(HALF OF) BOREL-WEIL FOR COMPLEXIFICATIONS OF COMPACT GROUPS NOTES FOR MATH 261, SPRING 2002

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This is so close to how we did things in $U(n)$ as to be essentially a review.

Throughout, K is a compact connected Lie group. Also fix a maximal torus T, and therefore a Weyl group $W = N(T)/T$, a complexification G homotopic to K,[1] a positive root system Δ_+ and therefore a Borel subgroup B and its commutator subgroup N.

Theorem (Weyl's unitary trick)**.** *Let K be compact, simple, simply connected. Each complex representation of K, i.e. a smooth homomorphism $K \to GL_n(\mathbb{C})$, extends uniquely to a representation of G, i.e. a complex-analytic homomorphism $G \to GL_n(\mathbb{C})$.*

Note that G will have plenty more smooth representations (forgetting G's complex structure). Also, each rep of G obviously restricts to one of K. So this says that there's a perfect correspondence.

Proof. A homomorphism $\phi : K \to GL_n(\mathbb{C})$ differentiates to the real-linear map $\mathfrak{k} \to \mathfrak{gl}_n(\mathbb{C})$, which extends uniquely to a complex-linear map $\mathfrak{g} \to \mathfrak{gl}_n(\mathbb{C})$. Uniqueness is now clear.

Assume the representation is nontrivial. Then it is faithful on the Lie algebra level. So $\phi(\mathfrak{g})$ is a centerless Lie algebra isomorphic to $\mathfrak{g}$.

We already know that G and K are homotopy equivalent, so G is simply connected. Therefore G covers the subgroup of $GL_n(\mathbb{C})$ with Lie algebra $\phi(\mathfrak{g})$. ☐

Exercise. Extend to the case that K has only discrete center. Then extend to the case of general K, but the complexification G still homotopic to K.

1. Dominant weights

Recall that given $X \in \mathfrak{t}$ with no W-stabilizer, we can use it to define a positive system Δ_+, and a positive Weyl chamber $\mathfrak{t}^*_+$. Call the weights in $\mathfrak{t}^*_+$ **dominant weights**.

Proposition. *Let V be a representation of G, and λ a weight of V maximizing $\langle \lambda, X\rangle$ (for X defined as above). Then λ is dominant.*

Also, each weight vector of weight λ is annihilated by all of $\mathfrak{n}$.

This first part is tricky, because of the subtle way we defined the positive Weyl chamber in $\mathfrak{t}^*$, as the only Weyl chamber containing the Weyl vector ρ. The second part is just a review of the $GL_n(\mathbb{C})$ proof.

Proof. The set of weights of V is a W-invariant subset of $\mathfrak{t}^*$, essentially because V is an $N(T)$-representation.

[1] We showed that if K is semisimple (discrete center), then a complexification G exists and is automatically homotopic to K. But sometimes we get lucky even if K not semisimple, like $U(n) \hookrightarrow GL_n(\mathbb{C})$.

Consider the interval from ρ to λ, $\{t\rho + (1-t)\lambda\}_{[0,1]}$; we want to show it lies entirely inside a Weyl chamber. Otherwise, for some $t \in (0,1)$ it intersects a wall of the positive Weyl chamber, which is to say $t\rho + (1-t)\lambda$ is invariant under r_α for α a simple root.

$$t\rho + (1-t)\lambda = r_\alpha \cdot (t\rho + (1-t)\lambda) = t r_\alpha \cdot \rho + (1-t) r_\alpha \cdot \lambda = t\rho - t\alpha + (1-t) r_\alpha \cdot \lambda$$

so

$$r_\alpha \cdot \lambda = \lambda + \frac{t}{1-t}\alpha$$

Since $t \in (0,1)$, this coefficient $\frac{t}{1-t} > 0$, so $r_\alpha \cdot \lambda$ is a weight of V with higher dot product with X. This contradicts its maximality in the set of weights.

To show the $\mathfrak{n}$-annihilation, it's enough to check on positive root spaces, and each of those raises the weight by the root (this is the statement that the action map $\mathfrak{n} \otimes V \to V$ is T-equivariant) unless it takes the vector to $\vec{0}$. Therefore it raises the dot product with X, which is impossible, or it takes it to zero. □

Given a rep V of the complex group G, call a vector $\vec{v} \neq \vec{0}$ a *high weight vector* if it is a weight vector ($\mathbb{C}\vec{v}$ is T-invariant), and annihilated by $\mathfrak{n}$. On the group level, this is equivalent to the weight vector $\vec{v}$ being N-invariant. In particular, the above proposition shows that every nonzero V has some high weight vectors.

We now abandon the purely algebraic side of the rep theory for a while to study the geometry.

2. Existence: all irreps come from line bundles

Theorem. *Let V be an irrep of G. Then V arises as a space of holomorphic sections of some G-equivariant line bundle over G/B.*

This doesn't say yet that it arises as the space of all sections (though that will turn out to be true).

Proof. Let $\mathbb{C}\vec{v}$ be a line of high weight vectors in V^*, and consider the G-equivariant map $G \to \mathbb{P}(V^*)$ taking $g \mapsto g \cdot [\vec{v}]$. Since $\vec{v}$ is N-invariant, this descends to a map $G/N \to \mathbb{P}(V^*)$; since $\mathbb{C}\vec{v}$ is projectively T-invariant, it actually descends to a map $\phi : G/B \to \mathbb{P}(V^*)$.

From there, we can pull back the antitautological line bundle $\mathcal{O}(1)$ on projective space, and any sections. We already know that the space of sections $\Gamma(\mathbb{P}(V^*); \mathcal{O}(1)) \cong V$ as a G-rep. That gives an equivariant map from $V \to \Gamma(G/B; \phi^*(\mathcal{O}(1)))$.

Since the elements of V separate the points of $\mathbb{P}(V^*)$, and in particular the points of the image of ϕ, this map of G-reps is nonzero. Since V is irreducible, it's an injection. □

3. Irreducibility of $\Gamma(G/B; \mathcal{L})$

In the $GL_n(\mathbb{C})$ case this followed from N having a dense orbit on G/B, which we now know to be a general phenomenon.

Proposition. *If $\mathcal{L}$ is a G-equivariant holomorphic line bundle on G/B, then the space of sections $\Gamma(G/B; \mathcal{L})$ is either 0 or has a 1-dimensional space of high weight vectors.*

Proof. Let p be a point in the big cell Nw_0B/B, and $\vec{v}_p$ a point in the line over p (this is the 1-d choice). Then there is a unique N-invariant section of $\mathcal{L}$ over the orbit, passing through the point $\vec{v}_p$. Since the orbit is dense, there is at most one way to extend it the section the one over all G/B. This shows there is at most a 1-dimensional space of N-invariant vectors.

Conversely, if the space of sections is nonzero, then it is a rep of G and those have high weight vectors. □

Corollary. *Every irrep of G has only a 1-d space of high weight vectors. For each G-equivariant holomorphic line bundle $\mathcal{L}$ on G/B, the space of sections is either 0 or an irrep of G. In particular, each irrep V arises as the full space of holomorphic sections of an equivariant line bundle on G/B.*

Proof. We already know that for each irrep V, there is a line bundle $\mathcal{L}$ on G/B such that $V \hookrightarrow \Gamma(G/B; \mathcal{L})$. Since the latter has only a 1-d space of high weight vectors, and V has at least a 1-d space thereof, but the map is injective, we find that V has *only* a 1-d space of high weight vectors.

Now given any line bundle $\mathcal{L}$, if the space of sections $\Gamma(G/B; \mathcal{L})$ were reducible, it would be a sum of several irreps each with their own 1-space of N-invariant vectors. But we know it only has a 1-d space thereof (if nonzero), so it must be irreducible. □

Note that the first claim in this corollary is a purely algebraic statement, so the geometry is teaching us something about the representation theory.

Also, now that we know that each irrep has only a 1-d high weight space, we can talk about the **highest weight of an irrep** V. Then the proposition from the last section says that highest weights are necessarily dominant, i.e. lie in the positive Weyl chamber.

4. Classifying line bundles on G/B

Recall that T is our compact torus, TA its complexification, and B is the semidirect product of TA and N.

Lemma. *The 1-d holomorphic reps of B correspond to the 1-d smooth reps of T (the correspondence given by restriction of reps).*

Proof. It's easy to extend any T-representation to a complex B-representation; the T-rep induces a $\mathfrak{t}$-rep, therefore an $i\mathfrak{t}$-rep, therefore an A-rep, making a holomorphic TA-representation. Then let N act trivially, and we have an action of B.

It remains to show that every 1-d rep of B actually factors through B/N. A 1-d rep of B is a map $B \to GL_1(\mathbb{C})$, an abelian group, hence it factors through the abelianization B/B'. We haven't calculated the commutator subgroup B' before, but it plainly contains the Lie group corresponding to the commutator subalgebra, which is $\mathfrak{n}$. □

Let V be a representation of B (which will again be 1-d in a moment). On $G \times V$ there is a diagonal action of B given by $b \cdot (g, \vec{v}) := (gb^{-1}, b \cdot \vec{v})$. Define the **associated vector bundle** $(G \times V)/B_{diag}$ as the quotient by this action. We now review some standard material on associated vector bundles.

Before quotienting, this space $G \times V$ had a well-defined map to the first factor G. Afterwards, we can't pick out an element of G in a well-defined way, but we still have a well-defined map to G/B. Also, since the action of B_{diag} commuted with the *left* action of G on G, the map $\pi : G \times^{B_{diag}} V \to G/B$ is G-equivariant. In particular, the fiber over a point $x \in G/B$ carries an action of $\mathrm{Stab}_G(x)$.

What do the fibers look like? Ordinarily one proves that each fiber can be identified with V, though not in any canonical way (but the different ways differ by a linear automorphism of V, hence the fibers are naturally vector spaces).

In our case, since G is acting transitively on G/B, it's enough to look at one fiber, so we look at that over B.

$$\pi^{-1}(B) = \{(b, \vec{v})\}/B_{diag} = (B \times V)/B_{diag} \cong V$$

This is an identification not just as vector spaces, but as $B = \mathrm{Stab}_G(B)$ representations.

Proposition. *The G-equivariant vector bundles on G/B are determined by the B-representation on the fiber over $B \in G/B$. In particular, they correspond $1:1$ to B-representations.*

Proof. Let $\mathcal{V}$ be a G-equivariant vector bundle on G/B, and V the fiber over $B \in G/B$, considered as a B-representation. We define a map from $G \times V \to \mathcal{V}$, given by

$$(g, v) \mapsto g \cdot v, \quad \text{a vector in the fiber over } gB.$$

This obviously descends to a map from the associated vector bundle $(G \times V)/B_{diag}$. It's then simple, and dull, to check that it's $1:1$ and onto. ☐

Putting these two together, we get

Corollary. *The G-equivariant line bundles on G/B correspond $1:1$ to weights of T. The correspondence takes a line bundle $\mathcal{L}$ to the weight of the T-action on the fiber $\mathcal{L}|_B$ over the unique B-fixed point $B \in G/B$.*

Moreover, each such line bundle carries a unique holomorphic structure.

This last statment is just coming from the fact that these associated vector bundles obviously carry a unique holomorphic structure.

We now have three connections: irreps have dominant weights, irreps give line bundles on G/B, and line bundles on G/B correspond to *all* weights. This triangle commutes in the following slightly tricky way:

Lemma. *Let V be an irrep with highest weight λ, and $\mathcal{L}$ the pullback of $\mathcal{O}(1)$ from $\mathbb{P}(V^*)$. Then the T-action on the fiber $\mathcal{L}|_B$ over $B \in G/B$ has weight $w_0 \cdot \lambda$.*

Proof. We first determine the T-action on the fiber $\mathcal{L}|_{w_0 B}$. using the T-equivariant restriction map

$$V = \Gamma(G/B; \mathcal{L}) \to \Gamma(\{w_0 B\}; \mathcal{L}) = \mathcal{L}|_{w_0 B}.$$

(It's worth noting that this point is only T-invariant, not B-invariant, unlike the basepoint $B \in G/B$.) Let $\vec{v} \in V^N \setminus \{\vec{0}\}$ be a high weight vector. Then $\vec{v}$ doesn't vanish at $w_0 B$, or else it would vanish on the big cell, and therefore be zero. Hence its image generates $\mathcal{L}|_{w_0 B}$, which therefore has weight λ.

Exercise. Show that weight vectors for any other weight do vanish at $w_0 B$.

Since $\mathcal{L}$ is G-equivariant, it's $N(T)$-equivariant, and therefore the T-action on the fiber B has weight $w_0 B$ times that on the fiber B, i.e. $w_0 \lambda$. ☐

Again, this has a purely algebraic corollary:

Theorem. *Two irreps are isomorphic if and only if their highest weights agree. In particular, the set of irreps corresponds* $1:1$ *to a subset of the dominant weights. Also, this subset is closed under addition. (We will show later that it's actually equal to the set of dominant weights.)*

Proof. If two irreps have the same highest weights, then they give rise to the same line bundle on G/B, and are each isomorphic to the space of sections of that line bundle.

To see the closure under addition, start with λ, μ two dominant weights such that the corresponding line bundles have sections. Then we can tensor a section of one with a section of the other to get a section of the $\lambda + \mu$ line bundle. □

It's going to be harder to show that given a dominant weight, the corresponding line bundle on G/B does indeed have holomorphic sections. (This will complete Borel-Weil.)

In the $GL_n(\mathbb{C})$ case, we found a generating set for the cone of dominant weights – the $\{(1,1,\ldots,1,0,0,\ldots,0)\}$s and $(-1,-1,\ldots,-1)$ – and showed that each of these was indeed the high weight of an irrep (the $\{\mathrm{Alt}^k\mathbb{C}^n\}$s and $\det^{-1}$ irreps). Therefore the subcone of dominant weights of irreps was in fact the whole cone. It's possible to extend this proof to the other groups (and Cartan did this) but it requires a lot of creativity to find the analogues of the $\mathrm{Alt}^k\mathbb{C}^n$.

Exercise. Let G_2 be the automorphism group of the octonions, obviously compact (since it lives inside $SO(\mathbb{O})$). You may assume known that G_2's root system is the exceptional root system we found before, and that G_2's root lattice equals its weight lattice. Find the highest weights of the fundamental representations, and then find the representations.