FROM COMPACT GROUPS TO ROOT SYSTEMS
NOTES FOR MATH 261, FALL 2001

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1. Generalities on compact connected Lie groups

Throughout, let K be a compact connected Lie group.

Theorem (Oracular fact #1). *There is a K-equivariant map $\exp : \mathfrak{k} \to K$ from the adjoint representation to the group acting on itself by conjugation, and its image is dense.*

Proof. There are a couple of different ways to see this. One is to put a K-invariant symmetric definite form on $\mathfrak{k}$, thereby a $K \times K$-invariant Riemannian metric on K, and use the geodesic spray to define the exponential map (as usual in Riemannian geometry). Another is to embed K in $U(n)$ via the Peter-Weyl theorem and restrict the standard exponential from $U(n)$.

(In fact for compact groups K, which is all we discuss, the map is onto.) $\square$

Corollary. *The kernel of the adjoint representation $\mathfrak{k}$ is the center Z of K.*

Proof. Since Z acts trivially on K by conjugation, it obviously acts trivially on $\mathfrak{k}$. Conversely, if $k \in K$ acts trivially on $\mathfrak{k}$, then it fixes a dense subset of K (by the exponential map), and by continuity it fixes all of K, so it's in the center. $\square$

Theorem (Oracular fact #2). *A closed subgroup of a Lie group is Lie.*

(This, we aren't going to prove right now.)

2. Maximal tori

Let T a **maximal torus** of K: a subgroup of K isomorphic to $(S^1)^n$, with n maximized. Our main example so far has been $K = U(n)$, T the diagonal unitary matrices. Note that this is *not* the same as a maximal abelian subgroup: the diagonal matrices in $SO(3)$ are a $Z_2 \times Z_2$ that do not live in any maximal torus (as those are $\cong S^1$).

Define the **rank** of K as the dimension of T. (In particular, if $K = T$, then $\operatorname{rank} T = \dim T$.) This is usually denoted l.

Define the **roots** Δ of K as the nonzero weights of the complexified adjoint representation $\mathfrak{k} \otimes \mathbb{C}$ of K on its Lie algebra, restricted to T. (We complexify only because we understand complex reps of T better than real ones.) This is a finite subset of the weight lattice T^*.

Lemma. *Let $\mathfrak{z}$ denote the Lie algebra of the center Z of K. Then $\mathfrak{z} \leq \mathfrak{t}$, and its perp inside $\mathfrak{t}^*$ is the linear span of the roots Δ.*

In particular, the roots span $\mathfrak{t}^*$ if and only if Z is discrete.

Proof. Let Z be the center of K, and Z_0 the identity component. Then we claim that $\mathsf{Z}_0 \leq \mathsf{T}$. For otherwise, we could take the closure of the group generated by $\mathsf{Z}_0 \cup \mathsf{T}$, which would be a compact, connected, abelian *Lie* subgroup, therefore a torus properly containing T. But this contradicts T's maximality.

In particular, $\mathsf{Z}_0 \leq \mathsf{Z} \cap \mathsf{T} \leq \mathsf{Z}$. On the Lie algebra level, this says $\mathfrak{z}_0 \leq \mathfrak{z} \cap \mathfrak{t} \leq \mathfrak{t}$. But $\mathfrak{z}_0 = \mathfrak{z}$. So we get $\mathfrak{z} \leq \mathfrak{t}$.

Now consider the kernel of T's action on $\mathfrak{k}\otimes\mathbb{C}$. This is just $\mathsf{T} \cap \mathsf{Z}$, so its Lie algebra is $\mathfrak{t} \cap \mathfrak{z} = \mathfrak{z}$. Put another way, the kernel of $\mathfrak{t}$'s action is $\mathfrak{z}$.

But we can calculate this kernel also from the roots Δ:

$$\mathfrak{z} = \ker = \bigcap_{\alpha\in\Delta} \alpha^{\perp} = \left(\bigoplus_{\alpha\in\Delta} \mathbb{R}\alpha\right)^{\perp}$$

(Here the $\perp$ of a root $\alpha \in \mathfrak{t}^*$ is a hyperplane in $\mathfrak{t}$.)

So the roots span $\mathfrak{t}^*$ if and only if $\mathfrak{z}$ is zero, which is true if and only if Z is discrete. $\square$

(In fact Z, not just Z_0, is always inside T, but this is not easy to show, since the example in $SO(3)$ above shows it's not true for arbitrary abelian subgroups.)

Proposition. *The multiplicity of the root $\alpha \in \Delta$ is the same as that of $-\alpha$.*

Proof. At this point we really need to think about T's action on $\mathfrak{k}\otimes\mathbb{C}$, and the hard part is to understand T's irreducible *real* representations (and how they complexify). Start with T acting on the real vector space $\mathfrak{k}$. This has a 1-d fixed subspace, $\mathfrak{t}$ (which complexifies to a complex 1-d fixed subspace). The rest breaks up into $\mathbb{R}^2$'s. Any one of them can be identified T-equivariantly with $\mathbb{C}$, acted on with some weight α. When we complexify, this irreducible action on $\mathbb{R}^2$ complexifies to the *reducible* representation $\mathbb{C}_\alpha \oplus \mathbb{C}_{-\alpha}$. Therefore the number of times $\mathbb{C}_\alpha$ shows up in $\mathfrak{k}\otimes\mathbb{C}$ is the same as the number of times $\mathbb{C}_{-\alpha}$ does. $\square$

(One of our goals is to show that this number is only 0 or 1.)

3. The Levi subgroups

Given a root $\alpha \in \Delta$, let $\ker\alpha$ denote the codimension-one subgroup of T. Note that it is *not* necessarily connected. Then define

$$\mathsf{K}_\alpha := \mathsf{C}_\mathsf{K}(\ker\alpha)$$

as the centralizer of this subgroup, i.e. the fixed-point set of the conjugation action of $\ker\alpha$ on K. This is called the **Levi subgroup** corresponding to α.

If $\mathsf{A} \leq \mathsf{B}$ is a subgroup, denote by $\mathsf{N}_\mathsf{B}(\mathsf{A})$ its normalizer in B.

Proposition. *Let $\mathsf{K}, \mathsf{T}, \Delta, \mathsf{K}_\alpha$ be as above.*

1. $\mathsf{K}_\alpha \geq \mathsf{T}$. *In particular K_α's roots are a subset of Δ.*
2. *The roots of K_α are $\Delta \cap \mathbb{Z}\alpha$.*
3. *Let $\phi : \mathsf{K}_\alpha \to \mathsf{K}_\alpha/\ker\alpha$. Then $\mathsf{S} := \phi(\mathsf{T})$ is a maximal torus of $\mathsf{K}_\alpha/\ker\alpha$, and $\mathsf{N}_\mathsf{K}(\mathsf{T}) = \phi^{-1}(\mathsf{N}_{\mathsf{K}_\alpha/\ker\alpha}(\mathsf{S}))$.*
4. S *is a circle, i.e. $\mathsf{K}_\alpha/\ker\alpha$ is a rank one group.*

Proof. 1. Since T is commutative, it centralizes any of its subgroups, so $T \leq K_\alpha$. Then $\mathfrak{k}_\alpha \otimes \mathbb{C}$ is a T-subrepresentation of $\mathfrak{k} \otimes \mathbb{C}$, and as such we have the containment on the weights.

2. The group $\ker \alpha$ acts trivially on K_α by conjugation, but the only weight spaces it acts trivially on are the multiples of α.
3. The action of T on $\mathfrak{k} \otimes \mathbb{C}$ descends to an action of $T/\ker \alpha$ on $\mathfrak{k}_\alpha/(\alpha^\perp) \otimes \mathbb{C}$, whose zero weight space is $\mathfrak{t}/(\alpha^\perp) \otimes \mathbb{C}$. So $S := T/\ker \alpha$ (obviously a torus) doesn't centralize any other subspace of $\mathfrak{k}/(\alpha^\perp)$. Hence it is a maximal torus.

   For the other statement, let $n \in K$:
   $$nTn^{-1} = T \implies \phi(n) S \phi(n)^{-1} = S$$
   therefore $\phi(N_K(T)) \leq N_{K/\ker \alpha}(S)$. Conversely, if $\phi(n)$ conjugates S to itself, then n conjugates $\phi^{-1}(S)$ to itself. Then only subtle point is that $\phi^{-1}(S)$ may be larger than T (and will be if $\ker \alpha$ is not connected), although dimensionally they must be the same.

   Therefore T is the identity component of $\phi^{-1}(S)$, and since n preserves $\phi^{-1}(S)$, it must preserve its identity component.
4. Since $\ker \alpha$ is codimension one in T, the quotient S is 1-dimensional.

□

Therefore we want to get a handle on these rank one groups.

4. Rank one groups

We want to prove the following:

Theorem. *Let K be a compact connected Lie group with a 1-dimensional maximal torus T. Then $K \cong S^1$, $SU(2)$, or $SO(3)$.*

In this section, K will always be such a group. We start with an easy case:

Proposition. *If K is three-dimensional, then $K \cong SU(2), SO(3)$.*

Proof. The adjoint action of K on itself is a map $K \to GL(\mathfrak{k}) \cong GL(\mathbb{R}^3)$, with kernel $Z(K)$. If we pick an invariant inner product on this space, the map lands inside $O(3)$. But since K's connected, it actually lands inside $SO(3)$.

If this real representation is reducible (hint: it's not), then it's got a 1-d invariant subspace. In that representation K is mapping to $SO(1) = 1$, i.e. it's the trivial representation. So it corresponds to a 1-dimensional central subgroup H of K. Pick a maximal torus S of the quotient K/H, and lift a Lie algebra element of it to $\mathfrak{k}$. That generates another circle in K, commuting with H, making a rank 2 torus (at least) in K, contradiction.

Therefore Z is finite, and the map factors as $K/Z \to SO(3)$, with image a 3-dimensional group, therefore all of $SO(3)$. So the map $K \to SO(3)$ is a covering space.

If we consider the case $SU(2)$ (which is simply connected), we get a double cover of $SO(3)$. Therefore $SO(3)$'s fundamental group is Z_2, and its only cover is S^3. It is easy to show that the group structure on the universal covering space of a connected Lie group is unique, so $SU(2)$ is the only possibility. □

Consider now the complexified adjoint representations of $SU(2), SO(3)$. Identify the maximal torus with S^1, so we can consider Δ as a subset of $\mathbb{Z}$.

Start with $SU(2)$. Then $\mathfrak{su}(2) \otimes \mathbb{C} \cong \mathfrak{sl}(2, \mathbb{C})$ has a basis $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, with weights $2, 0, -2$.

This action factors through $SO(3)$, but we have to use a different identification of $SO(3)$'s (different) maximal torus with S^1, and all the odd weights of $SU(2)$ disappear (they aren't weights of $SO(3)$'s torus). So $SO(3)$'s weights are $1, 0, -1$.

Call a 3-dimensional subgroup H of K a **high root subgroup** if $H \geq T$, H's roots are $\pm n$, and K's are contained in $[-n, n]$. This definition is used more generally than K being rank 1, where it turns out to be stupid:

Proposition. *If K has a high root subgroup H, then $K = H$.*

Proof. Consider $\mathfrak{k} \otimes \mathbb{C}$ as a representation of H, which we already know to be isomorphic to either $SO(3)$ or $SU(2)$.

If $H \cong SO(3)$, then $n = 1$, and Δ's only possible weights are $1, 0, -1$. So $\mathfrak{k} \otimes \mathbb{C}$ is a sum of a copies of $SO(3)$'s 3-d rep and b of its trivial rep, giving it a zero weight space of dimension $a + b$. But since T only centralizes itself, $a + b = 1$. So $a = 1$, $b = 0$, and K is three-dimensional. Therefore $K = H$.

If $H \cong SU(2)$, then $n = 2$, and it's slightly trickier. Now $\mathfrak{k} \otimes \mathbb{C}$ is a sum of a copies of $SU(2)$'s 3-d rep, b of its trivial rep, and c of its 2-d rep (which doesn't descend to $SO(3)$). By the previous analysis $a = 1, b = 0$, but it may still be that the weight spaces $\mathbb{C}_1 \oplus \mathbb{C}_{-1}$ is c copies of the standard representation of $SU(2)$ acting on $\mathbb{C}^2$.

However, that action does not arise as a complexification (i.e. $SU(2)$ isn't a subgroup of $SO(2)$!), contradiction. Hence $c = 0$, K is three-dimensional, so $K = H$. □

So it remains to show that K does indeed have a high root subgroup (unless it's S^1, of course). We do this via the Lie algebra, in a couple of steps.

Proposition. *Let n be the highest root in Δ. Let $V \leq \mathfrak{k}$ be a real T-irrep, complexifying to $\mathbb{C}_n \oplus \mathbb{C}_{-n}$. Then $V \oplus \mathfrak{t}$ is a Lie subalgebra.*

This almost deserves to be called a "high root subalgebra"; unfortunately its exponential might not be closed, and the closure might be more than three-dimensional. (Of course, this doesn't actually happen – in reality this is the whole algebra!) So even with this, we won't be done.

Proof. Consider $\mathfrak{g} := \mathfrak{k} \otimes \mathbb{C}$ as a complex Lie algebra, containing $V_{\mathbb{C}} := V \otimes \mathbb{C}$. It is enough to show that $V_{\mathbb{C}} \oplus \mathfrak{t} \otimes \mathbb{C}$ is a subalgebra of $\mathfrak{g}$.

Let $\mathfrak{g}_\alpha$, $\mathfrak{g}_\beta$ be two root spaces in $\mathfrak{g}$. Since the Lie bracket $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ is K-equivariant, it's T-equivariant, and takes $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \hookrightarrow \mathfrak{g}_{\alpha+\beta}$. For us, the important inclusion is $[\mathfrak{g}_n, \mathfrak{g}_{-n}] \leq \mathfrak{g}_0$ which is $\mathfrak{t} \otimes \mathbb{C}$, and 1-d.

Then for any weight vector $\vec{v} \in \mathfrak{g}_k$ we have $[\mathfrak{g}_0, \vec{v}] = \mathbb{C}\vec{v}$. In particular, writing $V_{\mathbb{C}} = \mathbb{C}\vec{v}_n \oplus \mathbb{C}\vec{v}_{-n}$ (its decomposition into weight spaces), we see that $[\vec{v}_n, \vec{v}_{-n}] \in \mathfrak{g}_0$, and therefore $\mathfrak{g}_0 \oplus \mathbb{C}\vec{v}_n \oplus \mathbb{C}\vec{v}_{-n}$ is closed under bracket. □

Proposition. *Let $V \oplus \mathfrak{t} \leq \mathfrak{k}$ be a high root subalgebra as in the last proposition, and H its stabilizer under the conjugation action. Then H is a high root subgroup.*

Proof. First off, H is at least 3-d, since it contains T and $\exp(V)$. Second, H maps to $O(V \oplus \mathfrak{t}) \cong O(3)$. If the kernel isn't finite, then it contains a circle. Lifting a Lie algebra generator of $O(3)$'s maximal torus and blah blah blah, we construct a 2-torus inside H, no good, therefore the kernel was finite, so H is three-dimensional. $\square$

We sum up:

Proof of the theorem. Let K be a rank one group. Look at $\Delta \subseteq [-n, n]$ (where n is the highest root). Then there is a 3-d subalgebra complexifying to the $0, \pm n$ root spaces, and it exponentiates to a 3-d subgroup (by checking its stabilizer). That subgroup must be a cover of $SO(3)$, and there's only two of those, namely $SO(3)$ and $SU(2)$. Then since we know those guys' representation theory, we see that $\mathfrak{k}$ can't have any other components. $\square$

5. Corollary: root systems

We're back to thinking about K arbitrary compact connected, but now we know a lot about its Levi subgroups K_α; when we quotient them by a central subgroup we get $SU(2)$ or $SO(3)$.

Lemma. *Let L be a compact, connected group, H a subgroup, such that L/H is rank one non-abelian. Then L's Lie algebra contains a subalgebra $\cong \mathfrak{su}(2)$ (in fact it is the commutator subalgebra).*

(This is pretty lame – in fact the commutator sub*group* L' of L is $SU(2)$ or $SO(3)$. Even better, $L \cong (L' \times H)/\Gamma$ where Γ is either trivial or Z_2.)

Proof. Let $\mathfrak{g} := \mathfrak{l} \otimes \mathbb{C}$. As a representation of T, its weights are $0, \pm n$ for $n = 1$ or 2, with multiplicities 1 at the weights $\pm n$, and $1 + \dim H$ at the weight 0. Let $\mathfrak{s}_\mathbb{C} := [\mathfrak{g}_n, \mathfrak{g}_{-n}]$. Then it is trivial to see that $\mathfrak{s}_\mathbb{C}, \mathfrak{g}_n, \mathfrak{g}_{-n}$ generate a subalgebra, and not very difficult to isomorph it to $\mathfrak{sl}(2, \mathbb{C})$.

Back in the real picture, instead of $\mathfrak{g}_{\pm n}$ we have a H-fixed $\mathbb{R}^2$, whose commutator defines a preferred line $\mathfrak{s} \leq \mathfrak{h}$, and that 3-space is our subalgebra $\mathfrak{su}(2)$. $\square$

Theorem (Oracular fact #3). *Every finite-dimensional representation of $\mathfrak{su}(2)$ comes from a unique representation of $SU(2)$.*

This isn't very hard, but we won't take the time for it. (The much better, harder, statement is that the same holds true for any connected simply-connected Lie group, compact or not.)

Theorem. *Pick a K-invariant positive definite bilinear form $(,)$ on $\mathfrak{k}$, and therefore on $\mathfrak{t}$ and $\mathfrak{t}^* \supset \Delta$. Then if $\alpha \in \Delta$ is a root,*

1. *the multiplicity of α is 1*
2. $\Delta \cap \mathbb{R}\alpha = \pm\alpha$
3. *if $\beta \in \Delta$ is another root, the reflection*
$$\beta - 2\frac{(\alpha, \beta)}{(\alpha, \alpha)}\alpha$$
of β through the hyperplane $\alpha^\perp$ is a root.
4. *that number $2\frac{(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer*

5. *each weight* $\beta - k\alpha$*, for* $k \in \left[0, 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\right]$*, is a root. ("Root strings are unbroken.")*

Proof. The first two of these just follow from the fact that $K_\alpha/\ker\alpha$ is a rank 1 group, and therefore $SU(2)$ or $SO(3)$, whose root systems we know. (In fact can only be $SO(3)$.)

For the third, pick a lift $n \in N(T)$ of the element

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

in $N_{K_\alpha/\ker\alpha}(S) \cong SO(3)$, where we have picked a basis so that S is the upper left $SO(2)$. This lift w normalizes T (as we proved before), preserves the invariant form, acts trivially on $Z(K_\alpha) \geq \ker\alpha$, and takes $\alpha \mapsto -\alpha$. Therefore it is the reflection in the hyperplane $\alpha^\perp$. (It's easy to see this must be the formula for the reflection – it takes $\alpha \mapsto -\alpha$, and fixes the hyperplane $\{\beta : (\beta, \alpha) = 0\}$.)

For the last two, use the fact that $\mathfrak{k}\otimes\mathbb{C}$ is a representation of the $\mathfrak{su}(2)$ found in the lemma. □

This suggests the definition of **abstract root system**: a subset Δ of a positive-definite real inner product space satisfying 2, 3, and 4 in the above theorem. (It turns out that 5 comes for free.)