REPRESENTATIONS OF U(N) – CLASSIFICATION BY HIGHEST WEIGHTS NOTES FOR MATH 261, FALL 2001

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1. Weight diagrams of T-representations

Let T be an n**-dimensional torus**, i.e. a group isomorphic to $(S^1)^n$. The T we will care about most is the subgroup of diagonal matrices in $U(n)$.

Lemma. *Every unitary matrix is conjugate to a diagonal matrix (using another unitary matrix).*

Proof. Let M be unitary, $\vec{v}$ a unit eigenvector, and $\vec{v}^\perp$ the subspace perpendicular to $\vec{v}$. Then M preserves that subspace (by unitarity) and acts unitarily on it, so by induction we can make an orthonormal basis of eigenvectors. □

Theorem. *Two reps of $U(n)$ are $U(n)$-isomorphic iff they are T^n-isomorphic, where T^n is the diagonal matrices in $U(n)$.*

Proof. Since $U(n)$ is compact, two reps are isomorphic iff they have the same character. Since T^n meets every conjugacy class in $U(n)$, we can determine the character (a class function) from elements of T alone. So if they're T^n-isomorphic, they have the same T-character, therefore the same $U(n)$-character, therefore they're $U(n)$-isomorphic. □

This lets us reduce lots of questions about reps of $U(n)$ to reps of tori. So we need to understand reps of tori.

Lemma. *Every torus in $U(n)$ can be conjugated into the diagonal matrices.*

Proof. Identify our torus S with $(S^1)^k$. Let $g = (\exp(2\pi i\alpha_1), \exp(2\pi i\alpha_2), \ldots, \exp(2\pi i\alpha_k)$ where the set $\{1, \alpha_1, \ldots, \alpha_k\}$ is linearly independent over the rationals. Then the powers of g are dense in S.

Since g lives in $U(n)$, it can be diagonalized. That diagonalizes all its powers, therefore all of S by continuity. □

Corollary. *All irreps of tori are 1-dimensional.*

Proof. Let $S \to U(n)$ be a representation. (The homomorphism may not be injective, but the same argument from the lemma still works.) Then by change of basis, S lands inside the diagonal matrices. Which means it preserves the subspaces $\mathbb{C}\vec{e}_i$ for each $i = 1, \ldots, n$. To be irreducible, we must have $n = 1$. □

Given an irrep $\phi : S \to U(1)$, associate a **weight** $d\phi|_{\mathbf{Id}} : T_{\mathbf{Id}}S \to T_{[1]}U(1) \cong \mathbb{R}$. Really, $d\phi|_{\mathbf{Id}}$ is an element of the dual space $(T_{\mathbf{Id}}S)^*$ to the Lie algebra $T_{\mathbf{Id}}S$ of S. Let S^* denote the **weight lattice** of S, i.e. the set of weights, which is a lattice living in the vector space $(T_{\mathbf{Id}}S)^*$, and corresponds 1:1 to the set of isomorphism classes of irreps of S.

Given a rep V of S, define the **weight diagram** of V as the $\mathbb{N}$-valued function on S^* taking a weight λ to the dimension of the λ-isotypic component of V. That subspace is called the λ **weight space**, and its dimension the **multiplicity of the weight** λ **in** V. Since V is finite-dimensional (always true for us), the weight diagram is compactly supported. The **weights of a representation** V are the support of this function.

In the case of $S = (S^1)^k$, the weight lattice is naturally isomorphic to $\mathbb{Z}^k$. In this way the weight

$$\lambda = (\lambda_1, \dots, \lambda_k)$$

corresponds to the representation

$$(t_1, t_2, \dots, t_k) \mapsto \left[\prod_{i=1}^{k} t_i^{\lambda_i} \right].$$

We'll write this as t^λ, which has the nice effect of making $t^\lambda t^\mu$ equal to $t^{\lambda+\mu}$.

2. Convolving weight diagrams

If V, W are reps of T, it's easy to compute the representation $V \otimes W$. Let $V = \oplus_\lambda V^\lambda$ denote the decomposition into weight spaces, likewise for W. Then

$$V \otimes W = (\oplus_\lambda V^\lambda) \otimes (\oplus_\mu W^\mu) = \oplus_{\lambda,\mu} V^\lambda \otimes W^\mu.$$

How does T act on one of these pieces? If $\vec{v} \in V^\lambda, \vec{w} \in W^\mu$,

$$t \cdot (\vec{v} \otimes \vec{w}) = (t \cdot \vec{v}) \otimes (t \cdot \vec{w}) = (t^\lambda \vec{v}) \otimes (t^\mu \vec{w}) = t^{\lambda+\mu} (\vec{v} \otimes \vec{w})$$

so the weights add. In particular,

$$(V \otimes W)^\nu = \oplus_{\lambda+\mu=\nu} V^\lambda \otimes W^\mu.$$

On the level of weight diagrams, this is convolution. On the level of characters, it is just multiplication. Which picture you should use depends on whether you think of multiplying Laurent polynomials as a pointwise multiplication of functions, or distributing over monomials and collecting terms. The relation between the two is the Fourier transform for tori.

3. Strongly dominated representations of $U(n)$

Let V be a representation of $U(n)$. Say that V is **strongly dominated by the weight** λ if

- every weight μ of V has
$$\sum_{i=1}^{n} \mu_i = \sum_{i=1}^{n} \lambda_i.$$
- every weight μ of V has
$$\sum_{i=1}^{k} \mu_i \leq \sum_{i=1}^{k} \lambda^i,$$
for $k = 1 \dots n$.
- the dimension of λ's weight space is 1.

Example. *The representations* $\mathrm{Sym}^m(\mathbb{C}^2) \otimes \det^n$ *and* $(\mathbb{C}^2)^{\otimes m} \otimes \det^n$ *are both strongly dominated by* $(m+n, n)$.

One of our goals will be to show that all irreps of $U(n)$ are strongly dominated. Condition #1 is the easiest:

Proposition. *Let V be an irrep of $U(n)$, λ, μ two weights of it. Then $\sum_{i=1}^n \mu_i = \sum_{i=1}^n \lambda_i$.*

Proof. Homework problem. $\square$

Example. *The representation $\mathrm{Alt}^k\mathbb{C}^n$ is strongly dominated by the weight $(1, 1, \ldots, 1, 0, 0, \ldots, 0)$ with k ones. (Also a homework problem.)*

Strongly dominated reps will lead us to irreps:

Theorem. *Let a $U(n)$-rep V be strongly dominated by λ. Then V contains a unique irrep also strongly dominated by λ (and contains just one copy).*

Proof. Let $V = \oplus_r V_r$ be a decomposition into irreducibles, where r runs over some indexing set. Inside each, let $(V_r)^\lambda$ denote the λ weight space.

Since each V_r is $U(n)$-invariant, it is T-invariant. So

$$V^\lambda = \oplus_r (V_r)^\lambda$$

so exactly one of these can be positive-dimensional, and then must be 1-dimensional. The other conditions for strong domination are inherited from V. $\square$

This gives us a healthy supply of irreps, once we can make enough strongly dominated representations. What constitutes "enough"?

Theorem. *Let V be strongly dominated by $\lambda = (\lambda_1, \ldots, \lambda_n)$. Then the $\{\lambda_i\}$ are a weakly decreasing sequence.*

Proof. If λ is not weakly decreasing, then there exists a permutation π to rearrange it into a decreasing sequence. Let π also denote the permutation matrix (an element of $U(n)$).

Then $\pi \cdot V^\lambda = V^{\pi\cdot\lambda}$. But obviously $\pi \cdot \lambda$'s partial sums beat those of λ, contradiction. $\square$

Call λ a **dominant weight (of $U(n)$)** if it is weakly decreasing. Another of our goals is to show that every dominant weight does actually strongly dominate some irrep of $U(n)$.

Proposition. *If V, W are strongly dominated by λ,μ respectively, then $V\otimes W$ is strongly dominated by $\lambda + \mu$.*

Proof.

$$(V\otimes W)^\nu = \oplus_{a+b=\nu} V^a \otimes W^b$$

If ν as a weight of $V\otimes W$, then there exist a with partial sums beaten by λ, b with partial sums beaten by μ, such that $a + b = \nu$. Therefore ν's partial sums are beaten by $\lambda + \mu$.

For the equality, note that if the center acts by scalars on V and W, then it will do so on $V\otimes W$.

Finally,

$$(V\otimes W)^{\lambda+\mu} = \oplus_{a+b=\lambda+\mu} V^a \otimes W^b$$

happens only if $a = \lambda$, $b = \mu$. $\square$

Lemma. *For each dominant λ, there exists an irrep strongly dominated by λ.*

Proof. Homework problem: construct a rep strongly dominated by λ. Then it contains an irrep also strongly dominated by λ. $\square$

4. The classification by highest weights

Theorem. *Fix n. For each dominant $\lambda \in \mathbb{Z}^n$, there exists a unique irrep strongly dominated by it. These are all the irreps of $U(n)$.*

Proof. Pick an irrep V_λ for each dominant λ (since we know they exist by the previous lemma). We will show that any rep W is isomorphic to a direct sum of these. Let D be the multiplicity diagram for W.

Pick a decreasing sequence $x = (x_1 > x_2 > \ldots > x_n)$ of reals that are linearly independent over the rationals. Then the dominant weights with a given sum are well-ordered by their dot product with x.

Let μ be a weight in d having highest dot product with x, and m its multiplicity. Call it the "top weight" (nonstandard notation, depends on x). By the S_n-symmetry argument from before, μ is dominant. Subtract m times the weight diagram for V_μ from d. This new d

- has zero μ-multiplicity
- is still S_n-symmetric
- has a top weight that is smaller in the well-ordering.

Now apply the same procedure to the new d, and repeat; eventually we get to zero. That shows that W's character was a linear combination of the characters of the V_λ, and therefore it was isomorphic to the corresponding direct sum. $\square$

(One doesn't really have to pick the x; it is enough to partially order by dot product with $(n, n-1, n-2, \ldots, 1)$. Even though it's just a partial order, it's still "well".)

To restate:

- Every irrep of $U(n)$ is strongly dominated by some weight $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n)$ (a "dominant weight"). This is called the **highest weight** of the representation.
- Every dominant weight appears as the highest weight of an irrep.
- Two irreps are isomorphic if and only if they have the same highest weight.

5. Reps of $GL_n(\mathbb{C})$

So far we've only thought about rep theory in terms of topology, i.e. our maps $G \to \mathrm{End}(\mathbb{C}^n)$ have been required to be continuous. But if our group G has more structure, like being a subgroup of $\mathrm{End}(\mathbb{C}^n)$, we can talk about "polynomial representations" or "rational representations".

If $G \leq \mathrm{End}(\mathbb{C}^n)$, call a representation $\phi : G \to \mathrm{End}(\mathbb{C}^k)$ **polynomial** if the matrix entries of $\phi(g)$ are polynomial functions of the entries of g. Call the representation **rational** if they are rational functions of the entries of g (ratios of polynomials).

Example. The representation $\mathrm{Alt}^k\mathbb{C}^n$ is a polynomial representation of $GL_n(\mathbb{C})$. The representation $\det^k$ is a rational representation of $GL_n(\mathbb{C})$, and is polynomial only if $k \geq 0$. The representation $GL_n(\mathbb{C}) \to GL_n(\mathbb{C})$, $M \mapsto \overline{M}$ is not rational.

Theorem. *Every representation of* $U(n)$ *is the restriction of a rational representation of* $GL_n(\mathbb{C})$.

Proof. It's enough to check on irreps, and we know how to make all of those out of the Alt^ks and $\det$. □

In fact it's the restriction of a unique representation of $GL_n(\mathbb{C})$, a fact we will see via Lie algebras.

Homework questions.

1. Let V be an irrep of $U(n)$, λ, μ two weights of it. Then $\sum_{i=1}^{n} \mu_i = \sum_{i=1}^{n} \lambda_i$. (Hint: consider the action of the center of $U(n)$, the scalar matrices.)

2. Show that the representation $\mathrm{Alt}^k \mathbb{C}^n$ of $U(n)$ is strongly dominated by the weight $(1, 1, \ldots, 1, 0, 0, \ldots, 0)$ with k ones.

3. Given a dominant weight λ, construct a representation strongly dominated by λ, built up out of the Alt^ks.

4. Assume the irreps of $U(2)$ are of the form $\mathrm{Sym}^m \mathbb{C}^2 \otimes \det^k$, $m \in \mathbb{N}, k \in \mathbb{Z}$. Into which irreducible representations does $(\mathrm{Sym}^a \mathbb{C}^2) \otimes (\mathrm{Sym}^b \mathbb{C}^2)$ decompose? (You don't have to actually find them inside there – just say which ones, and how many times.)