b. We have that f, g are L-integrable functions, and define:

$$F(x) = \int_0^x f(t)|dt| \qquad G(x) = \int_0^x g(t)|dt|.$$

These functions are obviously L-integrable on any bounded domain since they are continuous, by part (a). It is clear folly to try and prove the integration by parts formula using the product rule of differential calculus, since F,G are only guaranteed to be continuous, not differentiable. One can prove the result by instead using Fubini's theorem for Lebesgue integrals.

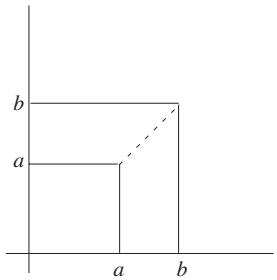

FIGURE FOR SOLUTION 4.11.8.

Consider the function $f(t)\cdot g(s)$ which is L-integrable since it is a product of L-integrable functions depending on different variables. We can think of the integration by parts formula geometrically, as shown in the margin.

The integral of $f \cdot g$ over the larger square

$$S_b = \{(t,s)|0 \le t \le b, 0 \le s \le b\}$$

minus the integral of $f \cdot g$ over the smaller square

$$S_a = \{(t,s)|0 \le t \le a, 0 \le s \le a\}$$

is equal to the sum of the integrals of $f \cdot g$ over the two "wedge" figures

$$W_{lower} = \{(t,s)|0 \le s \le t, a \le t \le b\} \qquad W_{upper} = \{(t,s)|0 \le t \le s, a \le s \le b\}$$

So much follows obviously from the indicator functions of those regions. Now,

$$\int_{S_b} f(t)g(s)|dt\,ds| - \int_{S_a} f(t)g(s)|dt\,ds| = \int_0^b f(t)|dt| \int_0^b g(s)|ds| - \int_0^a f(t)|dt| \int_0^a g(s)|ds|;$$

this follows immediately by applying Fubini's theorem and the fact that f is strictly a function of t and g of s. Using the function definitions for F, and G this becomes:

$$F(b)G(b) - F(a)G(a)$$

The integral

$$\int_{W_{lower}} f(t)g(s)|dt\,ds|$$

is by Fubini's theorem again equal to

$$\int_a^b f(t) \left(\int_0^t g(s)|ds| \right) |dt|$$

Since $f(t)$ is not a function of s. In terms of G, this equals

$$\int_a^b f(t)G(t)|dt|$$

In exactly the same fashion, one sees that

$$\int_{W_{upper}} f(t)g(s)|dt\,ds| = \int_a^b F(t)g(t)\,|dt|,$$

where the dummy variable s has been replaced by t.

Thus the geometric equality above implies that

$$\int_a^b f(t)G(t)\,|dt| + \int_a^b F(t)g(t)\,|dt| = F(b)G(b) - F(a)G(a),$$

which is equivalent to the integration by parts formula.

c. We have f and F as in part a. We also assume that F is L-integrable. We can define the following series representation of F, for fixed n:

$$F(t) = \sum_{k=0}^{\infty} \left(\mathbf{1}_{[\frac{-k-1}{n}, \frac{-k}{n}]}(t)F(t) + \mathbf{1}_{[\frac{k}{n}, \frac{k+1}{n}]}(t)F(t) \right).$$

Note that each term in the series is Riemann integrable since $F(t)$ is continuous by part (a). The L-integrability of F implies that

$$\sum_{k=0}^{\infty} \left| \int_{\mathbb{R}} \left(\mathbf{1}_{[\frac{-k-1}{n}, \frac{-k}{n}]}(t)F(t) + \mathbf{1}_{[\frac{k}{n}, \frac{k+1}{n}]}(t)F(t) \right) |dt| \right| < \infty.$$

This implies (since the indicator functions do not overlap—the triangle inequality is an equality in this case)

$$\sum_{k=0}^{\infty} \left| \int_{\mathbb{R}} \mathbf{1}_{[\frac{-k-1}{n}, \frac{-k}{n}]}(t)F(t)\,|dt| \right| + \sum_{k=0}^{\infty} \left| \int_{\mathbb{R}} \mathbf{1}_{[\frac{k}{n}, \frac{k+1}{n}]}(t)F(t)\,|dt| \right| < \infty$$

For each of these sums to converge, as they must, their individual terms must go to zero as $k \to \infty$:

$$\left| \int_{\mathbb{R}} \mathbf{1}_{[\frac{-k-1}{n}, \frac{-k}{n}]}(t)F(t)\,|dt| \right| \to 0 \qquad \left| \int_{\mathbb{R}} \mathbf{1}_{[\frac{k}{n}, \frac{k+1}{n}]}(t)F(t)\,|dt| \right| \to 0$$

Since each term is a normal Riemann integral, we can apply the mean value theorem to each, to get the conditions

$$\lim_{k\to\infty} \frac{1}{n} F(x_k) = 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{1}{n} F(x_k^{'}) = 0,$$

where $x_k \in [\frac{k}{n}, \frac{k+1}{n}]$ and $x_k^{'} \in [\frac{-k-1}{n}, \frac{-k}{n}]$. Since this is true for n arbitrarily large, and since $F(t)$ is continuous, we see that $\lim_{x\to\pm\infty} F(x) = 0$.

d. We have f,g,F,G as in part (b). We also have that F is L-integrable and that G is bounded. We begin with

$$\int_a^b f(t)G(t)\,|dt| + \int_a^b F(t)g(t)\,|dt| \;=\; F(b)G(b) - F(a)G(a)$$

and take the limit as $b \to \infty$ and $a \to -\infty$ on both sides. For the left side, this simply means integrating the integrands over all of $\mathbb{R}$, which for $F \cdot g$ is immediately allowable, since the product of two L-integrable functions is L-integrable. For $f \cdot G$ we have the product of an L-integrable function with a continuous (by part (a)), bounded function. Thus

$$\left| \int_a^b f(t)G(t)\,|dt| \right| \le \int_a^b |f(t)|C\,|dt| = C \int_a^b |f(t)|\,|dt|,$$

which is guaranteed to converge, since f is L-integrable. For the right side we take the limit

$$0 \le \lim_{a\to-\infty}\lim_{b\to\infty}|F(b)G(b)-F(a)G(a)| \le \lim_{a\to-\infty}\lim_{b\to\infty}|F(b)||G(b)|+|F(a)||G(a)|$$

$$\le \lim_{a\to-\infty}\lim_{b\to\infty} C(|F(b)|+|F(a)|) = C(\lim_{b\to\infty}|F(b)| + \lim_{a\to-\infty}|F(a)|).$$

By the result of part (c), this last expression goes to 0. Thus we may write

$$\int_{-\infty}^{\infty} f(t)G(t)|dt| + \int_{-\infty}^{\infty} F(t)g(t)|dt| = 0.$$

which is equivalent to the desired result.

4.11.9 There are no difficulties associated with infinities in this case:

$$\widehat{\mathbf{1}_{[-1,1]}}(\xi) = \int_{\mathbb{R}} \mathbf{1}_{[-1,1]}(x)e^{i\xi x}dx = \int_{-1}^{1} e^{i\xi x}dx$$

$$= \left[\frac{e^{i\xi x}}{i\xi}\right]_{-1}^{1} = \frac{e^{i\xi}-e^{-i\xi}}{i\xi} = 2\frac{\sin\xi}{\xi}.$$

The only fishy part is the case $\xi = 0$; but note that

$$\lim_{\xi\to 0} 2\frac{\sin\xi}{\xi} = 2 = \widehat{\mathbf{1}_{[-1,1]}}(0).$$

4.11.10 Set $\dfrac{x-\bar{x}}{\sqrt{2\sigma}} = u$. Then $du = \dfrac{dx}{\sqrt{2\sigma}}$ and

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma}}\, e^{-\frac{(x-\bar{x})^2}{2\sigma}}\, dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma}}\, e^{-u^2}\sqrt{2\sigma}\, du = \frac{1}{\sqrt{\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-u^2}\, du}_{1 \text{ by equation 4.11.73}} = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1.$$

4.11.11 It is clearly enough to show the result for any monomial

$$x_1^{k_1}\dots x_n^{k_n} \quad \text{and} \quad |x_1^{k_1}\dots x_n^{k_n}| \le (1+|\mathbf{x}|)^k,$$

where $k = k_1 + \cdots + k_n$. Thus it is enough to show that the integral

$$\int_{\mathbb{R}^n} (1+|\mathbf{x}|)^k e^{-|\mathbf{x}|^2}\, |d^n\mathbf{x}|$$

exists for every $k \ge 0$. Developing out the power, it is enough to show that

$$\int_{\mathbb{R}^n} |\mathbf{x}|^k e^{-|\mathbf{x}|^2}\, |d^n\mathbf{x}| \quad \text{exists for every } k \ge 0.$$

Solution 4.11.11: This is a matter of pinning down the fact that exponentials win over polynomials.

Let us break up $\mathbb{R}^n$ into the unit ball B and the sets

$$A_m = \left\{ \mathbf{x}\in\mathbb{R}^n \;\middle|\; 2^m < |\mathbf{x}| \le 2^{m+1} \right\}.$$

We can then write

$$|\mathbf{x}|^k = |\mathbf{x}|^k\mathbf{1}_B(\mathbf{x}) + \sum_{m=0}^{\infty} |\mathbf{x}|^k\mathbf{1}_{A_m}(\mathbf{x}) \stackrel{\text{def}}{=} g(\mathbf{x}) + \sum_{m=0}^{\infty} f_m(\mathbf{x}),$$