. If $v \in V$, we set $l(v) = \dim \mathcal{K}(v) - \dim \mathcal{B}(K^0)$. We say that $l(v)$ is the *length* of v. In the case $G = H \times H$ and $\theta(x, y) = (y, x)$, which was discussed in the Introduction, the orbits are parametrized by the Weyl group $W(H)$ and the length function l on the set V of orbits corresponds to the usual length function on the Coxeter group $W(H)$.

Since N is θ-stable, it is clear that $\mathcal{V}$ is stable under left multiplication by N. Thus we get an action of W on $V = T_0\backslash\mathcal{V}/K$. We let $w \cdot v$ denote the action of $w \in W$ on $v \in V$. A geometric interpretation of the action of W on the set V of orbits in terms of the canonical Weyl group $\mathcal{W}$ is given in 1.7.

The W action on V and the twisted W action on $\mathcal{I}$ are related by the following proposition:

PROPOSITION 1.4.4. (1) *If* $w \in W$ *and* $v \in V$, *then* $\phi(w \cdot v) = w * \phi(v)$. *Thus* $\phi : V \to \mathcal{I}$ *is* W*-equivariant.* (2) *If* $\phi(v) = \phi(v')$, *then* v *and* v' *lie in the same* W*-orbit.* (3) *There are canonical bijections between the following three sets of orbits:* (i) $W\backslash V$; (ii) $W\backslash \mathrm{image}(\phi)$; *and* (iii) $K\backslash\mathcal{T}^\theta$.

1.5. Examples. (See [**RS**, §10].)

(1) Let H be reductive, let $G = H \times H$ and let θ be defined by $\theta(x, y) = (y, x)$. In this case we may identify $W(G)$, the Weyl group of G, with $W(H) \times W(H)$ and the set $\mathcal{I}$ of twisted involutions is given by $\mathcal{I} = \{\, (w, w^{-1}) \mid w \in W(H)\,\}$. The map $\phi : V \to \mathcal{I}$ is a bijection.

(2) Let $G = GL(n, F)$ and define $\theta : G \to G$ by $\theta(g) = {}^tg^{-1}$. Thus K is equal to the orthogonal group $O(n, F)$ and W is the symmetric group S_n. Let $\mathcal{J} = \mathcal{J}_n$ be the set of all involutions in S_n. Then $\mathcal{I} = \mathcal{J}w_0$. In this case also, $\phi : V \to \mathcal{I}$ is a bijection.

(3) Let $G = SL(n, F)$ and let θ be as in (2). Then $K = SO(n, F)$. Again $W = S_n$ and $\mathcal{I} = \mathcal{J}w_0$. The map $\phi : V \to \mathcal{I}$ is surjective. Let $\mathcal{J}^0 \subset \mathcal{J}$ be the set of fixed point free involutions in S_n and let $\mathcal{J}^1 \subset \mathcal{J}$ be the set of involutions which have a fixed point. If $a \in \mathcal{J}^0$, then $|\phi^{-1}(aw_0)| = 2$ and if $a \in \mathcal{J}^1$, then $|\phi^{-1}(aw_0)| = 1$. Thus $|V| = 2|\mathcal{J}^0| + |\mathcal{J}^1|$. So, in particular, ϕ is bijective if n is odd but is not bijective if n is even.

(4) Let $G = SL(2n, F)$ and let $J \in G$ be defined by: $J(e_i) = -e_{n+i}$ and $J(e_{n+1}) = e_i$, $i = 1, \ldots, n$, where $e_1, \ldots, e_{2n}$ is the standard basis of F^{2n}. Define the involutive automorphism $\theta : G \to G$ by $\theta(g) = J\,{}^tg^{-1}J^{-1}$. Then the fixed point subgroup $K = G^\theta$ is the symplectic group $Sp(2n, F)$. The Weyl group W is the symmetric group S_{2n}. Let $\mathcal{J} = \mathcal{J}_{2n}$ and $\mathcal{J}^0$ be as above. Then the set $\mathcal{I}$ of twisted involutions is equal to $\mathcal{J}w_0$ and the image of ϕ is equal to $\mathcal{J}^0w_0$.

We note that in general the map $\phi : V \to \mathcal{I}$ is not necessarily either injective or surjective. It follows from Proposition 1.4.4 that the image of ϕ is a union of twisted W-orbits of $\mathcal{I}$.

1.6. Since $B_0\backslash G/K$ is finite, there exists a unique open $(B_0 \times K)$-orbit of G. Let $v_{max} \in V$ be such that $\mathcal{O}(v_{max})$ is open in G and let $a_{max} = \phi(v_{max})$. One can describe a_{max} in terms of the Araki diagram associated to (G, θ). (See [**S2**] for the Araki diagram.) Let $J \subset S$ correspond to the set of black dots in the Araki diagram. Then a_{max} is equal to w_Jw_0, where w_J and w_0 are as in 1.2.

We have the following criteria for the map ϕ to be surjective:

PROPOSITION 1.6. *The following four conditions are equivalent:* (i) ϕ *is surjective*; (ii) $a_{max} = w_0$; (iii) *the Araki diagram of* (G, θ) *does not contain any black dots*; *and* (iv) *there exists a* θ*-split Borel subgroup of* G.

Recall that a Borel subgroup B is θ-split if $B \cap \theta(B)$ is a maximal torus of G.

1.7. The canonical Weyl group. The map $\phi : V \to W$ and the W-action on V are defined in terms of the standard pair (B_0, T_0). If (B_1, T_1) is another standard pair, then it can be shown that T_0 and T_1 are K-conjugate (in fact K^0-conjugate), but it is not necessarily true that B_0 and B_1 are K-conjugate. We will show that ϕ and the W-action on V are canonically defined. To discuss this, we need the *canonical Weyl group* $\mathcal{W}$. As a set $\mathcal{W}$ is equal to $G\backslash(\mathcal{B} \times \mathcal{B})$, the set of G-orbits on $\mathcal{B} \times \mathcal{B}$. For each (B, T) in $\mathcal{C}$, there is a bijective map $\eta_{B,T} : W(T) \to \mathcal{W}$ defined by $\eta_{B,T}(w) = p(B, {}^wB)$, where $p : \mathcal{B} \times \mathcal{B} \to G\backslash(\mathcal{B} \times \mathcal{B}) = \mathcal{W}$ is the canonical map. Let $\eta_0 = \eta_{B_0,T_0}$. We define the group structure on $\mathcal{W}$ by requiring that η_0 be an isomorphism of groups. This implies that each $\eta_{B,T}$ is an isomorphism of groups. If $(B, B') \in \mathcal{B} \times \mathcal{B}$ and if $\overline{w} = p(B, B')$, then we say that $\overline{w} \in \mathcal{W}$ is the *relative position* of (B, B'). Let $\overline{S} = \eta_0(S)$. Then $\mathcal{W} = (\mathcal{W}, \overline{S})$ is a Coxeter group; this Coxeter group structure on $\mathcal{W}$ is canonical. As usual, we let l denote the length function on $\mathcal{W}$.

There is a canonical action of $\mathcal{W}$ on $\mathcal{C}$ defined as follows:

Let $\overline{w} \in \mathcal{W}$ and let $(B, T) \in \mathcal{C}$. Choose w in $W(T)$ such that $\eta_{B,T}(w) = \overline{w}$. Then $\overline{w} \cdot (B, T) = ({}^{w^{-1}}B, T)$. It is a straightforward exercise to prove that this defines an action of $\mathcal{W}$ on $\mathcal{C}$. The projection $\mathcal{C} \to \mathcal{T}$, $(B, T) \mapsto T$, is a Galois covering and $\mathcal{W}$ acts on $\mathcal{C}$ as the group of "deck transformations". (For the case $F = \mathbb{C}$, the projection $\mathcal{C} \to \mathcal{T}$ is a covering map in the usual sense.)

We note that $\mathcal{C}_\theta$ is stable under the $\mathcal{W}$-action and it is clear that the $\mathcal{W}$-action on $\mathcal{C}$ commutes with the action of G on $\mathcal{C}$ by conjugation. Let $\zeta : T_0\backslash G \to \mathcal{C}$ be as in 1.2. Left multiplication by N gives an action of $W = W(T_0)$ on $T_0\backslash G$. An easy argument shows that $\zeta(w \cdot x) = \eta_0(w) \cdot \zeta(x)$, for $w \in W$ and $x \in T_0\backslash G$, so that ζ is equivariant (with respect to η_0). Now ζ maps $T_0\backslash\mathcal{V}$ equivariantly onto $\mathcal{C}_\theta$. Thus the bijection of $V = T_0\backslash\mathcal{V}/K$ onto $K\backslash\mathcal{C}_\theta$ determined by ζ is equivariant with respect to η_0. This proves that the W-action on the set of orbits is canonical. (This was not done in [**RS**].)

As regards the map $\phi : V \to \mathcal{I}$, we have the following proposition.

PROPOSITION 1.7.1. *Let* $v \in V$ *and let* $x \in \mathcal{K}(v)$, *so that* $\mathcal{K}(v) = K \cdot x$. *Let* $\overline{a} = \eta_0(\phi(v))$. *Then* $\overline{a}$ *is the relative position of* $(B_x, \theta(B_x))$.

The proof is straightforward.

It follows from Proposition 1.7.1 that $\overline{a} = \eta_0(\phi(v))$ depends only on the orbit $\mathcal{K}(v) = K \cdot x$ and is independent of the choice of standard pair (B_0, T_0).

§2. The product of a minimal parabolic and an orbit

Let $s \in S$ and let $P_s = B_0 \cup B_0sB_0$ be the corresponding standard minimal parabolic subgroup. If $v \in V$, then the product $P_sB_0vK = P_svK$ is a union of a finite number of $(B_0 \times K)$-orbits. In this section, we will analyze the decomposition of $P_s vK$ into $(B_0 \times K)$-orbits. In order to do this, it is easier to work with the K-orbits on X rather than the $(B_0 \times K)$-orbits.

Most of the results of this section are contained in [**LV**].

2.1. Real, complex and imaginary roots. Let T be a θ-stable maximal torus and let $\Phi(T, G)$ be the set of roots of G relative to T. If $\alpha \in \Phi(G, T)$, let G_α be the subgroup of G generated by the root subgroups U_α and $U_{-\alpha}$; the subgroup G_α is semisimple of rank one. Let $T_\alpha = T \cap G_\alpha$; then T_α is a maximal torus of G_α.

There are three cases to consider.

(a) $\theta(\alpha) = \alpha$. In this case we say that α is *imaginary* (relative to θ). If α is imaginary, the subgroup G_α is θ-stable. There are two subcases. If the restriction of θ to G_α is trivial, so that $G_\alpha \subset K$, then α is *compact imaginary*. If $G_\alpha \not\subset K$, then α is *non-compact imaginary*. In the latter case, T_α is the identity component of G_α^θ.

(b) $\theta(\alpha) = -\alpha$. Then we say that α is *real*. In this case, G_α is θ-stable and $(G_\alpha^\theta)^0$ is a maximal torus of G_α.

(c) $\theta(\alpha) \neq \pm\alpha$. Then α is *complex*. In this case G_α is not θ-stable.

REMARK. The terminology of real, complex and imaginary roots is taken from the theory of real semisimple Lie groups.

2.2. The $\mathbb{P}^1$ approach. Let $Y = \mathcal{P}_s$ denote the variety of all conjugates of P_s. Let $\pi_s : X \to Y$ denote the morphism which assigns to every $B \in \mathcal{B}$ the unique $P \in \mathcal{P}_s$ which contains B. Let $y_0 = \pi_s(x_0)$ (recall that $B_{x_0} = B_0$). Now let $y \in Y$ and let $P_y \in \mathcal{P}_s$ be the corresponding parabolic subgroup. Let X_y denote $\mathcal{B}(P_y)$, the variety of Borel subgroups of P_y; note that $X_y = \pi_s^{-1}(y)$. Then X_y is a complete subvariety of X which is isomorphic to $\mathbb{P}^1(F) = \mathbb{P}^1$. Let $A_y = \operatorname{Aut}(X_y)$, the group of automorphisms of the algebraic variety X_y and let $h : P_y \to A_y$ be the canonical homomorphism. Then A_y is an algebraic group isomorphic to $PGL(2, F)$. Let $K_y = K \cap P_y$; then K_y is the isotropy subgroup at y for the action of K on Y. The following lemma is elementary:

LEMMA 2.2.1. *Let $x \in X_y$ and $g \in G$ be such that $g \cdot x \in X_y$. Then $g \in P_y$.*

Let $x \in X_y$. Then it follows easily from Lemma 2.2.1 that $K \cdot x \cap X_y = K_y \cdot x$. Furthermore $K \cdot x$ is a homogeneous fiber bundle over $K \cdot y$ with fiber $K_y \cdot x$. In particular, we have a bijective correspondence between the K-orbits on $K \cdot X_y$ and the K_y-orbits on X_y. Since $K \backslash X$ is finite, K_y has a finite number of orbits on X_y. Since $h(K_y)$ is an algebraic subgroup of $A_y \cong PGL(2, F)$, it is easy to analyze the possibilities for the K_y-orbits on X_y. There are four cases to consider.

Case 1. $h(K_y) \neq A_y$ and $h(K_y)$ contains a non-trivial unipotent subgroup. Then there are two K_y-orbits on X_y, one of which is a fixed point.

Case 2. $h(K_y) = A_y$. Then K_y is transitive on X_y.

Case 3. $h(K_y)$ is a maximal torus of A_y. There are three orbits, two fixed points and one open dense orbit.

Case 4. $h(K_y)$ is the normalizer of a maximal torus of A_y. There are two orbits. There are two fixed points of K_y^0, which are permuted by K_y, and there is an open dense orbit.

2.3. Further analysis of the K_y-orbits on X_y. Let $x \in X_y$ and let $B = B_x$. Let T be a θ-stable maximal torus of B, and let $\Phi(T, G)$ and $\Delta(T, B)$ be defined as usual. Then there exists $\alpha \in \Delta(T, B)$ such that $P = P_y$ is equal to $P_\alpha = B \cup Bs_\alpha B$. Note that, since T is θ-stable, θ acts on $\Phi(T, G)$.

Case A. α is complex (relative to θ). In this case $h(K_y)$ is a solvable group with non-trivial unipotent radical. Thus we are in Case 1 above.

Case B. α is compact imaginary. Then $G_\alpha \subset P \cap K$ and hence $h(K_y) = A_y$. Thus we are in Case 2 above and K_y is transitive on X_y.

Case C. α is non-compact imaginary. In this case $(G_\alpha^\theta)^0 = T_\alpha$ and $h(T_\alpha) = h(K_y)^0$. Then we are in either Case 3 or Case 4 above. We have $h(K_y)^0 \cdot x = x$, and there are either two or three K_y-orbits. If $h(K_y)$ is connected, there are three K_y-orbits on X_y and if $h(K_y)$ is not connected, there are two orbits.

Case D. α is real. Then $(G_\alpha^\theta)^0$ is a maximal torus of G_α and $h((G_\alpha^\theta)^0) = h(K_y)^0$. We are in either Case 3 or 4 above. There are either two or three K_y-orbits on X_y and $K_y \cdot x$ is the unique dense open orbit in X_y.

2.4. Case analysis for the $(B_0 \times K)$-orbits on $P_s vK$. Let $\alpha \in \Delta = \Delta(T_0, B_0)$. Let $s = s_\alpha$ and let $v = T_0 gK \in V$. We wish to describe the decomposition of $P_s vK$ into $(B_0 \times K)$-orbits. Let $x = g^{-1} \cdot x_0$, let $T = {}^{g^{-1}}T_0$ and let $B = {}^{g^{-1}}B_0$. Thus $T \in \mathcal{T}^\theta$ and $B = B_x$. Let $y = \pi_s(x)$ and let $P = P_y = {}^{g^{-1}}P_s$. The map $g' \mapsto g'^{-1} \cdot x_0$ from G to X determines a bijection from $(B_0 \times K)$-orbits on $P_s vK$ to K-orbits on $K \cdot X_y$ and it follows from the discussion above that there is a bijection from K_y-orbits on X_y to K-orbits on $K \cdot X_y$. Thus we may use the case analysis of 2.3 above for the $(B_0 \times K)$-orbits on $P_s vK$.

First we need some definitions. The inner automorphism $\operatorname{Int}(g^{-1})$ of G maps T_0 to T and maps $\Phi = \Phi(T_0, G)$ onto $\Phi(T, G)$. Let $\alpha' = \operatorname{Int}(g^{-1})(\alpha)$. We say that α (or $s = s_\alpha$) is complex, compact imaginary, ... , for v if α' is complex, compact imaginary, ... , relative to θ in the sense of 2.1. These definitions are independent of the choice of $g \in v$. Let $\phi(v) = a$. Then it is easy to check that: (i) α is complex for v if $a\theta(\alpha) \neq \pm\alpha$; (ii) α is imaginary for v if $a\theta(\alpha) = \alpha$; and (iii) α is real for v if $a\theta(\alpha) = -\alpha$. We observe that if $s = s_\alpha$ is real (respectively imaginary) for v, then $l(sa) < l(a)$ (respectively $l(sa) > l(a)$).

Now we can apply the results of 2.2 and 2.3 to the $(B_0 \times K)$-orbits of $P_s vK$.

Case A. s is complex for v. Then $P_s vK = \mathcal{O}(v) \cup \mathcal{O}(s \cdot v)$ and $s \cdot v \neq v$. Thus there are two $B_0 \times K$-orbits on $P_s vK$. If $sa > a$ (respectively $sa < a$) then $\mathcal{O}(v)$

(respectively $\mathcal{O}(s\cdot v)$) is closed in P_svK and $\mathcal{O}(s\cdot v)$ (respectively $\mathcal{O}(v)$) is open and dense in P_svK.

Case B. s is compact imaginary. There is only one orbit, so that $\mathcal{O}(v) = P_svK$.

Case C. s is non-compact imaginary for v. Then there exists $v' \in V$ such that $P_svK = \mathcal{O}(v) \cup \mathcal{O}(s\cdot v) \cup \mathcal{O}(v')$. The orbits $\mathcal{O}(v)$ and $\mathcal{O}(s\cdot v)$ are closed in P_svK and $\mathcal{O}(v')$ is open and dense in P_svK. If $s\cdot v \neq v$, there are three orbits and if $s\cdot v = v$, then there are two orbits. Both cases can occur.

Case D. s is real for v. Then there exists $v' \in V$ such that $P_svK = \mathcal{O}(v) \cup \mathcal{O}(v') \cup \mathcal{O}(s\cdot v')$. The orbits $\mathcal{O}(v')$ and $\mathcal{O}(s\cdot v')$ are closed in P_svK and $\mathcal{O}(v)$ is open and dense in P_svK. If $s\cdot v' \neq v'$, there are three orbits, otherwise there are two orbits. Both cases can occur.

In Case C (respectively Case D), s is real (respectively non-compact imaginary) for v' and $P_svK = P_sv'K$.

§3 The monoid $M(W)$

3.1. The monoid M(W). In our analysis of the orbits, an important role is played by a certain monoid $M = M(W)$ which is canonically associated to the Coxeter group $W = (W, S)$. (See [**RS**, §3].) As a set, M consists of symbols $m(w)$, one for each $w \in W$. Multiplication in M is determined by the following rule: if $w \in W$ and $s \in S$, then $m(s)m(w)$ is equal to $m(sw)$ if $l(sw) > l(w)$ and is equal to $m(w)$ if $l(sw) < l(w)$. Note that $m(1)$ is the identity element of M and $m(w_0)$ is the "final element" of M, i.e. $m(w)m(w_0) = m(w_0) = m(w_0)m(w)$ for every $w \in W$. If $w \in W$ and if $\mathbf{s} = (s_1, \ldots, s_k)$ is a reduced decomposition of w, then $m(w) = m(s_1)m(s_2)\cdots m(s_k)$. If $s \in S$, then $m(s)^2 = m(s)$.

The monoid algebra $\mathbb{Z}[M(W)]$ can be viewed as a degeneration of the Hecke algebra $\mathcal{H}$ of W (see Section 7).

There is a geometric interpretation of the multiplication in M in terms of the product of $(B_0 \times B_0)$-orbits of G. By the Bruhat Lemma, we have $G = \coprod_{w\in W} B_0wB_0$, where the symbol $\coprod$ denotes the disjoint union. Let w, w' and $w'' \in W$. Then $m(w)m(w') = m(w'')$ if and only if $B_0w''B_0$ is the unique dense open $(B_0 \times B_0)$-orbit in the product $B_0wB_0 \cdot B_0w'B_0$.

3.2. Action of M on V. There is an action of M on the set V (or, equivalently, an action on $B_0\backslash G/K$ or $\mathcal{K}\backslash\mathcal{B}$) defined as follows: If $v \in V$ and $w \in W$, then $m(w)\cdot v$ is the unique element $v' \in V$ such that $B_0v'K$ is the dense open $(B_0 \times K)$-orbit in the product $B_0wB_0 \cdot B_0vK$. It follows from the geometric description of the multiplication in M that this defines an action of M on V. If $v \in V$, then $M\cdot v = \{\, m(w)\cdot v \mid w \in W \,\}$ denotes the M-orbit of V.

Let $s \in S$, let $v \in V$ and let $a = \phi(v)$. Then $B_0m(s)\cdot vK$ is the dense open $(B_0 \times K)$-orbit in P_svK. If $m(s)\cdot v \neq v$, then we write $v \rightarrow m(s)\cdot v$. It follows from 2.5 that $v \rightarrow m(s)\cdot v$ if and only if one of the following two conditions holds: (i) s is complex for v and $l(sa) > l(a)$; or (ii) s is non-compact imaginary for v.

REMARK 3.2.1. If $v \rightarrow m(s)\cdot v$, then it is clear that $\dim \mathcal{O}(m(s)\cdot v) = \dim \mathcal{O}(v) + 1$ and $\dim \mathcal{K}(m(s)\cdot v) = \dim \mathcal{K}(v) + 1$.

LEMMA 3.2.2. *Let $s \in S$, let $v \in V$ and let $a = \phi(v)$. Assume that $l(sa) < l(a)$. Then there exists $v' \in V$ such that $v' \rightarrow m(s)\cdot v' = v$. Furthermore, if $v'' \in V$ is such that $v'' \rightarrow m(s)\cdot v'' = v$, then either* (i) *$v'' = v'$ or* (ii) *s is non-compact imaginary for v and $v'' = s\cdot v' \neq v'$.*

The proof follows easily from 2.4.

DEFINITION 3.2.3. Let $v \in V$. A *reduced decomposition* of v is a pair $(\mathbf{v}, \mathbf{s})$, where $\mathbf{v} = (v_0, \ldots, v_k)$ is a sequence in V and $\mathbf{s} = (s_1, \ldots, s_k)$ is a sequence in S, which satisfies the following conditions: (1) $v_0 \in V_0$ and $v_k = v$; and (2) for each $i = 0, 1, \ldots, k-1$, we have $v_i \rightarrow m(s_{i+1})\cdot v_i = v_{i+1}$. We say that k is the *length* of the reduced decomposition $(\mathbf{v}, \mathbf{s})$.

It follows from Lemma 3.2.2 that every $v \in V$ has a reduced decomposition. It follows from Remark 3.2.1 that every reduced decomposition of v has length equal to $l(v)$. If $(\mathbf{v}, \mathbf{s})$ is a reduced decomposition of v, it is clear that v_0 and $\mathbf{s}$ determine $(\mathbf{v}, \mathbf{s})$. It is not necessarily the case that $\mathbf{v}$ determines $\mathbf{s}$.

Reduced decompositions of elements of V should be considered as the analogue of reduced decompositions of elements of the Weyl group. For the case in which $G = H \times H$ and $\theta(h, h') = (h', h)$, the orbits are parametrized by the Weyl group $W(H)$ and reduced decompositions of elements of V correspond to reduced decompositions of the corresponding elements of $W(H)$.

3.3. Action of M on $\mathcal{I}$. (See [**RS**, §3].) There is an action of the monoid M on the set $\mathcal{I}$ of twisted involutions which, in a sense, parallels the twisted action of the Weyl group W on $\mathcal{I}$. If $s \in S$ and $a \in \mathcal{I}$, we define $s \circ a \in \mathcal{I}$ as follows: if $s * a = a$, then $s \circ a = sa$; if $s * a \neq a$, then $s \circ a = s * a$. For each $s \in S$, the map $a \mapsto s \circ a$ is a fixed point free bijection of $\mathcal{I}$ of order two. We now define the twisted action of elements $m(s)$, $s \in S$, on $\mathcal{I}$. If $s \in S$ and $a \in \mathcal{I}$, we set $m(s) * a$ equal to $s \circ a$ if $l(sa) > l(a)$ and equal to a if $l(sa) < l(a)$. The proof that this extends to give an action of M on $\mathcal{I}$ is a bit tricky. If $m \in M$ and $a \in \mathcal{I}$, then $m * a$ denotes the twisted action of m on a.

LEMMA 3.3.1. *If $a \in \mathcal{I}$, then there exists $m \in M$ such that $m * 1 = a$.*

DEFINITION 3.3.2. Let $a \in \mathcal{I}$. Then $L(a)$, *the length of a as a twisted involution*, is the smallest integer k for which there exists $w \in W$ with $l(w) = k$ and $m(w) * 1 = a$.

It is clear that $L(a) \leq l(a)$, where $l(a)$ is the length of a as an element of the Coxeter group $W = (W, S)$, but it seldom happens that $L(a) = l(a)$. We can also define $L(a)$ in terms of the -1 eigenspace of the involution $a\theta$, acting on

$E = X(T) \otimes_{\mathbb{Z}} \mathbb{R}$. If $a \in \mathcal{I}$, let $E_-(a)$ denote the -1 eigenspace of $a\theta$ on E. Then $2L(a) = l(a) + \dim E_-(a) - \dim E_-(1)$.

Let $s \in S$ and $a \in \mathcal{I}$. If $a \neq m(s) * a$, then we write $a \to m(s) * a$. In this case $L(a) + 1 = L(m(s) * a)$.

The action of M on V and the twisted action of M on $\mathcal{I}$ are related by the following proposition.

PROPOSITION 3.3.3. *Let v, $v' \in V$ and let $s \in S$. (1) If $v' \to m(s) \cdot v' = v$, then $\phi(v') \to m(s) * \phi(v') = \phi(v)$. (2) Assume that $\phi(v') \to m(s) * \phi(v')$. Then $v' \to m(s) \cdot v'$ unless s is compact imaginary for v', in which case $v' = m(s) \cdot v'$.*

Note that it is not necessarily the case that the map $\phi : V \to \mathcal{I}$ is M-equivariant.

Now let $(\mathbf{v}, \mathbf{s} = (s_1, \ldots, s_k))$ be a reduced decomposition of $v \in V$ and let $a = \phi(v)$. Then it follows from Proposition 3.3.3 that $a = m(s_k) * m(s_{k-1} * \cdots * m(s_1) * 1$ and that $L(a) = k$. Thus we obtain:

PROPOSITION 3.3.4. *Let $v \in V$ and let $a = \phi(v)$. Then $l(v) = L(a)$. Consequently $\dim \mathcal{K}(v) = L(a) + \dim \mathcal{B}(K^0)$ and $\dim \mathcal{O}(v) = L(a) + \dim \mathcal{B}(K^0) + \dim B$.*

We see from Proposition 3.3.4 that the dimensions of the orbits $\mathcal{K}(v)$ and $\mathcal{O}(v)$ are determined by $L(\phi(v))$, the length of $\phi(v)$ as a twisted involution.

We define the *weak order* on $\mathcal{I}$, denoted by $\vdash$, as follows: Let a, $b \in \mathcal{I}$. Then $a \vdash b$ if $b \in M * a$, where $M * a$ denotes the (twisted) M-orbit of a.

The next proposition describes the image of the map ϕ.

PROPOSITION 3.3.5. *The following three conditions on $a \in \mathcal{I}$ are equivalent: (i) $a \in \mathrm{image}(\phi)$; (ii) $a \vdash a_{max}$; and (iii) there exists $b \in W * a$ such that $E_-(b) \subset E_-(a_{max})$.*

3.4. $(P \times K)$**–orbits on** G. Let I be a subset of Δ and let $P = P_I$ be the corresponding standard parabolic subgroup. Then each $(B_0 \times K)$–orbit of G is contained in a unique $(P \times K)$–orbit, so that we have a surjective map $f_I : B_0 \backslash G/K \to P \backslash G/K$ of the orbit sets. It is of interest to describe the fibres of the map f_I. We shall how indicate how to handle this problem in terms of the $M(W)$ formalism. Since this problem is not discussed in [**RS**], we shall give proofs in this subsection.

PROPOSITION 3.4.1. *Let v', $v \in V$. Then the following conditions are equivalent:*

(1) $Pv'K = PvK$.
(2) $m(w_I) \cdot v' = m(w_I) \cdot v$.

PROOF. (1) $\Rightarrow$ (2). Since $B_0 w_I B_0$ is a dense open subset of P, it is clear that the product $B_0 w_I B_0 \cdot B_0 v' K$ (respectively $B_0 w_I B_0 \cdot B_0 v K$) is open and dense in $Pv'K$ (respectively PvK). Consequently we see that the sets $B_0 w_I B_0 \cdot B_0 v' K$ and $B_0 w_I B_0 \cdot B_0 v K$ intersect in a dense open subset of $Pv'K = PvK$, which implies that (2) holds.

(2) $\Rightarrow$ (1). Let $v'' = m(w_I) \cdot v'$. Then $B_0 v'' K$ is a dense open subset of $B_0 w_I B_0 \cdot B_0 v' K$, hence a dense open subset of $Pv'K$. Similarly it follows that $B_0 v'' K$ is a dense open subset of PvK. Thus we see that $Pv'K = Pv''K = PvK$.

It is clear that each $(P \times K)$–orbit of G contains a (unique) open dense $(B_0 \times K)$–orbit. Let V_I be the set of all $v \in V$ such that $B_0 v K$ is an open dense subset of PvK. Thus the map $v \mapsto PvK$ is a bijection from V_I to $P \backslash G/K$.

PROPOSITION 3.4.2. *The following conditions on $v \in V$ are equivalent:*

(1) $v \in V_I$.
(2) $m(s_\alpha) \cdot v = v$ *for every* $\alpha \in I$.
(3) $m(w_I) \cdot v = v$.
(4) $v = m(w_I) \cdot v'$ *for some* $v' \in V$.

PROOF. (1) $\Rightarrow$ (2). If (1) holds, then $B_0 v K$ is dense in PvK. If $\alpha \in I$ and $s = s_\alpha$, then $P_s \subset P$, so that $P_s v K \subset PvK \subset \overline{B_0 v K}$. Thus (2) holds.

(2) $\Leftrightarrow$ (3) If (2) holds, then (3) holds, since $m(w_I)$ is a product of terms of the form $m(s_\alpha)$, $\alpha \in I$. Assume (3) holds. If $\alpha \in I$, then $s_\alpha w_I < w_I$, so that $m(s_\alpha) m(w_I) = m(w_I)$. Hence (2) holds.

(3) $\Rightarrow$ (4). Trivial

(4) $\Rightarrow$ (1). If (4) holds, then $B_0 v K$ is dense in $B_0 w_I B_0 \cdot B_0 v' K$, and hence is dense in $Pv'K$, so that $v \in V_I$.

COROLLARY 3.4.3. *For each $v \in V_I$, let $\Gamma(v) = \{ v' \in V \mid m(w_I) \cdot v' = v \}$. Then we have:*

(1) $G = \coprod_{v \in V_I} PvK$.
(2) *For each $v \in V_I$, we have $PvK = \coprod_{v' \in \Gamma(v)} B_0 v' K$.*

We note that Proposition 3.4.2 and Corollary 3.4.3 give a description of the fibres of the map f_I in terms of the action of M on V.

Let $\mathcal{P}_I$ be the variety of all conjugates of the parabolic subgroup P. Then there is an obvious correspondence between the K–orbits on $\mathcal{P}_I$ and the $(P \times K)$–orbits on G. Thus all of the results of this subsection have an interpretation in terms of the K–orbits on $\mathcal{P}_I$.

In a number of concrete examples, one can give quite precise descriptions of the sets V_I and $\Gamma(v)$. See the comments in §5.4.

4. The partial order on the orbits

(See [**RS**, §7].) We define a partial order on the set V of orbits as follows: $v' \leq v$ if and only if $\mathcal{O}(v') \subset \overline{\mathcal{O}(v)}$. We call this the Bruhat order on V because of the analogy with the usual Bruhat order on W. It is also useful to define another partial order on V, the *weak order*, denoted by $\vdash$. If v', $v \in V$, we write $v' \vdash v$ if v belongs to $M \cdot v'$, the M-orbit of v'. This is equivalent to the condition that there exists a reduced decomposition $(\mathbf{v} = (v_0, \ldots, v_k), \mathbf{s})$ of v and an integer j, $0 \leq j \leq k$, such that $v' = v_j$. It is clear that $\vdash$ is a partial order on V and that $v' \vdash v$ implies that $v' \leq v$.

The following result is the key to the combinatorial description of the Bruhat order on V.

PROPOSITION 4.1. *Let $s \in S$ and $v \in V$ and assume that $v \to m(s) \cdot v$. Then*

$$\overline{\mathcal{O}(m(s) \cdot v)} = \bigcup_{v' \leq v} P_s \mathcal{O}(v').$$

The proof of Proposition 4.1 is by an easy induction on $l(v)$.

It will be convenient to translate Proposition 4.1 into a combinatorial framework involving the action of M on V. If $s \in S$ and $v \in V$, we let

$$p(s,v) = \{ v' \in V \mid \mathcal{O}(v') \subset P_s\mathcal{O}(v) \} = \{ v' \in V \mid m(s) \cdot v' = m(s) \cdot v \}.$$

It follows from the case analysis of 2.5 that $p(s,v)$ contains either one, two or three elements. If s is compact imaginary for v, then $p(s,v) = \{v\}$. If s is complex for v, then $p(s,v) = \{v, s \cdot v\}$ and $s \cdot v \neq v$. If s is either real or noncompact imaginary for v, then $p(s,v)$ contains either two or three elements. We observe that $\{v, m(s) \cdot v\} \subset p(s,v)$ and that, if $v' \in p(s,v)$, then $p(s,v) = p(s,v')$.

The following result is a reformulation of Proposition 4.1:

PROPOSITION 4.2. *Let $s \in S$ and $v \in V$ and assume that $v \to m(s) \cdot v$. Then*

$$\{ v' \in V \mid v' \leq m(s) \cdot v \} = \bigcup_{v' \leq v} p(s, v').$$

Thus, for $y \in V$, we have $y \leq m(s) \cdot v$ if and only if there exists $v' \leq v$ such that $m(s) \cdot y = m(s) \cdot v'$.

We note that Proposition 4.2 gives an elementary inductive description of the partial order on V in terms of the M-action on V. In [**RS**], the property of the partial order $\leq$ on V given by Proposition 4.2 is called the "one-step property". It describes how the partial order behaves as we go up one step from v to $m(s) \cdot v$. (The description of the set $p(s,v)$ in [**RS**] is slightly different from the one we have given here.)

The Bruhat order on V has a number of properties which are generalizations of standard properties of the Bruhat order on the Coxeter group W = (W,S). We list below a number of these properties. The proofs are in [**RS**, §5-§7] and involve a considerable amount of combinatorial formalism.

DEFINITION 4.3. Let $(\mathbf{v} = (v_0, \ldots, v_k), \mathbf{s} = (s_1, \ldots, s_k))$ be a reduced decomposition of v. A sequence $\mathbf{u} = (u_0, \ldots, u_k)$ in V is a *subexpression* of $(\mathbf{v}, \mathbf{s})$ if $u_0 = v_0$ and, for every $i = 0, \ldots, k-1$, one of the following three alternatives holds: (a) $u_{i+1} = u_i$; (b) $u_i \to m(s_{i+1}) \cdot u_i = u_{i+1}$; or

$$\text{(c)} \qquad \begin{aligned} u_i \to m(s_{i+1}) \cdot u_i, \qquad & u_{i+1} \to m(s_{i+1}) \cdot u_{i+1} \\ \text{and} \quad m(s_{i+1}) \cdot u_i &= m(s_{i+1}) \cdot u_{i+1}. \end{aligned}$$

If $\mathbf{u} = (u_0, \ldots, u_k)$ is a subexpression of $(\mathbf{v}, \mathbf{s})$, then u_k is the *final term* of $\mathbf{u}$.

In Definition 4.3, if alternative (c) holds and if $u_i \neq u_{i+1}$, then s_{i+1} is noncompact imaginary for u_i and $u_{i+1} = s_{i+1} \cdot u_i$.

PROPOSITION 4.4. *Let v', $v \in V$ and let $(\mathbf{v}, \mathbf{s})$ be a reduced decomposition of v. Then $v' \leq v$ if and only if there is a subexpression $\mathbf{u}$ of $(\mathbf{v}, \mathbf{s})$ with final term v'.*

PROPOSITION 4.5 (The exchange property). *Let $(\mathbf{v}, \mathbf{s} = (s_1, \ldots, s_k))$ be a reduced decomposition of v. Let s, v' be such that $v' \to m(s) \cdot v' = v$. Then there exists $i \in [1,k]$ and a reduced decomposition $(\mathbf{u} = (u_0, \ldots, u_k), \mathbf{s}')$ of v such that $u_{k-1} = v'$ and $\mathbf{s}' = (s_1, \ldots, \widehat{s}_i, \ldots, s_k, s)$.*

PROPOSITION 4.6 (Property $Z(s,u,v)$). *Let u, $v \in V$ and $s \in S$ be such that $u \to m(s) \cdot u$ and $v \to m(s) \cdot v$. Then the following three properties are equivalent:* (i) *either $u \leq v$ or there exists u' with $u' \to m(s) \cdot u' = m(s) \cdot u$ such that $u' \leq v$;* (ii) *$m(s) \cdot u \leq m(s) \cdot v$; and* (iii) *$u \leq m(s) \cdot v$.*

PROPOSITION 4.7. *Let u, $v \in V$ be such that $v \to m(s) \cdot v$ and $u \leq m(s) \cdot v$. Then one of the following three conditions holds:* (i) *$u \leq v$;* (ii) *$u \to m(s) \cdot u$ and there exists u' such that $u' \to m(s) \cdot u' = m(s) \cdot u$ and $u' \leq v$; and* (iii) *there exists $u'' \leq v$ such that $u'' \to m(s) \cdot u'' = u$.*

PROPOSITION 4.8 (The chain condition). *If $u \leq v$, then there exists a sequence $u = u_0 < u_1 < \cdots < u_k = v$ with $l(u_{i+1}) = l(u_i) + 1$ for $i = 0, \ldots, k-1$.*

DEFINITION 4.9. A partial order $\preceq$ on V is *compatible* with the M-action on V if the following three conditions are satisfied for all u, $v \in V$ and $s \in S$: (i) $v \preceq m(s) \cdot v$; (ii) if $u \preceq v$, then $m(s) \cdot u \preceq m(s) \cdot v$; and (iii) if $u \preceq v$ and $l(v) \leq l(u)$, then $u = v$.

PROPOSITION 4.10. *The Bruhat order on V is the weakest partial order on V which is compatible with the M-action on V.*

A direct description of the weakest partial order on V compatible with the M-action on V is given in [**RS**, §5.2].

4.11. The Bruhat order on $\mathcal{I}$. Using the M-action on $\mathcal{I}$ and the length function L on $\mathcal{I}$, we can define precise analogues of all of the earlier results of this section for the set $\mathcal{I}$ of twisted involutions. The definition of a reduced decomposition of an element of $\mathcal{I}$ is essentially the same as that given in 4.3 for reduced decompositions of elements of V. We also have an obvious definition of a partial order $\preceq$ on $\mathcal{I}$ being compatible with the M-action. The Bruhat order on $\mathcal{I}$ is defined to be the weakest partial order on $\mathcal{I}$ which is compatible with the M-action. It is a surprising fact (at least to the authors) that this Bruhat order on $\mathcal{I}$ agrees with the restriction to $\mathcal{I}$ of the usual Bruhat order on W. In [**RS**], it was incorrectly stated that these two partial orders on $\mathcal{I}$ were not necessarily equal. A proof that they are equal appears in [**RS1**]. We refer the reader to [**RS**, §5 and §8] for the precise formulation of analogues for the Bruhat order on $\mathcal{I}$ of 4.3-4.10 above. We will only formulate the appropriate exchange condition:

PROPOSITION 4.11.1 (The exchange condition). *Let $a \in \mathcal{I}$, let $L(a) = k$ and let $\mathbf{s} = (s_1, \dots, s_k)$ be a sequence in S such that $a = s_k \circ s_{k-1} \circ \cdots \circ s_1 \circ 1$. Let $s \in S$ be such that $s \circ a < a$ (or, equivalently, such that $sa < a$). Then there exists $i \in 1, \dots, k$ such that $s \circ a = s_k \circ \cdots \circ \widehat{s_i} \circ \cdots \circ s_1 \circ 1$.*

It seems to be a non-trivial exercise to give a direct proof of 4.11.1.

§5 Combinatorial parametrization of the orbits. The Hermitian symmetric case

It follows from the Bruhat Lemma that the $(B_0 \times B_0)$–orbits on G (or, equivalently, the G–orbits on $\mathcal{B} \times \mathcal{B}$) can be canonically parametrized by the Weyl group W. We would like a similar parametrization for the K–orbits on $\mathcal{B}$. For one class of pairs (G, θ), those of "Hermitian symmetric type", there exists an elementary parametrization which is very satisfactory. We shall describe this parametrization in this section. A detailed exposition will appear in a paper now in preparation by one of us (RWR).

5.1. The hermitian symmetric type. We will assume in §5 that G is simple (over its center) and simply connected. We consider first the case $F = \mathbb{C}$. In this case, G is a simple complex Lie group and has an underlying structure of a simple real Lie group. By a Cartan involution of G we mean a Cartan involution of the underlying simple real Lie group. It is known that there exists a Cartan involution τ of G which commutes with θ. Let $\sigma = \theta\tau = \tau\theta$. Then the fixed point subgroups $U = G^\tau$ and $G_\mathbb{R} = G^\sigma$ are real forms of G and U is a maximal compact subgroup of G. Let $K_\mathbb{R} = U^\theta = U \cap K = (G_\mathbb{R})^\theta = G_\mathbb{R} \cap U$. Then $K_\mathbb{R}$ is a maximal compact subgroup of both K and $G_\mathbb{R}$, and the coset spaces $U/K_\mathbb{R}$ and $G_\mathbb{R}/K_\mathbb{R}$ are dual irreducible Riemannian symmetric spaces. We say that the pair (G, θ) is of Hermitian symmetric type if $G_\mathbb{R}/K_\mathbb{R}$ is a Hermitian symmetric space; this is equivalent to the condition that $U/K_\mathbb{R}$ be a Hermitian symmetric space. It is known that $G_\mathbb{R}/K_\mathbb{R}$ is Hermitian symmetric if and only if the center of $K_\mathbb{R}$ (or of K) has positive dimension (see [**H**, Chap. 8]).

We return to the case where F is algebraically closed of characteristic $\neq 2$.

DEFINITION 5.1.1. Let G be simple and simply connected and let θ be an involutive automorphism of G. Then we say that (G, θ) is of *Hermitian symmetric type* if the center of $G^\theta = K$ has positive dimension.

Roughly speaking, involutions θ of G such that (G, θ) is of Hermitian symmetric type correspond to parabolic subgroups of G with abelian unipotent radical. A precise statement is given in Theorem 5.1.2 below.

In Theorem 5.1.2, G is simple and simply connected and we do not assume that we are given in advance an involution θ and a standard pair (B_0, T_0).

THEOREM 5.1.2. (1) *Let P be a parabolic subgroup of G with abelian unipotent radical. Let B_0 be a Borel subgroup of P and let T_0 be a maximal torus of B_0. Let $\Phi = \Phi(G, T_0)$, let $\Phi^+ = \Phi(T_0, B_0)$ and let $\Delta = \Delta(B_0, T_0)$. Let $\widetilde{\alpha} = \sum_{\alpha \in \Delta} n_\alpha(\widetilde{\alpha})\alpha$ denote the highest root. Let $J \subset \Delta$ be such that $P = P_J$. Then J is equal to $\Delta \setminus \{\gamma\}$, where γ is an element of Δ such that $n_\gamma(\widetilde{\alpha}) = 1$. In particular, P is a maximal (proper) parabolic subgroup of G. Let $c_\gamma \in T_0$ be such that $\gamma(c_\gamma) = -1$ and $\alpha(c_\gamma) = 1$ for $\alpha \in J$ and let $\theta = \mathrm{Int}(c_\gamma)$ denote the inner automorphism induced by c_γ. Then θ is an involutive automorphism of G and $G^\theta = K$ is the unique Levi subgroup of P containing T_0. In particular, K has a center of positive dimension, so that (G, θ) is of Hermitian symmetric type.*

(2) *Let B_0 be a Borel subgroup of G and let T_0 be a maximal torus of B_0. Let Φ, Δ and $\widetilde{\alpha}$ be as in (1) above. Let $\gamma \in \Delta$ be such that $n_\gamma(\widetilde{\alpha}) = 1$ and let $J = \Delta \setminus \{\gamma\}$. Then the standard parabolic subgroup P_J has abelian unipotent radical.*

(3) *Let θ be an involutive automorphism of G such that the center of $K = G^\theta$ has positive dimension. Then there exist P, B_0, T_0, c_γ, ... as in (1) above such that $\theta = \mathrm{Int}(c_\gamma)$ and $K = G^\theta$ is the unique Levi subgroup of P containing T_0.*

See [**RRoSt**] for (1) and (2).

Using the classification of involutions [**S2**], one can show that the classification (over F) of pairs (G, θ) of Hermitian symmetric type corresponds exactly to the classification of simply connected, irreducible Hermitian symmetric spaces of compact type. See [**H**, p. 518] for this classification.

REMARK 5.1.3. Let $\Delta = \{\alpha_1, \dots, \alpha_n\}$ and let the simple roots be indexed as in [**Bou**, Planches I-IX]. We list below all simple roots γ such that $n_\gamma(\widetilde{\alpha}) = 1$. The corresponding (maximal) parabolic subgroups P_J, where $J = \Delta \setminus \{\gamma\}$, give all standard parabolic subgroups with abelian unipotent radical.

(1) Type A_n. $\gamma = \alpha_1, \alpha_2, \dots, \alpha_n$.
(2) Type B_n. $\gamma = \alpha_1$.
(3) Type C_n. $\gamma = \alpha_n$.
(4) Type D_n. $\gamma = \alpha_1, \alpha_{n-1}, \alpha_n$.
(5) Type E_6. $\gamma = \alpha_1, \alpha_6$.
(6) Type E_7. $\gamma = \alpha_7$.

For types E_8, F_4 and G_2, there are no proper parabolic subgroups with abelian unipotent radical.

5.2. The parameter set $\mathcal{E}$. For the rest of §5, we will assume that (G, θ) is of hermitian symmetric type. Let P, B_0, T_0, Δ, $J = \Delta \setminus \{\gamma\}$, ... , be as in Theorem 5.1.2(1). We note that, since θ is an inner automorphism, $\mathcal{I}$ is the set of involutions of W. For each $v \in V$, the involution $\phi(v)$ is an invariant associated to the orbit $\mathcal{O}(v)$. In the case at hand, there is a second natural invariant which one can associate to $\mathcal{O}(v)$. Since $K \subset P$, each $(B_0 \times K)$–orbit of G is contained in a unique $(B_0 \times P)$–orbit and it is known that the set of $(B_0 \times P)$–orbits is canonically parametrized by W/W_J. Thus we obtain a surjective map $\nu : V \to W/W_J$ defined by: $\nu(v) = dW_J$ if $B_0vK \subset B_0dP$. Let $D = D_J = \{d \in W \mid d(J) \subset \Phi^+\}$ be the set of minimal left coset representatives for W/W_J. For $d \in D$, let $V(d) = \{v \in V \mid \nu(v) = dW_J\}$. Thus, for $v \in V$

and $d \in D$, we have $v \in V(d)$ if and only if $B_0 vK \subset B_0 dP$. Consequently, $B_0 dP = \coprod_{v \in V(d)} \mathcal{O}(v)$.

If $sd < d$, then $sd \in D$ and we have

$$P_s B_0 sdP = B_0 sdP \cup B_0 s B_0 sdP = B_0 sdP \cup B_0 dP.$$

The following theorem gives detailed information on the corresponding relationship between the sets of orbits $V(sd)$ and $V(d)$.

THEOREM 5.2.1. *Let $s \in S$ and $d \in D$, with $sd < d$.*

(1) *Let $v \in V(sd)$. Then:* (i) $v \to m(s) \cdot v$ *and* $v \neq s \cdot v$; (ii) $BsB\mathcal{O}(v) = \mathcal{O}(s \cdot v) \cup \mathcal{O}(m(s) \cdot v) \subset BdP$; *and* (iii) $P_s\mathcal{O}(v) = \mathcal{O}(v) \cup \mathcal{O}(s \cdot v) \cup \mathcal{O}(m(s) \cdot v)$. *Furthermore s is either complex for v, in which case $s \cdot v = m(s) \cdot v$, or noncompact imaginary for v, in which case $s \cdot v \neq m(s) \cdot v$.*

(2) *Let v, $v' \in V(sd)$, with $v \neq v'$. Then $P_s\mathcal{O}(v)$ and $P_s\mathcal{O}(v')$ are disjoint.*

(3) $V(d) = s \cdot V(sd) \cup m(s) \cdot V(sd) = \coprod_{v \in V(sd)} \{s \cdot v, m(s) \cdot v\}$.

We now have the following key theorem:

THEOREM 5.2.2. *Define $\eta : V \to \mathcal{I} \times W/W_J$ by $\eta(v) = (\phi(v), \nu(v))$. Then η is injective.*

SKETCH OF PROOF. In order to prove Theorem 5.2.2, it suffices to prove that the restriction of ϕ to each subset $V(d)$, $d \in D$, is injective. The proof is by induction on $l(d)$. Let $sd < d$. In the inductive step of the proof, one needs to analyze the passage from $V(sd)$ to $V(d)$, and the necessary information for doing this is given by Theorem 5.2.1.

It follows from Theorem 5.2.2 that the subset set $\mathcal{E} = \text{image}(\eta)$ of $\mathcal{I} \times W/W_J$ is a parameter set for the set V of orbits. If $d \in D$, let $\mathcal{I}(d) = \{ a \in \mathcal{I} \mid (a, d) \in \mathcal{E} \}$. Thus $\mathcal{E} = \{(a, d) \in \mathcal{I} \times W/W_J \mid a \in \mathcal{I}(d)\}$. Note that $1 \in D$. Since $B_0 K = B_0 P = P$, it is easy to see that $\mathcal{I}(1) = \{1\}$. The following lemma gives an elementary inductive description of the sets $\mathcal{I}(d)$, and hence of the parameter set $\mathcal{E}$.

LEMMA 5.2.3. *Let $d \in D$ and $s \in S$ be such that $sd < d$. Then*

$$\mathcal{I}(d) = s * \mathcal{I}(sd) \cup m(s) * \mathcal{I}(sd).$$

The proof follows from 1.4.4, 3.3.3 and 5.2.1(3).

Next we describe the image of η. Let Ψ denote $\Phi(T_0, R_u(P))$, the support of $R_u(P)$.

THEOREM 5.2.4. *Let $(a, dW_J) \in \mathcal{I} \times W/W_J$, with $d \in D$. Then $(a, dW_J) \in$ image(η) if and only if there exists a sequence $(\beta_1, \dots, \beta_k)$ of mutually orthogonal roots in $\Phi^+ \cap d(-\Psi)$ such that $a = s_{\beta_1} s_{\beta_2} \cdots s_{\beta_k}$.*

Briefly, the proof goes as follows. For each $d \in D$, let $\mathcal{A}(d)$ be the set of all $a \in \mathcal{I}$ such that $a = s_{\beta_1} s_{\beta_2} \cdots s_{\beta_k}$, where $(\beta_1, \dots, \beta_k)$ is a sequence of mutually orthogonal roots in $\Phi^+ \cap d(-\Psi)$. Let $sd < d$. Then one proves that

$$\mathcal{A}(d) = s * \mathcal{A}(sd) \cup m(s) * \mathcal{A}(sd).$$

Since it is clear that $\mathcal{A}(1) = \{1\}$, it follows from 5.2.3 that $\mathcal{A}(d) = \mathcal{I}(d)$, which proves 5.2.4.

5.3. An example. Let $G = SL(n, F)$ and let $\gamma = \alpha_k$ (we follow the notation of Remark 5.1.3). In this example, the coset space G/P is the Grassmann variety of k-planes in F^n and

$$K = \{ (g, h) \in GL(k, F) \times GL(n-k, F) \mid \det(g)\det(h) = 1 \},$$

where we embed $GL(k, F) \times GL(n-k, F)$ in $GL(n, F)$ in the obvious way. Assume that $k \le n - k$.

We will give an elementary description of the parameter set $\mathcal{E}$.

We identify the Weyl group with the symmetric group S_n. Then W_J is the stabilizer of $[1, k]$ in S_n. Each involution a in S_n is a product of disjoint two-cycles:

$$a = (i_1, j_1)(i_2, j_2) \cdots (i_r, j_r) \quad \text{with} \quad i_p < j_p, \quad p = 1, \dots, r.$$

We set $\text{Hi}(a) = \{j_1, \dots, j_r\} = \{j \in [1, n] \mid a(j) < j\}$ and $\text{Lo}(a) = \{i_1, \dots, i_r\} = \{i \in [1, n] \mid a(i) > i\}$; we say that Hi$(a)$ (resp. Lo(a)) is the set of *high points* (resp. *low points*) of a. Let $\Sigma = \Sigma(k, n)$ denote the set of all k-element subsets of $[1, n]$. If $w \in W$, let $\underline{w} \in \Sigma$ denote $\{w(1), \dots, w(k)\}$. Then the map $w \to \underline{w}$, $W \to \Sigma$, is constant on left coset of W_J and induces a bijection from W/W_J to Σ.

The set D of minimal left coset representatives is given by

$$D = \{ d \in W \mid d(1) < d(2) \cdots < d(k) \text{ and } d(k+1) < d(k+2) \cdots < d(n) \}.$$

We have $\Sigma = \{ \underline{d} \mid d \in D \}$. The subset Ψ (the support of $R_u(P)$) is equal to $\{ \varepsilon_i - \varepsilon_j \mid 1 \le i \le k < j \le n \}$. (We follow the notation of [**Bou**, Planche I].) Let $d \in D$. Then an easy argument shows that

$$\Phi^+ \cap d(-\Psi) = \{ \varepsilon_i - \varepsilon_j \mid i < j,\ j \in \underline{d} \text{ and } i \notin \underline{d} \}.$$

It follows from this that

$$\mathcal{I}(d) = \{ a \in \mathcal{I} \mid \text{Hi}(a) \subset \underline{d} \text{ and } \text{Lo}(a) \cap \underline{d} = \emptyset \}.$$

Thus the set

$$\mathcal{M} = \{ (a, \underline{d}) \in \mathcal{I} \times \Sigma \mid \text{Hi}(a) \subset \underline{d} \text{ and } \text{Lo}(a) \cap \underline{d} = \emptyset \}$$

is a parameter set for the set V of orbits.

We discuss in more detail the case $n = 4$, $k = 2$. In this case, G/P is the set of 2-planes in F^4 and Σ is the set of two-element subsets of $[1, 4]$. For each $\underline{d} \in \Sigma$, we list below the involutions in $\mathcal{I}(d)$.

$\underline{d}$	$\mathcal{I}(d)$
$\{1,2\}$	1
$\{1,3\}$	$1,\ (2,3)$
$\{1,4\}$	$1,\ (2,4),\ (3,4)$
$\{2,3\}$	$1,\ (1,2),\ (1,3)$
$\{2,4\}$	$1,\ (1,2),\ (1,4),\ (3,4),\ (1,2)(3,4)$
$\{3,4\}$	$1,\ (1,3),\ (1,4),\ (2,3),\ (2,4),\ (1,3)(2,4),\ (1,4)(2,3)$

Thus we see that there are 21 orbits. Note that this list agrees with the list in [**C**].

These examples for type A_n were worked out by P. D. Ryan [**Ry**] in his M.Sc. thesis at the Australian National University. Ryan also gives similar concrete models of the parameter sets for the other pairs (G, θ) of Hermitian symmetric type whenever G is a classical group (i.e., for the examples of type B_n, C_n and D_n in Remark 5.1.3).

5.4. The actions of M and W on the parameter set $\mathcal{E}$. We define actions of M and W on the parameter set $\mathcal{E} = \text{image}(\eta)$ by requiring that $\eta : V \to \mathcal{E}$ be an isomorphism of M-sets and of W-sets. Thus, if $w \in W$ and $v \in V$, then we have $\eta(m(w) \cdot v) = m(w) \cdot \eta(v)$ and $\eta(w \cdot v) = w \cdot \eta(v)$.

In order to describe the action of M on $\mathcal{E}$, we first need to define an action of M on the set W/W_J. Let $s \in S$ and let $d \in D$. Then we define $m(s) \cdot dW_J$ by the following rules: (i) if $sd < d$, then $m(s) \cdot dW_J = dW_J$; (ii) if $sd > d$, then $m(s) \cdot dW_J = sdW_J$. This determines an action of M on W/W_J.

REMARK. In the definition above, if $sd > d$, then $sd \in D$ if and only if $sdW_J \neq dW_J$. If $sdW_J = dW_J$, then we have $m(s) \cdot dW_J = dW_J$.

The following theorem determines the actions of M and W on $\mathcal{E}$:

THEOREM 5.4.1. *Let $s \in S$ and $(a, dW_J) \in \mathcal{E}$.*

(1) *$m(s) \cdot (a, dW_J)$ is equal to $(m(s) * a, m(s) \cdot dW_J)$ if $(m(s) * a, m(s) \cdot dW_J) \in \mathcal{E}$ and is equal to (a, dW_J) if $(m(s) * a, m(s) \cdot dW_J) \notin \mathcal{E}$. The latter case can only occur if s is imaginary for a and $sdW_J = dW_J$.*

(2) *$s \cdot (a, dW_J)$ is equal to (sas, sdW_J) if $(sas, sdW_J) \in \mathcal{E}$ and is equal to (a, dW_J) if $(sas, sdW_J) \notin \mathcal{E}$. The latter case occurs if and only if s is real for a and $sd < d$.*

5.5. The opposite parabolic P^-. Let P^- be the unique parabolic subgroup opposite to P which contains T_0. Then $K = P \cap P^-$. If $J' = -w_0(J)$, then $P^- = w_0 P_{J'} w_0$. In particular, P^- is conjugate to P if and only if $J = J'$.

LEMMA 5.5.1. (1) *The map $c \to BcP^-$ is a bijection from D to $B_0\backslash G/P^-$.* (2) *Let c, $d \in D$. Then $BdP^- \subset \overline{BcP^-}$ if and only if $c \le d$.* (3) *Let c, $d \in D$. Then $BdP \cap BcP^- \neq \emptyset$ if and only if $c \le d$. If $c \le d$, then $BdP \cap BcP^-$ is a smooth irreducible variety of dimension $l(d) - l(c) + \dim P$.*

PROOF. The proof of (1) follows from [**RRoSt**, §5]. The proof of (2) and (3) follows from [**R2**, §3] (see, in particular, 3.6, 3.7 and 3.9.1).

Note the reversal of order in (2) of Lemma 5.6.1.

Since $K \subset P^-$, every $(B_0 \times K)$-orbit is contained in a unique $(B_0 \times P^-)$-orbit. It follows from Lemma 5.6.1 that the $(B_0 \times P^-)$-orbits are also parametrized by W/W_J. We define a map $\nu' : V \to W/W_J$ as follows: $\nu'(v) = cW_J$ if $B_0 vK \subset B_0 cP^-$. We let $\rho : V \to \mathcal{I} \times W/W_J \times W/W_J$ denote the map $v \mapsto (\phi(v), \nu(v), \nu'(v))$. We let $\mathcal{E}^*$ denote the subset image(ρ) of $\mathcal{I} \times W/W_J \times W/W_J$. Thus ρ maps V bijectively onto $\mathcal{E}^*$ and we may also use $\mathcal{E}^*$ as a parameter set for the orbits.

The relation between the maps ν and ν' is given by the following lemma:

LEMMA 5.5.2. *Let $v \in V$ and let $\rho(v) = (a, dW_J, cW_J)$. Then $dW_J = acW_J = m(a) \cdot cW_J$.*

5.6 Partial order on the orbits. The following lemma is easy to prove:

LEMMA 5.6.1. *Let v', $v \in V$, let $\rho(v') = (a', d'W_J, c'W_J)$ and let $\rho(v) = (a, dW_J, cW_J)$, with c', d', c, $d \in D$. If $v' \le v$, then $a' \le a$ and $c \le c' \le d' \le d$.*

It has been conjectured by P. D. Ryan that the converse of Lemma 5.6.1 holds.

CONJECTURE 5.6.2 (P. D. Ryan). *Let v', $v \in V$, let $\rho(v') = (a', d'W_J, c'W_J)$ and let $\rho(v) = (a, dW_J, cW_J)$, with c', d', c, $d \in D$. Then $v' \le v$ if and only if $a' \le a$ and $c \le c' \le d' \le d$.*

This conjecture is implicit in Ryan's M.Sc. thesis [**Ry**], although it is not explicitly stated there. In loc. cit., Ryan gives considerable evidence for this conjecture. We (PDR and RWR) now have a proof of the following slightly weaker result:

THEOREM 5.6.3. *Let the notation be as in Conjecture 5.6.2 and assume further that $l(v) = l(v') + 1$. Then $v' \le v$ if and only if $a' \le a$ and $c \le c' \le d' \le d$.*

Since the partial order on V satisfies the chain condition (see Proposition 4.8), we see that Theorem 5.6.3 gives an elementary description of the partial order on the set V of orbits in terms of the parameter set $\mathcal{E}^*$.

If $d \in D$, it is not difficult to see that $B_0 dK$ is a closed subset of G. Let $v(d)$ be the corresponding element of V, so that $B_0 v(d) K = B_0 dK$. Then V_0, the set of closed orbits, is equal to $\{\, v(d) \mid d \in D \,\}$ and $v(d) \neq v(d')$ if $d \neq d'$. If $v \in V$, let $\mathcal{C}_0(v) = \{\, v_0 \in V_0 \mid v_0 \le v \,\}$. We can also prove the following special case of Ryan's conjecture (which is not covered by Theorem 5.6.3).

THEOREM 5.6.4. *Let $v \in V$ and let $\rho(v) = (a, dW_J, cW_J)$, with c, $d \in D$. Then $\mathcal{C}_0(v) = \{\, v(d') \mid c \le d' \le d \,\}$.*

Here is an equivalent form of Ryan's conjecture:

CONJECTURE 5.6.5. *Let v', $v \in V$, let $a' = \phi(v)$ and let $a = \phi(v)$. Then $v' \leq v$ if and only if $a' \leq a$ and $\mathcal{C}_0(v') \subset \mathcal{C}_0(v)$.*

§6. Orbits of real forms of G and duality of orbits

In §6 the base field F will be the field $\mathbb{C}$ of complex numbers. Other than this, the notation is as in §1–§4. We will have occasion to consider two distinct topologies on algebraic varieties and their subsets, the classical (Hausdorff) topology and the Zariski topology. Unless explicitly indicated otherwise, in §6 all references to topological concepts will refer to the classical topology. Thus a closed set is closed in the classical topology and a Zariski-closed subset is closed in the Zariski topology. By a torus, we mean a compact Lie group isomorphic to a product of circle groups, as opposed to an algebraic torus, which is an algebraic group isomorphic to a product of multiplicative groups $\mathbb{C}^*$.

Let τ be a Cartan involution of (the underlying real Lie group of) G which commutes with θ. Let $\sigma = \theta\tau = \tau\theta$ and let $G_{\mathbb{R}} = G^\sigma$ and $U = G^\tau$ be the fixed point subgroups. Then U and $G_{\mathbb{R}}$ are real forms of G and U is a maximal compact subgroup of G. We wish to study the orbits of the real Lie group $G_{\mathbb{R}}$ on the flag manifold $\mathcal{B}$ or, equivalently, the $(B_0 \times G_{\mathbb{R}})$–orbits on G. In general, the $G_{\mathbb{R}}$–orbits on $\mathcal{B}$ will not be complex submanifolds of $\mathcal{B}$. We also study orbits of $G_{\mathbb{R}}$ on generalized flag manifolds $\mathcal{P}_I \cong G/P_I$. Matsuki has shown that there is a natural duality between K–orbits and $G_{\mathbb{R}}$–orbits on $\mathcal{B}$ (or on $\mathcal{P}_I$) which reverses the natural partial order on these orbits. It turns out that all of the machinery of the paper **[RS]** applies to the $G_{\mathbb{R}}$ orbits in a natural way. In §6, we will review (and sometimes reformulate) a number of Matsuki's results and indicate how to apply the methods of [**RS**] to these problems. We will only sketch the results here. A detailed exposition will appear in a paper by one of us (RWR).

6.1. Duality of orbits. Let $K_{\mathbb{R}} = G_{\mathbb{R}} \cap K$. Then $K_{\mathbb{R}}$ is a maximal compact subgroup of both $G_{\mathbb{R}}$ and K. In particular every connected component of $G_{\mathbb{R}}$ and of K meets $K_{\mathbb{R}}$. Let $\Gamma = \{\mathrm{Id}_G, \theta, \tau, \sigma\}$. Then Γ is a subgroup of the group of all Lie group automorphisms of G. It is clear that $K_{\mathbb{R}} = G^\Gamma$. We note that Γ acts on $\mathcal{T}$ and $\mathcal{B}$; recall that $\mathcal{T}$ denotes the set of all maximal algebraic tori of G. We let $\mathcal{T}^\Gamma$ denote the set of all Γ–stable maximal algebraic tori.

PROPOSITION 6.1.1. *Each K-orbit on $\mathcal{T}^\theta$ meets $\mathcal{T}^\Gamma$ in a unique $K_{\mathbb{R}}$-orbit and each $G_{\mathbb{R}}$-orbit on $\mathcal{T}^\sigma$ meets $\mathcal{T}^\Gamma$ in a unique $K_{\mathbb{R}}$-orbit. Thus we have canonical bijections between the three following sets of orbits*: (i) $K\backslash\mathcal{T}^\theta$; (ii) $G_{\mathbb{R}}\backslash\mathcal{T}^\sigma$; *and* (iii) $K_{\mathbb{R}}\backslash\mathcal{T}^\Gamma$.

Note that Proposition 6.1.1 also gives a bijective correspondence between the following sets: (i) $G_{\mathbb{R}}$ conjugacy classes of Cartan subalgebras of $\mathrm{Lie}(G_{\mathbb{R}})$; (ii) K conjugacy classes of θ–stable Cartan subalgebras of $\mathrm{Lie}(G)$; and (iii) $K_{\mathbb{R}}$ conjugacy classes of θ-stable Cartan subalgebras of $\mathrm{Lie}(G_{\mathbb{R}})$.

For the rest of §6, we assume that the standard pair (B_0, T_0) of 1.2 is chosen such that $T_0 \in \mathcal{T}^\Gamma$. Let $C = U \cap T_0$. Then C is the unique maximum compact subgroup of T_0 and is a maximal torus of the compact connected Lie group U. Let $W_K(T_0)$ (respectively $W_{G_{\mathbb{R}}}(T_0)$), denote the image of $N_K(T_0)$ (respectively $N_{G_{\mathbb{R}}}(T_0)$) in $W = W(T_0)$. Similarly, let $W_{K_{\mathbb{R}}}(C)$ denote the image of $N_{K_{\mathbb{R}}}(C)$ in $W(C) = N_U(C)/C$.

LEMMA 6.1.2. (1) *The inclusion map $U \hookrightarrow G$ maps $N_U(C)$ into $N_G(T_0)$ and induces an isomorphism of $W(C)$ onto $W(T_0)$.* (2) *The inclusion map $K_{\mathbb{R}} \hookrightarrow K$ induces an isomorphism of $W_{K_{\mathbb{R}}}(C)$ onto $W_K(T_0)$.* (3) *The inclusion map $K_{\mathbb{R}} \hookrightarrow G_{\mathbb{R}}$ induces an isomorphism of $W_{K_{\mathbb{R}}}(C)$ onto $W_{G_{\mathbb{R}}}(T_0)$.*

COROLLARY 6.1.3. $W_K(T_0) = W_{G_{\mathbb{R}}}(T_0)$.

We define subsets $\mathcal{V}_\theta$, $\mathcal{V}_\sigma$ and $\mathcal{V}_\Gamma$ of G by:

(6-1) $\mathcal{V}_\theta = \{ g \in G \mid g\theta(g^{-1}) \in N_G(T_0) \}$.

(6-2) $\mathcal{V}_\sigma = \{ g \in G \mid g\sigma(g^{-1}) \in N_G(T_0) \}$.

(6-3) $\mathcal{V}_\Gamma = \{ u \in U \mid u\theta(u^{-1}) \in N_U(C) \}$.

We note that $g \in \mathcal{V}_\sigma$ if and only if the maximal algebraic torus ${}^{g^{-1}}T_0$ is σ–stable. Similarly, if $u \in U$, then $u \in \mathcal{V}_\Gamma$ if and only if the maximal torus ${}^{u^{-1}}C$ of U is θ–stable. Note also that $\mathcal{V}_\Gamma = \mathcal{V}_\theta \cap U = \mathcal{V}_\sigma \cap U$. We observe that $\mathcal{V}_\Gamma$ (respectively $\mathcal{V}_\sigma$) is stable under the action of $C \times K_{\mathbb{R}}$ (respectively $T_0 \times G_{\mathbb{R}}$). We have the following orbit sets:

(6-4) $V_\Gamma = C\backslash\mathcal{V}_\Gamma/K_{\mathbb{R}}$.

(6-5) $V_\theta = T_o\backslash\mathcal{V}_\theta/K$.

(6-6) $V_\sigma = T_0\backslash\mathcal{V}_\sigma/G_{\mathbb{R}}$.

REMARK. In the earlier sections, the set $\mathcal{V}_\theta$ was denoted by $\mathcal{V}$ and V_θ was denoted by V.

THEOREM 6.1.4. (1) *Each $(T_0 \times K)$–orbit on $\mathcal{V}_\theta$ meets $\mathcal{V}_\Gamma$ in a unique $(C \times K_{\mathbb{R}})$–orbit. Thus the inclusion $\mathcal{V}_\Gamma \hookrightarrow \mathcal{V}_\theta$ induces a bijection of orbit sets $V_\Gamma \to V_\theta$.* (2) *Each $(T_0 \times G_{\mathbb{R}})$–orbit on $\mathcal{V}_\sigma$ meets $\mathcal{V}_\Gamma$ in a unique $K_{\mathbb{R}}$–orbit. Consequently the inclusion map $\mathcal{V}_\Gamma \hookrightarrow \mathcal{V}_\sigma$ induces a bijection from V_Γ to V_σ.* (3) *The inclusion $\mathcal{V}_\Gamma \hookrightarrow G$ induces bijections from V_Γ onto $B_0\backslash G/K$ and $B_0\backslash G/G_{\mathbb{R}}$.*

It follows from Theorem 6.1.4 that the set V_Γ naturally parametrizes the following sets of orbits: (a) $B_0\backslash G/K$; (b) $B_0\backslash G/G_{\mathbb{R}}$; (c) $K\backslash\mathcal{B}$; and (d) $G_{\mathbb{R}}\backslash\mathcal{B}$.

If $v \in V_\Gamma$, we let $\mathcal{O}(v) = B_0vK$ and let $\mathcal{R}(v) = B_0vG_{\mathbb{R}}$. We let $\mathcal{K}(v)$ (respectively $\mathcal{G}(v)$) denote the K–orbit on X (respectively the $G_{\mathbb{R}}$–orbit on X) corresponding to v. Thus, if $v = CuK_0 \in V_\Gamma$ and if $x = u^{-1} \cdot x_0$ (where $x_0 \in X$ corresponds to $B_0 \in \mathcal{B}$), then $\mathcal{K}(v) = K \cdot x$ and $\mathcal{G}(v) = G_{\mathbb{R}} \cdot x$. We have:

(6-7) $B_0\backslash G/K = \{ \mathcal{O}(v) \mid v \in V_\Gamma \}$.

(6-8) $B_0\backslash G/G_{\mathbb{R}} = \{ \mathcal{R}(v) \mid v \in V_\Gamma \}$.

(6-9) $K\backslash X = \{ \mathcal{K}(v) \mid v \in V_\Gamma \}$.

(6-10) $G_{\mathbb{R}}\backslash X = \{ \mathcal{G}(v) \mid v \in V_\Gamma \}$.

It is well-known that U acts transitively on $\mathcal{B}$ and that $U \cap B_0 = U \cap T_0 = C$. It follows easily that, for every $B \in \mathcal{B}$, the intersection $U \cap B$ is a maximal torus of U. We also note that $\sigma(B_0) = \tau(B_0) = {}^{w_0}B_0$. The intersection $B \cap \tau(B)$ is a (τ-stable) maximal algebraic torus of G and $B \cap U$ is the unique maximal compact subgroup of $B \cap \tau(B)$. We let $\mathcal{B}^*$ denote the set of all $B \in \mathcal{B}$ such that $B \cap U$ is θ-stable, or, equivalently, such that $B \cap \tau(B)$ is θ-stable. We observe further that $B \in \mathcal{B}^*$ if and only if $B \cap \theta(B) \cap \tau(B) \cap \sigma(B)$ is of maximal rank.

THEOREM 6.1.5. (1) *Each K-orbit on $\mathcal{B}$ meets $\mathcal{B}^*$ in a unique $K_{\mathbb{R}}$-orbit. Thus the inclusion map $\mathcal{B}^* \hookrightarrow \mathcal{B}$ induces a bijection of orbit sets $K_{\mathbb{R}}\backslash\mathcal{B}^* \to K\backslash\mathcal{B}$.* (2) *Each $G_{\mathbb{R}}$-orbit on $\mathcal{B}$ meets $\mathcal{B}^*$ in a unique $K_{\mathbb{R}}$-orbit, so that the inclusion $\mathcal{B}^* \hookrightarrow \mathcal{B}$ induces a bijection $K_{\mathbb{R}}\backslash\mathcal{B}^* \to K\backslash\mathcal{B}$.*

The proof follows easily from Theorem 6.1.4.

We let $X^* = \{x \in X \mid B_x \in \mathcal{B}^*\}$. Then X^* is a $K_{\mathbb{R}}$-stable, closed differentiable submanifold of the projective variety X.

PROPOSITION 6.1.6. *Let $x \in X^*$. Then $G_{\mathbb{R}} \cdot x \cap K \cdot x = K_{\mathbb{R}} \cdot x$.*

DEFINITION 6.1.7. (1) Let x, $y \in X$. Then the orbits $K \cdot x$ and $G_{\mathbb{R}} \cdot y$ are *dual orbits* if $K \cdot x \cap G_{\mathbb{R}} \cdot y$ meets X^*. If this happens, then there exists $z \in X^*$ such that $K \cdot x \cap G_{\mathbb{R}} \cdot y = K_{\mathbb{R}} \cdot z$. (2) Let g, $g' \in G$. Then the $(B_0 \times K)$-orbit B_0gK and the $(B_0 \times G_{\mathbb{R}})$-orbit $B_0g'G_{\mathbb{R}}$ are *dual orbits* if there exists $h \in G$ such that $B_0gK \cap B_0g'G_{\mathbb{R}} = B_0hK_{\mathbb{R}}$.

PROPOSITION 6.1.8. *Let v, $v' \in V_\Gamma$. Then the orbits $\mathcal{K}(v')$ and $\mathcal{G}(v)$ (respectively $\mathcal{O}(v')$ and $\mathcal{R}(v)$) are dual orbits if and only if $v = v'$.*

6.2. The map ϕ and the action of M and W on the $G_{\mathbb{R}}$-orbits. For the moment, we let f denote the bijection $V_\Gamma \to V_\theta$ given by Theorem 6.1.4. We use the bijection f to transfer the actions of M and W on V_θ to actions on V_Γ. Similarly, we may consider ϕ as a map from V_Γ to $\mathcal{I}$. Thus let $v \in V_\Gamma$, let $v_1 = f(v)$ and let $w \in W$. Then we set $\phi(v) = \phi(v_1)$, $w \cdot v = f^{-1}(w \cdot v_1)$ and $m(w) \cdot v = f^{-1}(m(w) \cdot v_1)$. Since we also have a bijection from V_Γ to V_σ, and since V_σ parametrizes the orbit sets $B_0\backslash G/G_{\mathbb{R}}$ and $G_{\mathbb{R}}\backslash\mathcal{B}$, we obtain actions of W and M on these orbit sets; we may also consider ϕ as a map from these orbits sets.

For $w \in W$, we let $\overline{w} \in \mathcal{W}$ denote $\eta_0(w)$, where $\eta_0 = \eta_{B_0,T_0} : W \to \mathcal{W}$ is as in Remark 1.7.

The following proposition gives a geometric interpretation of the map ϕ in terms of the action of σ on $\mathcal{B}$.

PROPOSITION 6.2.1. *Let $v \in V_\Gamma$, let $a = \phi(v)$ and let $x \in \mathcal{G}(v)$ (so that $\mathcal{G}(v) = G_{\mathbb{R}} \cdot x$). Then $\overline{a}\,\overline{w}_0 = \eta_0(aw_0)$ is the relative position of $(B_x, \sigma(B_x))$.*

We define a partial order on V_Γ by requiring that the bijection $V_\Gamma \to V$ be an isomorphism of posets.

6.3. Case analysis for the $(B_0 \times G_{\mathbb{R}})$-orbits on $P_svG_{\mathbb{R}}$. We follow the notation of the earlier sections as regards real, complex and imaginary roots.

Thus, let $v \in V_\Gamma$, let $v_1 = f(v)$ and let $s \in S$. Then we say that s is real (respectively compact imaginary, ...) for v if s is real (respectively compact imaginary, ...) for v_1.

Let $s \in S$ and $v \in V_\Gamma$. We can carry out an analysis of the $(B_0 \times G_{\mathbb{R}})$-orbits on $P_svG_{\mathbb{R}}$ as in §2. For the $(B_0 \times G_{\mathbb{R}})$-orbits, the analysis of the corresponding orbits on $\mathbb{P}^1$ is a bit different from that of §2 and we will discuss it first. Let $Y = \mathcal{P}_s$, let $y \in Y$ and let $P = P_y$. Then $G_{\mathbb{R}} \cap P = (P \cap \sigma(P))^\sigma$. As before, let $X_y = \mathcal{B}(P) \cong \mathbb{P}^1$. Let H_y denote the image of $G_{\mathbb{R}} \cap P$ in $A_y = \mathrm{Aut}(X_y)$; note that H_y is a (real) Lie subgroup of A_y. There are essentially three cases:

Case 1. s is complex for v. In this case, there are two H_y-orbits on X_y, one of which is a fixed point.

Case 2. s is compact imaginary for v. In this case H_y is a maximum compact subgroup of A_y and acts transitively on X_y.

Case 3. s is either real or non-compact imaginary for v. Then H_y^0, the identity component of H_y, is isomorphic to $PSL(2,\mathbb{R})$. There are three H_y^0-orbits on X_y, the "equator", the "upper hemisphere" and the "lower hemisphere". The equator is a circle and the two hemispheres are open orbits. If H_y is not connected, then it permutes the two hemispheres, and there are only two H_y orbits.

We have the following analysis for the $(B_0 \times G_{\mathbb{R}})$-orbits on $P_svG_{\mathbb{R}}$.

Case A. s is complex for v. Then $P_svG_{\mathbb{R}} = \mathcal{R}(v) \cup \mathcal{R}(s \cdot v)$ and $s \cdot v \neq v$. If $sa > a$ (respectively $sa < a$), then $\mathcal{R}(v)$ (respectively $\mathcal{R}(s \cdot v)$) is open and dense in $P_svG_{\mathbb{R}}$ and $\mathcal{R}(s \cdot v)$ (respectively $\mathcal{R}(v)$) is closed of codimension two in $P_svG_{\mathbb{R}}$.

Case B. s is compact imaginary for v. In this case there is only one orbit, so that $P_svG_{\mathbb{R}} = \mathcal{R}(v)$.

Case C. s is non-compact imaginary for v. Then there exists $v' \in V_\Gamma$ such that $\mathcal{R}(v')$ is closed and of codimension one in $P_svG_{\mathbb{R}}$. We have $P_svG_{\mathbb{R}} = \mathcal{R}(v') \cup \mathcal{R}(v) \cup \mathcal{R}(s \cdot v)$. The orbits $\mathcal{R}(v)$ and $\mathcal{R}(s \cdot v)$ are open in $P_svG_{\mathbb{R}}$. There are either two or three orbits, depending on whether or not $v = s \cdot v$.

Case D. s is real for v. The orbit $\mathcal{R}(v)$ is closed of codimension one in $P_svG_{\mathbb{R}}$. There exists $v' \in V_\Gamma$ such that $P_svG_{\mathbb{R}} = \mathcal{R}(v) \cup \mathcal{R}(v') \cup \mathcal{R}(s \cdot v')$. The orbits $\mathcal{R}(v')$ and $\mathcal{R}(s \cdot v')$ are open in $P_svG_{\mathbb{R}}$ and there are either two or three orbits, depending on whether or not $s \cdot v' = v'$.

We note that the case analysis for the $(B_0 \times G_{\mathbb{R}})$-orbits on $P_svG_{\mathbb{R}}$ is a bit different from the corresponding analysis for the $B_0 \times K$-orbits. In the first place, all of the closure relations get reversed. Furthermore, the dimensions of the orbits behave somewhat differently.

6.4. Further results. Using the case analysis of 6.3, we can now prove a number of theorems for the $(B_0 \times G_{\mathbb{R}})$-orbits on G or, equivalently, for the $G_{\mathbb{R}}$-orbits on X. We can also obtain a number of results relating to the duality of orbits. We list below some of these results. In the results below, all dimensions referred to will be the real dimensions.

LEMMA 6.4.1. *Let* $v \in V_\Gamma$ *and* $s \in S$. *Then* $\mathcal{R}(m(s) \cdot v)$ *is the unique closed* $(B_0 \times G_\mathbb{R})$*-orbit in* $P_s v G_\mathbb{R}$.

THEOREM 6.4.2. *Let* $v \in V_\Gamma$ *and let* $a = \phi(v)$. *Then the codimension of* $\mathcal{R}(v)$ *in* G *is equal to* $l(a)$. *Equivalently, the codimension of* $\mathcal{G}(v)$ *in* X *is equal to* $l(a)$.

COROLLARY 6.4.3. *Let* $x \in X$ *and let* $\bar{b} \in \mathcal{W}$ *be the relative position of* $(B_x, \sigma(B_x))$. *Then* $\dim G_\mathbb{R} \cdot x = |\Phi^+| + l(\bar{b})$.

Note that the dimension of the orbit $G_\mathbb{R} \cdot x$ is explicitly determined by the relative position of $(B_x, \sigma(B_x))$.

THEOREM 6.4.4. *Let* v', $v \in V_\Gamma$, *let* $a' = \phi(v')$ *and let* $a = \phi(v)$. *Then the orbits* $\mathcal{G}(v')$ *and* $\mathcal{K}(v)$ *intersect transversally in* X. *In particular, each connected component of* $\mathcal{G}(v') \cap \mathcal{K}(v)$ *is a smooth locally closed submanifold of* X *whose dimension is* $2L(a) + \dim \mathcal{B}(K^0) - l(a')$.

THEOREM 6.4.5. *Let* v', $v \in V_\Gamma$. *Then the following conditions are equivalent*:

(1) $v' \leq v$.
(2) $\mathcal{K}(v') \subset \overline{\mathcal{K}(v)}$.
(3) $\mathcal{G}(v) \subset \overline{\mathcal{G}(v')}$.
(4) $\mathcal{G}(v') \cap \mathcal{K}(v) \neq \emptyset$.

THEOREM 6.4.6. *Let* v', $v \in V_\Gamma$ *with* $v' \leq v$. *Then*

$$\begin{aligned}\overline{\mathcal{G}(v') \cap \mathcal{K}(v)} &= \overline{\mathcal{G}(v')} \cap \overline{\mathcal{K}(v)} \\ &= \coprod_{v' \leq v_2 \leq v_1 \leq v} \mathcal{G}(v_2) \cap \mathcal{K}(v_1).\end{aligned}$$

All of the results of §4 concerning the partial order on the $(B_0 \times K)$–orbits of G (or the K–orbits of X) apply equally well to the $(B_0 \times G_\mathbb{R})$–orbits of G (or the $G_\mathbb{R}$–orbits of X). However all of the inclusion relations are reversed. In this case the length function on the set V_Γ (carried over from the length function on V_θ via the bijection $V_\Gamma \to V_\theta$) does not relate quite so directly to the dimension of the corresponding orbit $\mathcal{R}(v)$ (or $\mathcal{G}(v)$). We give below the analogue for $(B_0 \times G_\mathbb{R})$–orbits of Proposition 4.1. First we need more notation. If $E \subset G$, we set $E^\frown = G \setminus (\overline{E} \setminus E)$.

PROPOSITION 6.4.7. *Let* $s \in S$ *and* $v \in V_\Gamma$ *and assume that* $v \to m(s) \cdot v$. *Then*

$$\mathcal{R}(m(s) \cdot v)^\frown = \bigcup_{v' \leq v} P_s \mathcal{R}(v').$$

6.5. The Hermitian symmetric case. Assume that (G, θ) is of Hermitian symmetric type, as defined in §5. This is equivalent to the condition that $G_\mathbb{R}/K_\mathbb{R}$ be a Hermitian symmetric space. The combinatorial classification of $(B_0 \times K)$–orbits on G given in §5 also gives a combinatorial classification of the $(B_0 \times G_\mathbb{R})$–orbits on G via the duality between these two sets of orbits. Note that, by Theorem 6.3.2, one can easily read off the dimensions of the $(B_0 \times G_\mathbb{R})$–orbits from the parameter set $\mathcal{E}$. Note also that we have a complete description of the closure relations between the $G_\mathbb{R}$–orbits in terms of $\mathcal{E}$. There does not seem to be any easy geometric interpretation of the map $\nu : V \to W/W_J$ of §5 in terms of the $G_\mathbb{R}$–orbits.

6.6. Duality of orbits on generalized flag manifolds. Let $I \subset \Delta$ and let $\mathcal{P} = \mathcal{P}_I$ be the variety of all conjugates of the standard parabolic subgroup P_I. We will briefly indicate how to set up a correspondence between K–orbits and $G_\mathbb{R}$-orbits on $\mathcal{P}$. If $Q \in \mathcal{P}_I$, set

$$Q_\Gamma = Q \cap \tau(Q) \cap \theta(Q) \cap \sigma(Q).$$

One can show that Q_Γ is a reductive algebraic subgroup of G. Let $\mathcal{P}^*$ denote the set of all $Q \in \mathcal{P}$ such that Q_Γ is of maximal rank. We note that $\mathcal{P}^*$ is stable under conjugation by elements of $K_\mathbb{R}$.

THEOREM 6.6.1. (1) *Each* K*–orbit on* $\mathcal{P}$ *meets* $\mathcal{P}^*$ *in a unique* $K_\mathbb{R}$*–orbit. Thus the inclusion* $\mathcal{P}^* \hookrightarrow \mathcal{P}$ *induces a bijection* $K_\mathbb{R}\backslash\mathcal{P}^* \to K\backslash\mathcal{P}$ *of orbit sets.* (2) *Each* $G_\mathbb{R}$ *orbit on* $\mathcal{P}$ *meets* $\mathcal{P}^*$ *in a unique* $K_\mathbb{R}$*–orbit. Hence the injection* $\mathcal{P}^* \to \mathcal{P}$ *induces a bijection* $K_\mathbb{R}\backslash\mathcal{P}^* \to G_\mathbb{R}\backslash\mathcal{P}$.

Let $Z = K_\mathbb{R}\backslash\mathcal{P}^*$. By Theorem 6.5.1, Z naturally parametrizes the sets of K–orbits and $G_\mathbb{R}$–orbits on $\mathcal{P}$, so that again we get a duality between these orbits. For each $z \in Z$, let $\mathcal{K}(z)$ (respectively $\mathcal{G}(z)$) denote the corresponding K-orbit (respectively $G_\mathbb{R}$–orbit) on $\mathcal{P}$.

THEOREM 6.6.2. *Let* z', $z \in Z$. *Then* $\mathcal{K}(z') \subset \overline{\mathcal{K}(z)}$ *if and only if* $\mathcal{G}(z) \subset \overline{\mathcal{G}(z')}$.

We note that, if (G, θ) is of Hermitian symmetric type, one can get quite explicit information on the $G_\mathbb{R}$–orbits on $\mathcal{P}$ if we combine the results of §5 and of 3.4. In particular, if G is a classical group, then **[Ry]** gives elementary and explicit combinatorial models for the set $K\backslash\mathcal{P}$ of K–orbits on $\mathcal{P}$ (and hence for the $G_\mathbb{R}$–orbits on $\mathcal{P}$).

6.7. Comments. (1) Most of the results of §6 are due to Matsuki. The work of Matsuki related to K–orbits and $G_\mathbb{R}$–orbits on flag manifolds (and generalized flag manifolds) appears in a series of six papers **[M1-M5, MO]**. Matsuki considers a somewhat more general problem, namely the orbits of minimal parabolic subgroups on (real) affine symmetric spaces. In addition, there is a considerable amount of cross referencing in this series of papers. Perhaps for these reasons, we have sometimes had difficulty in understanding the precise statements of his theorems. The proof that there is a natural bijective correspondence between the K-orbits and the $G_{\mathbb{R}}$-orbits is in Matsuki's original paper [**M1**], although the proof that this bijection is order reversing does not seem to appear until a later paper. As nearly as we can tell, most of the results of §6.1 are essentially due to Matsuki. The results of Theorem 6.4.2, Corollary 6.4.3 and Theorem 6.4.4 on the orbit dimensions and the transversality of intersections seem to be new. Theorems 6.4.5 and 6.4.6 are due to Matsuki. Our proofs of these theorems, which rely on Theorem 6.4.4, are quite different. The duality of K-orbits and $G_{\mathbb{R}}$-orbits on generalized flag manifolds is due to Matsuki.The application of the techniques of [**RS**] to the $G_{\mathbb{R}}$-orbits is new, as are the more explicit results for the Hermitian symmetric case.

(2) Throughout §6, we have assumed that $G_{\mathbb{R}} = G^{\sigma}$. This is not an essential assumption, and all of the results of this section go through if one only assumes that $G_{\mathbb{R}}$ is a subgroup of G^{σ} containing $(G^{\sigma})^0$. In this case, one lets $K_{\mathbb{R}} = G_{\mathbb{R}} \cap U$ and lets K denote the Zariski closure of $K_{\mathbb{R}}$ in G. Then we have $(G^{\theta})^0 \subset K \subset G^{\theta}$, and the arguments of Remark 1.1 apply.

§7. A Hecke algebra representation

7.1. Introduction. We keep the notations of the preceding sections. Denote by $\mathcal{H}$ the Hecke algebra of the Weyl group (W, S). Recall that $\mathcal{H}$ is an algebra over the ring of Laurent polynomials $\mathbb{Z}[q, q^{-1}]$ with a basis e_w $(w \in W)$. The multiplication is defined by the following rules, where $w \in W$, $s \in S$,

$$e_s e_w = \begin{cases} e_s e_w & \text{if } l(sw) > l(w), \\ (q-1)e_w + q e_{sw} & \text{if } l(sw) < l(w). \end{cases}$$

See [**Hu**, Ch. 7]

To our data G, θ, B_0, T_0 one can associate a representation of $\mathcal{H}$, i.e. an $\mathcal{H}$-module $\mathcal{M}$. This representation was found by Lusztig and Vogan. It is described in geometric terms in [**LV**]. We shall describe it here in purely combinatorial terms, following [**MS**]. We shall not go into the details of the construction, which invokes the powers of l-adic cohomology. But in order to describe the module $\mathcal{M}$ we need to say a bit about the fine structure of the $(B_0 \times K)$-orbits $\mathcal{O}(v)$.

Namely, one ingredient of the construction is a finer geometric analysis of the product morphism $P_s \times \mathcal{O}(v) \to P_s vK$. The analysis leads one to consider sheaves on $\mathcal{O}(v)$ which are locally constant for the étale topology (briefly: local systems), which moreover are $(B_0 \times K)$-equivariant. Let $x \in \mathcal{O}(v)$ and let I_x be its isotropy group in $(B_0 \times K)$. The local systems in question are classified by the characters of the finite group I_x/I_x^0. Therefore we need some information about these groups.

7.2. Component groups. Let $x \in \mathcal{V}$, let $v = T_0 x K$ and write write $n = x(\theta x)^{-1}$. Let $a = \phi(v)$, so that a is the image of n in W. Write $T_v = \{ t \in T_0 \mid a\theta(t) = t \}$ and denote by A_v the component group T_v/T_v^0. Let $U = R_u(B_0)$.

LEMMA 7.2.1. *(1) I_x is isomorphic to the semi-direct product of T_v and a connected subgroup of U. (2) A_v is an elementary abelian 2-group.*

The proof is easy.

Let $y = tuxz \in \mathcal{O}(v)$, where $t \in T_0$, $u \in U$, $z \in K$. Then the coset tT_v depends only on y. We define a morphism $\mu_x : \mathcal{O}(v) \to T_0/T_v$ by $\mu_x(y) = tT_v$. We have an isogeny of tori $T_0/T_v^0 \to T_0/T_v$ which is a Galois covering with group A_v. A character χ of that group defines a local system of rank one $\mathcal{L}_\chi$ on T_0/T_v. The inverse image $\mathcal{L}_{x,\chi} = \mu_x{}^*(\mathcal{L}_\chi)$ is a $(B_0 \times K)$-equivariant local system on $\mathcal{O}(v)$, whose isomorpism class does not depend on the choice of the representative x.

The Hecke algebra module $\mathcal{M}$ which we are going to describe has a basis indexed by such isomorphism classes. In the geometric analysis alluded to before one has to relate the groups A_v and $A_{m(s)\cdot v}$. In relating these groups, we have to distinguish cases (see §2).

LEMMA 7.2.2. *Assume that s is complex for v. Then s induces an isomorphism $A_v \cong A_{m(s)\cdot v}$.*

This is straightforward. Recall that in this case $m(s) \cdot v = s \cdot v$.

We assume (as we may) that $m(s) \cdot v > v$. If s is not complex for v then s is non-compact imaginary for v. We then have $\phi(m(s) \cdot v) = sa = a\theta(s)$. Let α be the simple root with $s = s_\alpha$. Then $sa\theta(\alpha) = -\alpha$, from which one sees that $\alpha(T_{m(s)\cdot v}) \subset \{1, -1\}$. It follows that α induces a character $\chi(\alpha)$ of $A_{m(s)\cdot v}$. Denote by $\alpha^\vee$ the co-root associated to α. Then $\alpha^\vee(-1) \in T_{m(s)\cdot v}$ and $\eta(\alpha^\vee) = \alpha^\vee(-1)T^0_{m(s)\cdot v}$ defines an element of $A_{m(s)\cdot v}$. From $\langle \alpha, \alpha^\vee \rangle = 2$ it follows that $\eta(\alpha^\vee)$ lies in the kernel of $\chi(\alpha)$.

LEMMA 7.2.3. *Assume that s is non-compact imaginary for v. Then*

$$A_v \cong \mathrm{Ker}\chi(\alpha)/\{1, \eta(\alpha^\vee)\}.$$

PROOF. One shows that

$$T_v = \mathrm{Im}(\alpha^\vee)\left(T_{m(s)\cdot v} \cap \mathrm{Ker}(\alpha)\right), \quad \text{and}$$
$$T_v^0 = \mathrm{Im}(\alpha^\vee)(T^0_{m(s)\cdot v}),$$

whence

$$\mathrm{Ker}(\chi(\alpha)) = \left(T_{m(s)\cdot v} \cap \mathrm{Ker}(\alpha)\right)/T^0_{m(s)\cdot v} \to$$
$$\to \left(\mathrm{Im}(\alpha^\vee)(T_{m(s)\cdot v} \cap \mathrm{Ker}(\alpha)\right) / \left(\mathrm{Im}(\alpha^\vee)T^0_{m(s)\cdot v}\right) = T_v/T_v^0,$$

from which the lemma follows.

This lemma shows that the quotient of the order of $A_{m(s)\cdot v}$ by the order of A_v equals 1, 2 or 4. Hence the order of A_v is increasing with respect to the weak order on V.

Denote by Γ_v the character group of A_v. The lemma implies that Γ_v is isomorphic to a quotient of the annihilator $\Gamma'_{m(s)\cdot v}$ of $\eta(\alpha^\vee)$ in $\Gamma_{m(s)\cdot v}$. We denote by ψ or $\psi_{m(s)\cdot v,s}$ the induced homomorphism $\Gamma'_{m(s)\cdot v} \to \Gamma_v$. Notice ψ is bijective if and only if $\chi(\alpha) = 0$.

The following criterion for the vanishing of $\chi(\alpha)$ follows readily from the definitions.

LEMMA 7.2.4. *$\chi(\alpha) = 0$ if and only if $s \cdot v \neq v$.*

Notice that in this case there is an obvious isomorphism $s : \Gamma_v \to \Gamma_{s\cdot v}$.

If s is real for v, then there exists $v' \in V$ such that s is non-compact imaginary for v' and $v = m(s)\cdot v'$. The preceding results then apply with v, $m(s)\cdot v$ replaced by v', v.

7.3. The Hecke algebra module. Denote by $\mathcal{M}$ the free $\mathbb{Z}[q, q^{-1}]$–module with a basis $f_{v,\chi}$, where v runs through V and $\chi \in \Gamma_v$. In [**LV**] $\mathcal{M}$ is endowed with a structure of $\mathcal{H}$-module. In order to describe this module structure, it suffices to describe the products $e_s f_{v,\chi}$, where $s \in S$. We have to distinguish several cases, to be described now.

(a) s is complex for v and $m(s) \cdot v = s \cdot v > v$. Then

$$e_s f_{v,\chi} = f_{m(s)\cdot v, s\cdot\chi}\,.$$

The notation $s.\chi$ is explained by Lemma 7.2.2.

(b) s is complex for v and $v = m(s) \cdot (s \cdot v) > s \cdot v$. Then

$$e_s f_{v,\chi} = (q-1) f_{v,\chi} + q f_{s\cdot v, s\cdot\chi}\,.$$

(c) s is compact imaginary for v. Then

$$e_s f_{v,\chi} = q f_{v,\chi}\,.$$

(d) s is non-compact imaginary for v and $s \cdot v \neq v$. Using Lemma 7.2.4 we see that now the homomorphism ψ of the preceding section is an isomorphism of a subgroup of $\Gamma_{m(s)\cdot v}$ onto Γ_v. We have

$$e_s f_{v,\chi} = f_{m(s)\cdot v, \psi^{-1}(\chi)} + f_{s\cdot v, s\cdot\chi}\,.$$

(e) s is non-compact imaginary for v and $s\cdot v = v$. Then here are two elements χ' and χ'' in $\Gamma'_{m(s)\cdot v}$ whose image under ψ is χ. We have

$$e_s f_{v,\chi} = f_{m(s)\cdot v,\chi'} + f_{m(s)\cdot v,\chi''} + f_{v,\chi}\,.$$

In the remaining cases s is real for v. Then there is $v' \in V$ such that $v = m(s) \cdot v' > v'$ and s is non-compact imaginary for v'. We denote by ψ' the homomorphism $\psi_{v,s}$ of of the preceding section. Let $v \in V$, $\chi \in \Gamma_v$.

(f) If $\chi \notin \Gamma'_v$ then

$$e_s f_{v,\chi} = -f_{v,\chi}\,.$$

(g) If $\chi \in \Gamma'_v$ and $s \cdot v' \neq v'$ then

$$e_s f_{v,\chi} = (q-2) f_{v,\chi} + (q-1)(f_{v',\nu} + f_{s\cdot v', s\cdot\nu}),$$

where $\nu = \psi'(\chi)$.

(h) If $\chi \in \Gamma'_v$ and $s \cdot v' = v'$ then

$$e_s f_{v,\chi} = (q-1) f_{v,\chi} - f_{v,\chi'} + (q-1) f_{v',\nu}\,,$$

where $\chi' \neq \chi$, $\psi'(\chi') = \psi'(\chi)$ and where ν is as in case (g).

That the preceding formulas do indeed define an $\mathcal{H}$-module is proved in [**LV**] by geometric means. Of course, a direct algebraic proof ought to be possible, but would be cumbersome. Our formulas look somewhat different from those of [**LV**, pp. 371-372] or [**V1**, p. 403], as we have formulated everything in combinatorial terms. Our formulation follows [**MS**], where a more general situation is considered. Our cases (a), ... ,(h) correspond to the cases labeled (b1),(b2),(a),(d1),(c1),(e),(d2),(c2) in [**LV**,**V1**]. In these references, Kazhdan-Lusztig polynomials are defined for this situation. We shall not go into this. See also [**MS**].

7.4. Specializations. The formulas describing the module structure on $\mathcal{M}$ show that they define, in fact, a representation of the $\mathbb{Z}[q]$-subalgebra $\mathcal{H}_0$ of $\mathcal{H}$ with basis e_w $(w \in W)$. We can therefore specialize q to some element of F, obtaining a representation of the specialization of the Hecke algebra.

(a) $q \to 1$. It is well-known that now $\mathcal{H}_0$ specializes to the group ring $\mathbb{Z}[W]$. Hence we obtain a representation of W in a free $\mathbb{Z}$-module with basis $(f_{v,\chi;1})$, the index set being as before. The action of a simple reflection $s \in W$ is read off from the formulas for the action of e_s, with $q = 1$.

(b) $q \to 0$. Now $\mathcal{H}_0$ specializes to the ring with a free $\mathbb{Z}$-basis $m(w)$ $(w \in W)$, the multiplication of the basis elements being as in the monoid $M = M(W)$. The element e_w specializes to $(-1)^{l(w)} m(w)$. We have a representation of the monoid M in a free module with basis $f_{v,\chi;0}$. The action of $m(s)$ $(s \in S)$, is obtained from the formulas for the action of e_s by changing the sign of the right-hand sides and specializing q to 0.

(c) $q \to \infty$. For $w \in W$, $v \in V$, and $\chi \in \Gamma_v$, put

$$\tilde{e}_w = q^{-l(w)} e_w, \quad \text{and} \quad \tilde{f}_{v,\chi} = q^{-d(v)} f_{v,\chi},$$

where $d(v)$ is the dimension of the orbit $\mathcal{O}_v$. One checks that the e_w span a $\mathbb{Z}[q^{-1}]$-subalgebra of $\mathcal{H}$ and that its elements stabilize the $\mathbb{Z}[q^{-1}]$-submodule of $\mathcal{M}$ with basis $\tilde{f}_{v,\chi}$ of $\mathcal{M}$. We can specialize q to ∞, obtaining a representation of $M(W)$ in a free module with basis $f_{v,\chi;\infty}$. We obtain that in cases (b),(c),(g),(h)

$$m(s) \cdot f_{v,\chi;\infty} = f_{v,\chi;\infty}\,.$$

In case (f) the left-hand side is 0, and it equals $f_{s\cdot v,s.\chi;\infty}$ in case (a),

$$f_{m(s)\cdot v,\psi^{-1}(\chi);\infty}$$

in case (d) and

$$f_{m(s)v,\chi';\infty} + f_{m(s)v,\chi'';\infty}$$

in case (e). The notations are as before.

7.5. Examples. We now give a few examples of the Hecke algebra representations in low rank. In the first two examples the groups Γ_v have order ≤ 2. We shall write $f_{v,\chi} = f_{di}$, where d is the dimension of the orbit $\mathcal{O}(v)$ and $i = 0$ (respectively 1) designates the trivial (respectively the non-trivial) element of Γ_v. If Γ_v is trivial we write $f_{d0} = f_d$. If there are several orbits with the same dimension we write $f_{di}, f'_{di}, \ldots$ for the corresponding basis elements. In these examples there will be at most two such orbits. Working out the details is a straightforward matter.

(a) $G = SL(2,\mathbb{C})$, $\theta(g) = cgc^{-1}$, where $c = \mathrm{diag}(1,-1)$. We write e for the only generator of $\mathcal{H}$. The representation is as follows.

$$\begin{aligned} ef_0 &= f'_0 + f_{10}\,, \\ ef'_0 &= f_0 + f_{10}\,, \\ ef_{10} &= (q-2)f_{10} + (q-1)(f_0 + f'_0)\,, \\ ef_{11} &= -f_{11}\,. \end{aligned}$$

(b) $G = PSL(2,\mathbb{C})$ and θ is induced by the automorphism of the previous example. The formulas are different.

$$\begin{aligned} ef_0 &= f_0 + f_{10} + f_{11}\,, \\ ef_{10} &= (q-1)f_{10} - f_{11} + (q-1)f_0\,, \\ ef_{11} &= (q-1)f_{11} - f_{10} + (q-1)f_0\,. \end{aligned}$$

Write

$$\widetilde{f_0} = -f^\star_{10}\,, \quad \widetilde{f'_0} = -f^\star_{11}\,, \quad \text{and} \quad \tilde{f}_{10} = f^\star_0\,,$$

where the star denotes the dual basis of the dual module. One checks that the in the dual representation $(q-1) - e$ acts on the elements $\widetilde{f_0}$, $\widetilde{f'_0}$, and $\widetilde{f_{10}}$ as e acts on the corresponding elements without a tilde in example (a). This is an example of a general duality phenomenon, studied at length in [**V2**].

(c) $G = SL(3,\mathbb{C})$, $\theta(g) = {}^tg^{-1}$. The Hecke algebra $\mathcal{H}$ has two generators, e_1 and e_2, corresponding to the generators (12) resp. (23) of the Weyl group S_3. The corresponding simple roots are α_1 and α_2.

There is one orbit of length 0 and 2, and there are two of length 1. See [**RS**, pp. 432-433], where the case of $SL(n,\mathbb{C})$ is dealt with. One shows that the groups Γ_v are trivial, except when $\mathcal{O}(v)$ is the maximal orbit, in which case it has order 4. In fact, the analysis of [loc. cit.] gives in that case an isomorphism of Γ_v onto the diagonal subgroup of $SL(3,\mathbb{C})$ with (diagonal) entries ± 1. The roots α_1, α_2, and $\alpha_1 + \alpha_2$ define characters of Γ_v. The corresponding basis elements of $\mathcal{M}$ are denoted by f_{21}, f_{22} and f_{23}, respectively. The other notations are as before. The representation is as follows.

$$e_1f_0 = f'_1, \quad e_2f_0 = f_1, \quad e_1f_1 = f_{20} + f_{21} + f_1,$$
$$e_2f_1 = (q-1)f_1 + qf_0, \quad e_1f'_1 = (q-1)f'_1 + qf_0,$$
$$e_2f'_1 = f_{20} + f_{22} + f_1, \quad e_1f_{20} = (q-1)f_{20} - f_{21} + (q-1)f_1,$$
$$e_2f_{20} = (q-1)f_{20} - f_{22} + (q-1)f'_1, \quad e_1f_{21} = (q-1)f_{21} - f_{20} + (q-1)f_1,$$
$$e_2f_{21} = -f_{21}, \quad e_1f_{22} = -f_{22}, \quad e_2f_{22} = (q-1)f_{22} - f_{20} + (q-1)f'_1,$$
$$e_1f_{23} = e_2f_{23} = -f_{23}.$$

Editors' note. In the final stages of the production of this volume the editors learned that one of the contributors, Roger W. Richardson, died in Canberra, Australia, on June 15, 1993. Roger was a valued colleague and a close friend of many of the contributors to this volume, as well as of many others who participated in the conference. His sudden death at a time when he was actively engaged in mathematical work is a sad loss.

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ON EXTRASPECIAL PARABOLIC SUBGROUPS

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0. Introduction

In this note we extend a result from joint work with R. Richardson and R. Steinberg [RRS] regarding the structure of abelian radicals of parabolic subgroups in Chevalley Groups to the so called 'extraspecial' case.

Let G be a simple, connected algebraic group over an algebraically closed field of characteristic $p > 2$ or 0. Let T be a maximal torus of G and Ψ the corresponding root system of G. We fix a Borel subgroup $B \supset T$ and denote the set of simple roots of Ψ with respect to B by Δ. Let ρ (ρ_s) denote the highest (short) root in Ψ. Assume that G is not of type A_n. There is a unique simple root α satisfying $\langle \alpha, \rho \rangle = 1$. (Here $\langle \beta, \gamma \rangle = (\beta, \gamma^\vee)$, where $\gamma^\vee = 2\gamma/(\gamma,\gamma)$ is the coroot of γ, for $\beta, \gamma \in \Psi$.) Let $P = P_\alpha = LV$ be the corresponding maximal standard parabolic subgroup of G with $V = R_u(P)$, the unipotent radical of P. The Levi complement of V in P, L, is generated by T and all root subgroups whose roots are orthogonal to ρ. There is a unique parabolic subgroup P^- of G satisfying $P^- \cap P = L$ with Levi decomposition $P^- = LV^-$, where $V^- = R_u(P^-)$. The derived subgroup of V is U_ρ, the root subgroup corresponding to the highest root ρ. Let V_1, respectively V_1^-, denote the commutator quotient of V, respectively V^-. Our goal is to study the conjugation action of L on V_1 or V_1^-.

By Ψ, Ψ_L, Ψ_V, etc. we denote the root system of the groups G, L, V, etc. We also write Ψ_{V_1} for the roots in the support of V_1, i.e., $\Psi_{V_1} = \Psi_V \setminus \{\rho\}$. The Weyl groups of G and L are W and W_L, respectively. We call a subset of roots of Ψ an *orthogonal set of roots* provided these roots are mutually orthogonal. Let $\mathcal{A}$ denote the set of all orthogonal sets of long roots in Ψ_{V_1} including the empty set. Note that W_L acts on $\mathcal{A}$.

It is known that L acts on V_1 with a finite number of orbits. This is a special case of a result due to R. W. Richardson [Ri2, Theorem E]. Our main theorem describes a connection between the L-orbits on V_1, the (P, P) double cosets of G, and the W_L-orbits on $\mathcal{A}$. This is the precise analogue for the case of an abelian radical [RRS]. Let $u_\beta \in U_\beta$ and $u_\beta \neq 1$.

Main Theorem. *Let G be a simple, connected algebraic group over an algebraically closed field of characteristic $p > 2$ or 0, not of type A_n. Let $P_\alpha = LV$ be*

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