THE BOTT-SAMELSON CRANKS AND BOTT-SAMELSON MANIFOLDS NOTES FOR MATH 261, SPRING 2002

ALLEN KNUTSON

Fix our usual notation (G, B, T, Δ_+). In this chapter, T is a *complex* torus, $(\mathbb{C}^\times)^n$.

1. The Bott-Samelson operations ∂^α

Let X be a space equipped with a map ϕ to G/B, and α a simple root. Then X has a forgetful map down to G/P_α, where P_α is the corresponding minimal parabolic. Define $\partial^\alpha X$ as the pullback of the diagram

$$\begin{array}{ccc} \partial^\alpha X & \longrightarrow & G/B \\ \downarrow & & \downarrow \\ X & \longrightarrow & G/P_\alpha \end{array}$$

Specifically, this means that

$$\partial^\alpha X := \left\{ (x, F) : \phi(x) = \overline{F} \right\}$$

where $F \in G/B$, and $\overline{F}$ is its image in G/P_α. In particular, since G/B is a $\mathbb{CP}^1$-bundle over G/P_α, $\partial^\alpha X$ is a $\mathbb{CP}^1$-bundle over X.

This is very easy to picture in the $GL_n(\mathbb{C})$ case. Each $x \in X$ is somehow being assigned a full flag. We then loosen that full flag by letting one subspace vary (which one, depends on α; there are $n-1$ choices). That subspace is still trapped between two others, defining a line in the 2-d quotient of one by the other, making only a $\mathbb{CP}^1$ freedom of choice.

We call α a **(right) Bott-Samelson operation**, on the category of spaces equipped with a map into G/B. (There will be left operations ∂_α later.)

There are a bunch of obvious properties to note:

1. If X is compact, so is $\partial^\alpha X$.
2. If X is smooth, so is $\partial^\alpha X$.
3. If some subgroup $H \leq G$ acts on X, and the original map $X \to G/B$ is H-equivariant, then the horizontal map $\partial^\alpha X \to G/B$ is too.

2. Iterating the Bott-Samelson crank

Define a **word** $(I, <, \alpha)$ **in the simple roots** as a set I totally ordered by $<$ with a map α to the simple roots, taking $i \mapsto \alpha_i$.

Then we define the **Bott-Samelson manifold**

$$BS^I := \left(\prod_{i \in I} \partial^{\alpha_i}\right) \cdot pt$$

where the original point pt had a map taking it to $B \in G/B$.

Proposition. *The Bott-Samelson manifold BS^I has a natural B-action.*

Proof. The map $pt \to G/B$ with image $\{B\} \subseteq G/B$ is B-equivariant w.r.t. the trivial B-action on the point. Then each turn of the Bott-Samelson crank preserves this. □

In the $GL_n(\mathbb{C})$ case it's easy to picture an element of a Bott-Samelson manifold. It's a long list of flags (the first being the base flag), such that each one agrees with the previous except possibly in one subspace. Taking the very last flag, but forgetting its origins, gives the map to the flag manifold. There's an obvious action of $GL_n(\mathbb{C})$ on this interconnected system of subspaces, preserving containment, but it doesn't preserve the base flag; that's why this is only a B-space really.

For example, take $\alpha_I = (1,2,1)$ for $G = GL_3$, and denote the base flag $0 < \mathbb{C}^1 < \mathbb{C}^2 < \mathbb{C}^3$. Then we're considering triples of subspaces (V_1, V_2, V_1') where $V_1 < \mathbb{C}^2$ (and $V_1 > 0$, which we needn't mention, except to stress that V_1 is 1-dimensional), $V_2 > V_1$, and $V_1' < V_2$. Then the map to the flag manifold takes

$$(V_1, V_2, V_1') \mapsto (0 < V_1' < V_2 < \mathbb{C}^3).$$

Exercise. Show that in this example:

1. the map is onto the flag manifold
2. it is also generically 1 : 1
3. when the fibers aren't points, they're $\mathbb{CP}^1$s
4. the subvariety of GL_3/B over which it is not 1 : 1 is a $\mathbb{CP}^1$
5. the induced $\mathbb{CP}^1$ bundle over that $\mathbb{CP}^1$ is trivial. (Warning: having a section is not enough, because this isn't a principal bundle.)

The next two exercises are about the general case. Let $\prod I$ be our shorthand for the product $\prod_I r_{\alpha_i}$ of the associated simple reflections.

Exercise. Show by induction that the T-fixed points on BS^I correspond to subwords $J \subseteq I$. Also, the image in G/B of a subword J is $(\prod J)B$.

Exercise. Given a subword $J \subseteq I$, find a natural B-equivariant inclusion $BS^J \hookrightarrow BS^I$.

3. The Bott-Samelson map

We need first a lemma about T-actions on projective varieties. Let V be a T-representation, and M a closed subset of $\mathbb{P}V$ invariant under the induced action of T on $\mathbb{P}V$. We will frequently assume that M linearly spans $\mathbb{P}V$, by replacing V by the smallest subspace W with $\mathbb{P}W \supseteq M$.

Lemma. *Let $M, \mathbb{P}V$ be as above, with M spanning $\mathbb{P}V$. If M is nonempty (i.e. $\mathbb{P}V$ nonempty), then M^T is nonempty. If M has more than one point, then M^T has more than one point.*

Proof. Let $\mathbb{C}\vec{v} \in M$, and write $\vec{v}$ as a sum $\sum_\lambda \vec{v}_\lambda$ of weight vectors (the subscript denoting the weight). Let $\phi : \mathbb{C}^\times \to T$ be a one-parameter subgroup, with $X \in \mathfrak{t}$ its generator, i.e. $X = \phi'(1)$. Then

$$\phi(z) \cdot \vec{v} = \sum_\lambda z^{\langle X,\lambda \rangle} \vec{v}_\lambda.$$

In particular, if we take the limit $z \to \infty$, then $\phi(z) \cdot \mathbb{C}\vec{v}$ converges to the line through the sum of the components with highest $\langle X, \lambda \rangle$. Whereas if we take the limit $z \to 0$, it converges to the line through the sum of the components with lowest $\langle X, \lambda \rangle$.

It seems instructive to break into two cases. If V is a single weight space for T, then T's action on $\mathbb{P}V$ is trivial, and so are the claims made.

In general, the set of weights $\{\lambda\}$ forms a finite set in the weight lattice; take its convex hull, and choose X such that $\langle X, \lambda \rangle$ has unique extrema λ_{max} and λ_{min} on this polytope. Choose $m_{max}, m_{min} \in M$ such that m_{max} (resp. m_{min}) has some λ_{max} (resp. λ_{min}) component, using the assumption that M spans V.

Then $\lim_{z\to\infty} \varphi(z) \cdot m_{max}$ lies in the λ_{max} weight space, and $\lim_{z\to\infty} \varphi(z) \cdot m_{max}$ lies in the λ_{min} weight space. Each is therefore a T-fixed point on $\mathbb{P}V$. Since M is closed in $\mathbb{P}V$, it contains these two T-invariant limits. Since there is more than one weight, these two fixed points are unequal, and M^T has more than one point. □

If M is a complex submanifold, there is also a symplectic proof: the Hamiltonian defined by X has a maximum and a minimum, each of which gives a fixed point.

Note that the statement fails for T-actions on projective varieties if the action is not assumed to be inherited from the ambient projective space; one can equivariantly compactify $\mathbb{C}^\times$ to a nodal cubic (looking essentially like $\mathbb{CP}^1$ with 0 and ∞ stuck together). Then there is only one fixed point.

Lemma. *Let* $\varphi : BS^I \to G/B$ *be the natural map of a Bott-Samelson. Then* $\varphi(BS^I)^T = \varphi((BS^I)^T)$.

Proof. Since the map is B-equivariant, it's T-equivariant, and therefore the image of any T-fixed point is T-fixed. We need to show the converse.

It's not true that any preimage of a T-fixed point is T-fixed; we just need the fiber to contain *some* T-fixed point. These fibers (which are usually singular) live in the Bott-Samelson, and are therefore T-invariant subvarieties of a product of G/Bs, therefore of a product of projective spaces. Then we apply the lemma from the last section to know that there are T-fixed points in the fiber. □

Let $X^w := \overline{BwB}/B$ denote the closure of the B-orbit through w; it is called an **opposite Schubert variety**. By the Bruhat decomposition, we know that means it's the union of finitely many other orbits, themselves just called **opposite Schubert cells**.

Theorem. *Let* I *be a reduced word for* w. *Then the image of* BS^I *is* X^w, *and the map is generically* $1:1$.

Proof. The image of BS^I is B-invariant – therefore it's a union of opposite Schubert cells. The image is also closed – therefore it's a union of opposite Schubert varieties. Finally, it's irreducible; therefore it's just one Schubert variety.

By the exercise, the subwords $J \subseteq I$ correspond to the T-fixed points on BS^I. In particular, I itself maps to w. So the image of BS^I contains w, and therefore also X^w. The image can't be any larger for dimension reasons. So the image is X^w.

Now look at the fiber over wB. Since it's a T-fixed point, T acts on the fiber, and the fixed points are the subwords of I with product w. Since I is a *reduced* expression for w, it's the only one with that product. So there's only one T-fixed point in the fiber, and by the lemma, the fiber is just a point.

Since the map $BS^I \to X^w$ is B-equivariant, and the B-orbit through w is dense, the map is generically $1:1$. □

This is our main reason for introducing Bott-Samelsons – they give a way of approximating a Schubert variety by a smooth manifold with easily-described fixed points.

4. THE BRUHAT ORDER

Define the **Bruhat order** on W (really due to Chevalley) by $v \leq w$ if $vB \in \overline{BwB}/B \subseteq G/B$. For example, w_0 is the maximum element, as we know that Bw_0B is dense in G/B.

Theorem. *The following are equivalent:*

1. $v \leq w$.
2. *There exists a reduced word* I *for* w *with a reduced subword* J *for* v *in it.*
3. *Given any word* I *for* w*, there is a reduced subword* J *for* v *in it.*

In a related result, given any word J' *for* v*, there exists a* reduced *subword* $J \subseteq J'$ *with product* v.

Proof. Obviously $3 \implies 2$. Also $2 \implies 1$, as follows: such a J gives a T-invariant element of BS^I, and its image $(\prod J)B$ is therefore in X^w.

Now we prove the "related result". Let I be a word with product v, and let I_m be its initial string of length m. Then the $\{BS^{I_m}\}$ form a strictly increasing series of varieties, and their images in G/B form a weakly increasing series of Schubert varieties. If we let J denote the subset of I where the image actually jumps up, then we get a subset whose length is the dimension of X^w, which is the length of w.

(The combinatorial version of this is to multiply out the generators, only keeping those that increase the length, but it is less clear why the resulting subword has the same product as the original.)

For the last implication $1 \implies 3$, let J' be a T-fixed point in the fiber over v (our lemma from before guarantees there is one), and J a reduced subword of J' (using the "related result"). $\square$

5. OTHER BOTT-SAMELSON STUFF

There is an alternate, mildly different, picture of Bott-Samelsons. Let $I = [1 \dots n]$ be a word $\alpha_1 \dots \alpha_n$ in the simple roots. Then

$$BS^I = P_{\alpha_n} \times^B P_{\alpha_{n-1}} \times^B \dots \times^B P_{\alpha_1}/B$$

where $\times^B$ means to divide by the diagonal B action coming from the right multiplication on the P to the left crossed with the left multiplication on the P to the left. (The notation $\times_B$ is also common, but is confusable with fiber product.)

Exercise. Relate this to the previous definition.

This alternate description has a pleasant consequence:

Proposition. *Let* BS^I *be a Bott-Samelson manifold. Then* BS^I *has a natural left action not just of* B*, but of* P_{α_n}.

This picture motivates a **left Bott-Samelson crank** ∂_α, defined for manifolds X equipped with a B-action and a B-equivariant map to G/B: let $\partial_\alpha X := P_\alpha \times^B X$. Where $\partial^\alpha X$ was a $\mathbb{CP}^1$-bundle over X, $\partial_\alpha X$ is an X-bundle over $\mathbb{CP}^1$.

Raoul Bott and Hans Samelson (both alive; Bott has even had papers in the last ten years) invented these in the 1950's. They had a compact-group picture of them that hid the algebraic structure: instead of multiplying together P_α's and dividing by B, they multiplied together the $K_\alpha \cong SU(2) \times T^{n-1}$ subgroups we defined last term and divided by T. In particular, this let them see a great big torus action (of half the real dimension) on the Bott-Samelson that is unfortunately not algebraic.

6. Non-minimal parabolics

Recall that a parabolic subgroup P is one that contains B.

Exercise. The parabolic sub*algebras* $\mathfrak{p} \geq \mathfrak{b}$ correspond $1:1$ to subsets of the (negative) simple roots.

Exercise. Each parabolic subgroup P is connected, so they correspond also to subsets of the simple roots. (Recall we proved this for $\mathfrak{p} = \mathfrak{b}$.)

6.1. Long elements of parabolics.

Proposition. *Given a subset S of the simple roots, there exists a unique Weyl group element that takes all of S's elements to negative roots and fixes all the simple roots not in S. Also, it is of order 2.*

Proof. First we show existence. Let w be a Weyl group element fixing the simple roots not in S. Then if $\alpha \in S$ has $w \cdot \alpha \in \Delta_+$, we claim wr_α is longer (it takes one more root negative), and negates α. In particular a longest such w must negate all the roots in S.

Now uniqueness. We first restrict to the sub-root-system of linear combinations of S. Since w is taking all the simple roots to negative roots, it is taking all positive roots to negative roots, and therefore must be taking the simple roots to negative simple roots (not necessarily taking each one to its negative; they may be permuted).

If there are two such w, w' that do this, then ww' takes the simple roots back to the simple roots, and in particular preserves the original positive system Δ_+. Therefore ww' is the identity. Doing this with $w = w'$ we see each such is order 2, and in general $w = w'$. □

In the $GL_n(\mathbb{C})$ case, the corresponding permutation matrices are direct sums of antidiagonal permutation matrices. In particular they do not take the simple roots directly to their negatives (except for 2×2 blocks); they do permute them.

Call this element w the **long element for the parabolic** P.

We defined an operator ∂^α for each minimal parabolic containing B. Why minimal? For $P \geq B$ any parabolic, define ∂^P by the same pullback diagram.

In the $GL_n(\mathbb{C})$ case, this is easy to picture; instead of forgetting one subspace and filling in a substitute, forget several subspaces simultaneously and fill in substitutes.

6.2. Generalized words.

Define a **generalized word** as a triple $(I, <, P)$ where $P : i \mapsto P_i$ takes an element of the totally ordered set I to a parabolic. Define $\prod I$ as the Weyl group element made by multiplying out the long elements of the P_i. Call I **reduced** if the length $l(\prod I)$ is the sum of the lengths of the long elements of the P_i.

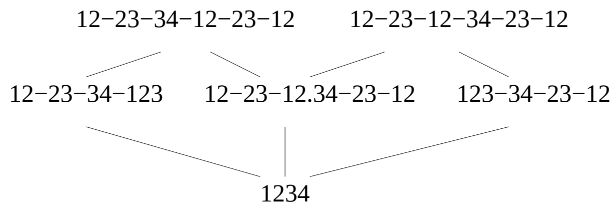
Exercise. Convince yourself that this generalizes the usual notion of reduced word.

With this, we can define a **generalized Bott-Samelson** BS^J as

$$P_{i_1} \times^B \ldots \times^B P_{i_n}/B$$

where the $\{P_i\}$ are not-necessarily-minimal parabolics.

Define a **simple refinement** of a generalized word I as a longer word, in which a letter $i \mapsto P_i$ has been replaced by a reduced (generalized) word for the long element of P_i. Then we transitively extend this notion to define a **refinement** of a generalized word. Here are some examples for the long element of S_4; a straight 2345 means "flip the interval $2-5$", etc.



Theorem. *Two reduced words for a Weyl group element W are connected through a series of simple refinements and un-refinements.*

Proof. We almost proved this in class, with the topological argument using paths transverse to the Weyl hyperplane arrangement. The only thing left to show is that one can stay reduced all the time, i.e. one need never cross a hyperplane twice. Put another way, we need to know that each of the intermediate paths can be assumed transverse to all the hyperplanes. This is left as an exercise. □

Exercise. If I is a refinement of J, then there is a natural B-equivariant map $BS^I \twoheadrightarrow BS^J$, and the composite $BS^I \twoheadrightarrow BS^J \to G/B$ gives the usual map $BS^I \to G/B$.