Prof. Allen Knutson's Math 2240 Practice Final, for May 15, 2012

Let

$$M = \begin{pmatrix} 1 & 0 & 5 & -3 & 2 & 8 & 0 \\ 4 & 4 & 0 & 3 & 0 & -3 & 1 \end{pmatrix}, N = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 2 & 0 \\ 1 & 1 \\ 3 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix}.$$

Compute $\det(MN)$ and $\det(NM)$.

A. $MN$ is an easily calculated $2 \times 2$ matrix whose determinant happens to be nonzero. $NM$ is a lengthily calculated $7 \times 7$ matrix that cannot be invertible, since its image has dimension at most $2$. (In fact it's exactly $2$.) So its determinant is zero.

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Let $Q = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$ be the first quadrant, and consider the map

$$\phi : Q \to Q, \qquad (x, y) \mapsto (x + y, x\sqrt{y}).$$

a. Compute the derivative of $\phi$.

A.

$$\begin{pmatrix} 1 & 1 \\ \sqrt{y} & x/(2\sqrt{y}) \end{pmatrix}$$

b. This isn't a parametrization, as the derivative sometimes drops rank. Find the equation of the curve $C$ inside $Q$ where this happens.

A. The determinant is zero where $\sqrt{y} - x/2\sqrt{y} = 0$, i.e. $x = 2y$.

c. Find the equation of its image $\phi(C)$.

A. If $u = x + y = 3y$, and $v = x\sqrt{y} = 2y\sqrt{y}$, then $v = 2(u/3)^{3/2} = \frac{2}{3^{3/2}} u^{3/2}$.

d. Say we wanted to Lebesgue integrate the function $e^{-x}$ on $Q$, in the (unbounded) region below the curve $\phi(C)$. The change of variable formula would let us turn it into a different integral, by pulling back along $\phi$; what would this integral be? (Don't evaluate it.)

A. It's easier to work in different coordinates on the two $\mathbb{R}^2$s, i.e. $(u,v) = (x+y, x\sqrt{y})$ on the target. Then the form we're pulling back is

$$\begin{aligned} e^{-u} du \wedge dv &= e^{-(x+y)} d(x+y) \wedge d(x\sqrt{y}) \\ &= e^{-(x+y)} (dx+dy) \wedge (\sqrt{y} dx + x\ dy/(2\sqrt{y}) \\ &= e^{-(x+y)} (dx+dy) \wedge (\sqrt{y} dx + x\ dy/(2\sqrt{y}) \\ &= e^{-(x+y)} \big( \sqrt{y} dx \wedge\ dx + x/(2\sqrt{y}) dx \wedge dy \\ &+ \sqrt{y} dy \wedge dx + x/(2\sqrt{y}) dy \wedge dy \big) \\ &= e^{-(x+y)} \big( x/(2\sqrt{y}) dx \wedge dy - \sqrt{y} dx \wedge dy) \\ &= e^{-(x+y)} \big( x/(2\sqrt{y}) - \sqrt{y} \big)\ dx \wedge dy \end{aligned}$$

At this point I should have had a practice question about setting up (not doing) each of these integrals via Fubini; first $x$ then $y$ or vice versa, in each coordinate system.

e. Let $\alpha = x\ dy$. Compute the pullback $\phi^*(\alpha)$.

A. Same sorta thing:

$$u\ dv = (x+y) d(x\sqrt{y}) = (x+y)(\sqrt{y} dx + x/(2\sqrt{y}) dy)$$

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Buncha true/false about whether continuous implies integrable and whatnot. Basically the answer is always "false", so the point is to give a counterexample.

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Let $\omega = \sum_{i=1}^{n} dx_i \wedge dx_{n+i}$. Let $v = \omega \wedge \omega \wedge \cdots \wedge \omega$ ($n$ factors).

a. What's $v$ when $n = 2$?

b. What's $v$ when $n = 3$?

c. What's $v$ for general $n$?

A. A priori when we wedge these $n$ factors together, each containing $n$ terms, we have $n^n$ terms to look at. But if there's any repetition in the factors, the wedge dies, so only the $n!$ terms with no repetition survive.

Also, those terms are all equal, because we can commute $2$-forms past each other, to make them all look like

$$(dx_1 \wedge dx_{n+1}) \wedge (dx_2 \wedge dx_{n+2}) \wedge \cdots$$

so we only need to figure out the signs involved in turning that into $dx_1 \wedge \cdots \wedge dx_{2n}$.

Imagine sorting the large $dx_i$ towards the end. The $dx_{2n}$ is already there, good, but each $dx_{i+n}$ needs to move past all the $dx_j$, $j > i$, each time incurring a factor of $-1$. That's $0+1+\ldots+(n-1)$ such factors, which adds up to $\binom{n}{2}$. Then $(-1)^{\binom{n}{2}}$ is $+1$ for $n \equiv 0, 3 \pmod 4$, and $-1$ for $n \equiv 1, 2 \pmod 4$.

Final answer: $(-1)^{\binom{n}{2}} n!\ dx_1 \wedge \cdots \wedge dx_{2n}$.

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Let $f : \mathbb{R} \to \mathbb{R}^2$ take $t \mapsto (\cos e^t, \sin e^t)$.

a. Compute the derivative.

b. Let $\alpha = b(x, y)\ dx + c(x, y)\ dy$ be a general $1$-form on $\mathbb{R}^2$. Compute $f^*(\alpha)$.

A. This is just like the question two above.

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Let $V$, $W$ be vector spaces with orientations.

a. Say how to define an orientation on $V \times W = \{(\vec{v} \in V, \vec{w} \in W)\}$.

A. To put an orientation on a vector space, it suffices to state a basis and then say "*that* one's positively oriented".

Here, the answer people usually like is to take a positive basis $(v_1, \ldots, v_a)$ of $V$, another one $(w_1, \ldots, w_b)$ of $W$, soup them up to vectors in $V \times W$, and concatenate the lists to get a basis

$$\left((v_1, \vec{0}), \ldots, (v_a, \vec{0}), (\vec{0}, w_1), \ldots, (\vec{0}, w_b)\right)$$

of $V \times W$.

b. Define

$$\sigma : V \times W \to W \times V, \qquad (\vec{v}, \vec{w}) \mapsto (\vec{w}, \vec{v}).$$

When is it orientation-preserving, when orientation-reversing?

A. We take a positive basis of $V \times W$ (such as the above), hit it with $\sigma$, and compare that to a positive basis of $W \times V$.

The positive basis we would expect to see of $W \times V$, following the recipe above, is

$$\big((w_1, \vec{0}), \ldots, (w_b, \vec{0}), (\vec{0}, v_1), \ldots, (\vec{0}, v_a)\big).$$

If we hit part (a)'s basis with $\sigma$, we get

$$\big((\vec{0}, v_1), \ldots, (\vec{0}, v_a), (w_1, \vec{0}), \ldots, (w_b, \vec{0})\big).$$

So how many switches does it take to turn one of these bases of $W \times V$ into the other? We need to carry each of these $a$ vectors past each of those $b$ vectors, so $ab$ switches.

Hence the map is orientation-preserving if $a$ or $b$ are even, and reversing if $a$ and $b$ are both odd.