THE BRUHAT DECOMPOSITION
NOTES FOR MATH 261, SPRING 2002

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Fix our usual notation $(G, T, \Delta_+, B, N, W)$ (here T is a *complex* torus, $\cong (\mathbb{C}^\times)^n$). Our main goal is to prove the **Bruhat decomposition**:

$$G/B = \coprod_{w \in W} N\widetilde{w}B/B \qquad \text{(disjoint union).}$$

We'll actually prove another version, $G = \coprod_{w \in W} N\widetilde{w}B$, though it's less natural for us as a final answer.

These $\widetilde{w}$ are irrelevant choices of lifts of $w \in N(T)/T$ up to elements of $N(T)$. It's important to remember that W is not actually a subset of G, even though in cases like the above one can be lax. Accordingly, we'll leave off the tildes where we can get away with it.

1. The Bruhat decomposition for minimal parabolics

For each simple root α, let P_α denote the connected subgroup with Lie algebra $\mathfrak{b} \oplus \mathfrak{g}_{-\alpha}$. These are called **minimal parabolics**. (We'll prove soon that they're closed.) More generally, a **parabolic subgroup** is one that contains a Borel.

Exercise. Show that the minimal parabolic subgroups are indeed the P_α. (Hint: if they contain B, they contain T.)

In the $G = GL_n(\mathbb{C})$ case, these minimal parabolics are the block-upper-triangular matrices with one 2×2 block, the others being 1×1s.

Lemma. *Let R_α denote the subgroup of P_α with Lie algebra $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. Then R_α is isomorphic to either $SL_2(\mathbb{C})$ or $PSL_2(\mathbb{C})$.*

Let $A := \ker \alpha$, which is a codimension-1 torus in T, and $N_{\overline{\alpha}}$ be the connected subgroup of N whose Lie algebra is $\oplus_{\Delta_+ \setminus \alpha} \mathfrak{g}_\beta$. Then A and R_α commute, each normalizes $N_{\overline{\alpha}}$, and the Lie algebra $\mathfrak{p}_\alpha$ splits as a direct sum:

$$\mathfrak{p}_\alpha = \mathfrak{r}_\alpha \oplus \mathfrak{a} \oplus \mathfrak{n}_{\overline{\alpha}}.$$

Proof. It's easy to check that this Lie algebra is isomorphic to that of $SL_2(\mathbb{C})$; therefore the simply-connected group $SL_2(\mathbb{C})$ maps onto R_α, and R_α is a quotient of it by a discrete central subgroup. But the center is just Z_2, so those are the only two possibilities.

By its definition, A commutes with $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$, therefore with R_α. The last statement is pretty obvious; one dimension of $\mathfrak{t}$ is in $\mathfrak{r}_\alpha$, and the rest is in $\mathfrak{a}$.

The only tricky statement is that $\mathfrak{g}_{-\alpha}$ normalizes $\mathfrak{n}_{\overline{\alpha}}$. Let $\beta \in \Delta_+ \setminus \alpha$; therefore β is a positive combination of simple roots with some other root used with a positive coefficient. Therefore $\beta - \alpha$ is a linear combination of simple roots with some coefficients positive, some negative. Therefore $\beta - \alpha$ is not a root. Consequently $\mathfrak{g}_{-\alpha}$ actually centralizes $\mathfrak{n}_{\overline{\alpha}}$. $\square$

In particular $AN_{\overline{\alpha}}$ is a normal subgroup of P_α, with quotient isomorphic to PSL_2: to see this, one notices that the Lie algebra is right, and that if R_α had a center, the central element ended up in A.

Proposition. *The P_αs are closed subgroups of G. For each simple root α, the homogeneous space P_α/B is a $\mathbb{CP}^1$.*

Proof. We prove the second statement first:

$$P_\alpha/B \cong (P_\alpha/AN_{\overline{\alpha}})/(B/AN_{\overline{\alpha}}) \cong PSL_2(\mathbb{C})/B_{PSL_2(\mathbb{C})} \cong \mathbb{CP}^1.$$

Then G/B maps to G/P_α with compact fibers P_α/B; in particular the quotient is Hausdorff. Therefore P_α was a closed subgroup. □

In the case of $GL_n(\mathbb{C})$, these spaces G/P_α are partial flag manifolds, where one has a not-quite-maximal chain of subspaces, in that one subspace is skipped.

Let N_α be the 1-dimensional subgroup with Lie algebra $\mathfrak{g}_\alpha$.

Corollary (The Bruhat decomposition of $\mathbb{CP}^1$). *If P_α is a minimal parabolic, and $\tilde{r}_\alpha$ a lift of the nontrivial element of R_α's Weyl group, then P_α is the disjoint union of B and $N_\alpha \tilde{r}_\alpha B$.*

Moreover, every element of P_α not in B is uniquely expressible as such a product.

Proof. Let N_α act on $P_\alpha/B \cong \mathbb{CP}^1$ by left multiplication. If we can show it has two orbits, the fixed point $B \in P_\alpha/B$ and the other orbit $N_\alpha \tilde{r}_\alpha B \subset P_\alpha/B$, then we're done.

We pass to the quotient by $AN_{\overline{\alpha}}$; then we're discussing $N_\alpha \leq PSL_2(\mathbb{C})$ acting on $\mathbb{CP}^1$, where we know this to be true. □

We will actually use this in a slightly different formulation, inverting and multiplying by r_α on the right:

$$P_\alpha = Br_\alpha \cup BN_{-\alpha}.$$

2. The Bruhat decomposition in general: existence

Lemma. *The Lie algebra $\mathfrak{n}$ is generated by the root spaces of the simple roots.*

Proof. Let $\alpha, \beta, \alpha+\beta$ be roots, and $\mathfrak{g}_\alpha, \mathfrak{g}_\beta, \mathfrak{g}_{\alpha+\beta}$ the corresponding root spaces. In particular β is not a multiple of α. Then the subspace $\sum_{n\in\mathbb{Z}} \mathfrak{g}_{\beta+n\alpha}$ is a multiplicity-free representation of the root $\mathfrak{sl}_2$ corresponding to α.

Therefore it's an irrep of that $\mathfrak{sl}_2$, and so we determine that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. (Containment is easy; we wanted to know that the bracket is nonzero.)

Given any $\beta \in \Delta_+$, either it's simple or it's a sum of two other positive roots, so by induction on the height of β we can find $\mathfrak{g}_\beta$ by bracketing simple root spaces together. □

We'll denote by N_β the 1-parameter subgroup corresponding to a root space $\mathfrak{g}_\beta$.

Lemma. *The group G is generated by the subgroup B and the subgroups $\{N_{-\alpha}\}$, as $-\alpha$ varies over the negative roots.*

Proof. The Lie algebra of the subgroup generated contains all the negative simple root spaces. Therefore by the previous lemma, it contains $\mathfrak{n}_-$. But that and $\mathfrak{b}$ add up to $\mathfrak{g}$. □

Theorem (Bruhat decomposition; existence). *Each element of* G *is of the form* $n\tilde{w}b$*, where* $\tilde{w}$ *is a lift to* G *of a Weyl group element,* $b \in B$*, and* $n \in N$*.*

Proof. This union NWB is some subset of G; what we want to show is that it is closed under left multiplication by G. For that, it's enough to show that it's closed under left multiplication by B, and also under left multiplication by $\tilde{r}_\alpha$.

Note that $BWB = NTWB = NWTB = NWB$, so we can work with BWB instead if we want. In particular the closure under left multiplication by B is clear.

1. Next we show that for each negative simple root $-\alpha$ and Weyl group element w, the subset $N_{-\alpha}wB = wN_{w^{-1}\cdot(-\alpha)}B \subseteq NWB$. This step is really the heart of the matter.

There are two cases: if $w^{-1} \cdot (-\alpha) \in \Delta_+$, then we're done, as $N_{w^{-1}\cdot(-\alpha)} \leq B$. Assume now that $w^{-1} \cdot \alpha \in \Delta_+$, and let R_α denote the α root SL_2.

Each $n \in N_{-\alpha}$ (or indeed anywhere in R_α), *by the Bruhat decomposition for* SL_2, can be written as $b\tilde{r}_\alpha n'$ for some element $b \in B \cap R_\alpha, n' \in N_\alpha$. (Or else just as b, though this second case only happens for $b = n = 1$.) So we can rewrite

$$nwB = br_\alpha n'wB \in br_\alpha N_\alpha wB = br_\alpha wN_{w^{-1}\cdot\alpha}B = bwr_\alpha B$$

since $N_{w^{-1}\cdot\alpha}\cdot \leq B$. Then this last is part of BWB.

2. Now we show that BWB is invariant under left multiplication by each $N_{-\alpha}$.

$$N_{-\alpha}BWB \subseteq P_\alpha WB = (Br_\alpha \cup BN_{-\alpha})WB = BWB \cup BN_{-\alpha}WB$$

by the Bruhat decomposition of P_α (actually, using the formulation that followed it). Since $N_{-\alpha}WB \subseteq BWB$, this second piece is BWB too. ☐

Corollary. *One of the* N*-orbits on* G/B *is open dense. (This was the hole in our proof that* W *acted simply transitively on the Weyl chambers.)*

Proof. When we rip out the finitely many non-open orbits, each of which is real codimension at least 2, we don't disconnect G/B. ☐

3. The Bruhat decomposition in general: uniqueness

We start with a technical lemma, but one with a rather interesting proof.

Proposition. *Let* $S \subseteq N$ *be a* T*-invariant nonempty closed subset. Then* $S \ni 1_N$*.*

Proof. Let $X \in \mathfrak{t}$ be a lattice point in the open positive Weyl chamber in $\mathfrak{t}$. So $\langle X, \beta \rangle > 0$ for all $\beta \in \Delta_+$. Since X is a lattice point, it arises as the differential of a map $\phi : \mathbb{C}^\times \to T$ (this is how we define the coweight lattice). Then the inequality tells us that

$$\lim_{z \to 0} \phi(z) \cdot Y = \vec{0}, \qquad \forall Y \in \mathfrak{n}.$$

That shows that any T-invariant closed subset of $\mathfrak{n}$ must contain $\vec{0}$. Exponentiating (T-equivariantly), we get the conclusion of the lemma. ☐

Let N_w denote the intersection of the subgroups $N \cap wNw^{-1}$. This is a T-normalized subgroup of N. If w is of order two, like a simple reflection, then it's also w-normalized.

Corollary. *Each group* N_w *is connected, and in fact contractible.*

Proof. Since N and wNw^{-1} are each T-invariant, so is their intersection. Since T is connected, it can't switch around components of the intersection, so each component is also T-invariant. By the proposition, each component contains 1. So there's only one component.

We know what Lie groups with nilpotent Lie algebras look like; their universal covering groups are contractible. One way to see that such a group is *not* contractible is whether it has an element of finite order. Since $N_w \leq N$, and N is contractible, N has no elements of finite order, therefore N_w doesn't either, and therefore N_w is contractible. $\square$

Lemma. *The multiplication map* $N_w \times N_{ww_0} \to N$ *is a* T*-equivariant bijection.*

Proof. The T-equivariance is clear. Also, by Lie algebra considerations $N_w \times N_{ww_0}$ acting by left and right multiplication on N has an open orbit (double coset). If it isn't the whole thing, then there are other, smaller-dimensional, double cosets $N_w u N_{ww_0}$. Equivalently, there exists a $u \in N$ such that $u\mathfrak{n}_w u^{-1} \cap \mathfrak{n}_{ww_0} > 0$.

The set of such u is obviously closed, and is also invariant under conjugation by t:

$$(tut^{-1})\mathfrak{n}_w(tut^{-1})^{-1} \cap \mathfrak{n}_{ww_0} = t\left(u(t^{-1}\mathfrak{n}_w t)u^{-1} \cap t^{-1}\mathfrak{n}_{ww_0}t\right)t^{-1} = t(u\mathfrak{n}_w u^{-1} \cap \mathfrak{n}_{ww_0})t^{-1}$$

By the proposition, if the set of u is nonempty, it contains 1, which is a contradiction.

So far we know that $N_w \times N_{ww_0}$ acts transitively on N by right and left multiplication, and that the dimensions are the same. Therefore it's a (connected) covering space of N. But N is simply connected, so the covering map is a diffeomorphism. $\square$

Theorem (Bruhat decomposition; uniqueness)**.** *Let* $\{\tilde{w}\}_{w\in W}$ *be a system of lifts of Weyl group elements to elements of* $N(T)$ *lying over them.*

Each element g *of* G *can be expressed as a product* $g = n\tilde{w}b$*, where* $\tilde{w}$ *is uniquely determined.*

This n *can be taken to be in* N_{ww_0}*, in which case* n *and* b *are also uniquely determined.*

Proof. We work in G/B rather than G.

The N-stabilizer of wB is N_w, so the orbit is $N/N_w \cong N_{ww_0}$. In particular, T has only one fixed point on NwB (since it has only one on N_{ww_0}). Therefore no other $w'B$ are in NwB.

The second statement is easy, now that we know that N_{ww_0} acts simply transitively on the orbit NwB. $\square$

4. Corollaries

By Morse theory, we already knew the odd Betti numbers of G/B were all 0. Now we know the indices of the fixed points:

Corollary. $\dim H_{2n}(G/B) =$ [*the number of* $w \in W$ *with length* n].

Corollary. *The Poincaré duals of the divisors* $\{\overline{Br_\alpha w_0 B}/B\}$ *give a basis for* $H^2(G/B)$*. The dual basis in* $H_2(G/B)$ *is given by the curves* $\{\overline{B_- r_\alpha w_0 B}/B\}$*.*

Proof. The dual basis statement just comes because $\overline{Br_\alpha w_0 B\}} \cap \overline{B_- r_\beta w_0 B}$ is a (transverse, complex) point if $\alpha = \beta$, and empty otherwise. $\square$

4.1. **The Bruhat decomposition for G/P.** For each parabolic P, let W_P denote the subgroup of W generated by the simple reflections through roots in $\mathfrak{p}$.

Exercise. Let $C \in W/W_P$ be a coset. Show that there is a unique shortest and a unique longest element in C.

Define the **length $l(wW_P)$ of the coset wW_P** as the length of the shortest element. Then there is a Bruhat decomposition for each G/P:

Corollary. $G/P = \coprod_{gW_P \in W/W_P} NgP$*, and the dimension of the orbit* NgP *is the length of the coset* gW_P*.*

In particular, if P *is a maximal parabolic, then* $H^2(G/P)$ *is* 1*-dimensional.*

From here, one can finish off Borel-Weil, by showing the following are equivalent:

- $\lambda \in \mathfrak{t}^*_+$
- the corresponding element of H^2 is a sum of (Poincaré duals of) opposite Schubert divisors
- the restriction of that element to each Schubert curve has positive integral
- the restriction of the corresponding line bundle to each Schubert curve has sections
- the N-invariant section of the line bundle over the big cell extends over the opposite Schubert divisors
- the N-invariant section of the line bundle over the big cell extends over the whole G/B.

We won't go through the details.