Prof. Allen Knutson's Math 2240 Final, May 15, 2012

Numbers in boxes are the point scores for their questions.

1.True or false? If false, give a counterexample. (If you think it's *really* obvious that it's a counterexample, don't bother proving it.) If true, you don't have to prove it.

[3 apiece]

Every continuous function is Lebesgue integrable.

A. False; $f = 1$ on $\mathbb{R}$ is not.

Every Riemann integrable function is continuous.

A. False; $f = 1_{[0,17]}$ is not.

Every Riemann integrable function is Lebesgue integrable.

A. True.

Every Lebesgue integrable function is Riemann integrable.

A. False; $f = 1_{\mathbb{Q}\cap[0,1]}$ is not.

Every continuous, bounded, Lebesgue integrable function of bounded support is Riemann integrable.

A. True; one doesn't even need to assume Lebesgue integrable (it follows).

Every Lebesgue integrable function has bounded support.

A. False; $f = 1_{\mathbb{Z}}$ is Lebesgue integrable.

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2. [10]

Let $S \subseteq \mathbb{R}^2$ be the half-disc defined by $x^2+y^2 \leq 1, x \geq 0$. Integrate the function $f(x, y) = x$ over it via Fubini's theorem. Do it in both orders – over x then y, and vice versa. Hint: the substitution $z = 1 - x^2$ will help more than trig substitutions.

A.

$$\begin{aligned}
\int_{x=0}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x\ dy\ dx &= \int_{x=0}^{1} x \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\ dy\ dx \\
&= \int_{x=0}^{1} x2\sqrt{1-x^2}\ dx \\
&= \int_{z=1}^{0} -\sqrt{z}\ dz \\
&= \int_{z=0}^{1} \sqrt{z}\ dz \\
&= \frac{z^{3/2}}{3/2}|_0^1 \\
&= 2/3.
\end{aligned}$$

$$\begin{aligned}
\int_{y=-1}^{1}\int_{x=0}^{\sqrt{1-y^2}} x\ dx\ dy \\
&= \int_{y=-1}^{1} \frac{x^2}{2}|_0^{\sqrt{1-y^2}}\ dy \\
&= \int_{y=-1}^{1} \frac{1-y^2}{2}\ dy \\
&= \frac{y-y^3/3}{2}|_0^1 \\
&= 2/3.
\end{aligned}$$

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3. [10]

Let $S \subseteq \mathbb{R}^n$ have Hausdorff dimension $d < n$, i.e. for any $\epsilon > 0$, we can cover it fully with cubes of side-lengths $r_1, r_2, \ldots$ all $< \epsilon$, such that $\sum_i r_i^{d+\epsilon} < \epsilon$.

Prove that S has measure 0.

A. Since $d < n$, there exists $\epsilon \in (0, n-d)$, so $d + \epsilon < n$. If $r_i < 1$, then $r_i^n < r_i^{d+\epsilon}$. Adding them up,

$$\sum_i r_i^n < \sum r_i^{d+\epsilon} < \epsilon$$

which is the bound we needed to show to say S has measure 0.

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4. Consider, if you will, two Lebesgue integrable functions f, g that aren't Riemann integrable, but whose product fg is.

a. [5] Give an example of such a pair, where f, g are bounded with bounded support.

A. $f = 1_{\mathbb{Q}\cap[0,1]}$, $g = 1_{\mathbb{Q}\cap[2,3]}$, so $fg = 0$.

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b. [5] Give an example of such a pair, where f, g are continuous almost everywhere.

A. $f = 1_{2\mathbb{Z}}$, $g = 1_{2\mathbb{Z}+1}$, i.e. on the evens vs. the odds. Again $fg = 0$.

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c. [5] Can we achieve (b) and (c) with the same pair f, g?

A. Nope. If f (or g) is bounded with bounded support and continuous almost everywhere, it's Riemann integrable.

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5. Let M be an $n \times n$ matrix, and N the matrix obtained by shaving off M's last row and column.

a. [5]

If M is upper triangular, show that N's characteristic polynomial $\det(N - tI_{n-1})$ divides M's.

A. The characteristic polynomial of an upper triangular matrix with diagonal entries $d_1, \ldots, d_m$ is $\prod_{i=1}^{m}(d_i - t)$, since the determinant of an upper triangular matrix is the product down the diagonal.

Hence if M's diagonal entries are $m_{11}, \ldots, m_{nn}$, the polynomials in question are $\prod_{i=1}^{n}(m_{ii} - t)$ and $\prod_{i=1}^{n-1}(m_{ii} - t)$.

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b. [5]

Does this divisibility hold without the upper triangularity assumption?

A. Nope. Perhaps the simplest nontriangular matrix is

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

whose characteristic polynomial is $t^2 - 1$, which is not a multiple of N's, which is just t, or I guess $-t$, as defined above. (Here $N = \begin{pmatrix} 0 \end{pmatrix}$.)

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6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function such that

(1) g, h are two more smooth functions, with the property that

$$f = \frac{\partial h}{\partial x} - \frac{\partial g}{\partial y}$$

(2) If $|\vec{v}| > 17$, then $|f(\vec{v})| < 17|\vec{v}|^{-17}$, and same for g and h.

Your goal: show the equality of (Lebesgue) integrals

$$\int_{x=1}^{\infty} \int_{y=-\infty}^{\infty} f(x, y)\ dx\ dy = \int_{y=-\infty}^{\infty} h(1, y)\ dy.$$

a. [10] Set this up as a formal (i.e. nonrigorous) application of Stokes' theorem: say what manifold-with-boundary M you're using, what are the forms, etc. Hint: g and (obviously) h will come into play.

A. Stokes' theorem says

$$\int_M d\alpha = \int_{\partial M} \alpha$$

so it looks like M should be the half plane $\{(x, y) : x \geq 1\}$, and $d\alpha$ should be $f\ dx \wedge dy$. What should α be? Some 1-form, certainly.

A general 1-form is a combination of dx and dy, say $\alpha = A\ dx + B\ dy$. Then

$$d\alpha = dA \wedge dx + dB \wedge dy = \frac{\partial A}{\partial y} dy \wedge dx + \frac{\partial B}{\partial x} dx \wedge dy = \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) dx \wedge dy.$$

(There are other terms in dA and dB, but they die upon being wedged with dx and dy.)

That begins to look like assumption (1), with $A = f$ and $B = g$. So now Stokes would say

$$\int_{(x,y):x \geq 1} f = \int_{(x,y):x=1} (g\ dx + h\ dy)$$

but as we traverse the line $x = 1$, the tangent vector has no horizontal component, so the $g\ dx$ term disappears.

Therefore it becomes $\int_{(x,y):x=1} h\ dy$, which is what we wanted.

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b. [5] Explain why Stokes' theorem doesn't *quite* apply as usually stated. Then give whatever additional argument is necessary to complete the proof. (Hint: look at the picture you drew, or should have drawn, in question #2.)

A. M is noncompact. (If we didn't include compactness in the assumptions of Stokes' theorem, we could e.g. just rip out all of ∂M, which wouldn't change the $\int_M$ side but would make the $\int_{\partial M}$ side vanish.) So Stokes' theorem doesn't actually apply.

Instead, let's apply it to a large half-disc M_R, where $|(x, y)| \leq R$. Then Stokes says

$$\int_{M_R} f\, dx \wedge dy = \int_{x=1,|(x,y)|\leq R} \alpha + \int_{x\geq 1,|(x,y)|=R} \alpha$$

where the RHS has a term for the left edge, and the arc, of the half-disc. Now consider the limit $R \to \infty$. The first two terms become what we want them to. The arc term goes to zero because of the bound (2) on g and h. (The length of the arc increases like R^1, while g and h decrease like R^{-17}.)

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7. We know that if M is a compact oriented manifold-with-boundary, then its boundary ∂M is a compact oriented manifold.

So let's consider the reverse. If N is a compact oriented manifold, is $N = \partial M$ for some M? If N is 1-dimensional, it's a bunch of oriented circles, and **Yes** this N is the boundary of a bunch of discs (or of a sphere with a bunch of holes chopped in it, like a Wiffle® ball).

The same holds if N is 2-d (easy to visualize) or even 3-d (less easy!). The questions below concern $\dim N = 0$.

[4] If you were thinking of a compact oriented 0-manifold N, and wanted to tell someone about it over the phone, what data would you tell them to describe it?

A. How many + points and how many − points there are.

[10] State the necessary and sufficient condition for such an N to be the boundary of a compact 1-manifold M with boundary.

A. A compact 1-manifold with boundary is a collection of intervals. The boundary of any one is a $+$ point and a $-$ point. So there should be the same number of each.

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8. Define $f : \mathbb{R}^2 \to \mathbb{R}^3$ by $f(x, y) = (x, e^x, 1+y^2)$. Let $\alpha = u\ du + w\ dv$ be a 1-form on $\mathbb{R}^3$, in its u, v, w coordinate system.

a. [3] Compute the 1-form $f^*(\alpha)$.

A. $u\ du + w\ dv \mapsto x\ dx + (1+y^2)\ d(e^x) = x\ dx + (1+y^2)e^x\ dx = (x + (1+y^2)e^x)dx.$

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b. [5] Set up and compute its integral over the interval $x \in [0, 1], y = 1$.

A.

$$\begin{aligned}\int_{x=0}^{1} (x + (1+y^2)e^x)dx &= \int_{x=0}^{1} (x + 2e^x)dx \\ &= \left(\frac{x^2}{2} + 2e^x\right)\Big|_0^1 \\ &= (1/2 + 2e) - (0 + 2) \\ &= 2e - 3/2.\end{aligned}$$