THE ATIYAH-BOTT FORMULA AND THE DEMAZURE CHARACTER FORMULA NOTES FOR MATH 261, SPRING 2002

ALLEN KNUTSON

1. Čech cohomology of a line bundle

Begin by recalling the Čech cohomology of a topological space, with values in an abelian group A (considered as a discrete topological space). The kth cochain group is defined as the continuous A-values functions on the k-fold intersections of open sets. (Since A is discrete, these are the locally constant functions.) Then there's the usual differential, and we take cohomology.

The intersection of 0 open sets gives us the whole space, and H^0 is then just the space of A-valued functions on the set of components, i.e. A to the number of components. In general it's hard to compute H^i, but if $A = \mathbb{R}$ it's easy to compute the **Euler characteristic** $\sum_i (-1)^i \dim H^i$.

The generalization we need of this picture is to replace "continous A-valued functions", which is a functor taking open subsets to abelian groups, to any other sheaf (a functor taking open subsets to somethings, satisfying a gluing property). The sheaf we'll be interested in is "holomorphic sections of a certain line bundle."

So given a holomorphic line bundle on a complex variety X, we can define a kth cochain group as the holomorphic sections of the line bundle restricted to k-fold intersections of open sets. They're no longer locally constant, but just holomorphic.

Then H^0 is the space of global holomorphic sections. We're very interested in this, of course, for Borel-Weil. But the above suggests that we should look at the higher cohomology of the line bundle too, if we want to get things that are easy to compute. We will call the alternating sum of the dimensions the **holomorphic Euler characteristic** of the line bundle.

For example:

Theorem (Riemann-Roch)**.** *Let Σ be a compact Riemann surface of genus g, and $\mathcal{L}$ a holomorphic line bundle on it, with a holomorphic section that has k simple zeroes and l simple poles. Then*

$$\dim H^0(\Sigma; \mathcal{L}) - \dim H^1(\Sigma; \mathcal{L}) = (k - l) + (1 - g).$$

(There is no higher cohomology, which turns out to follow from Σ being 1 complex dimensional.)

For example, the trivial line bundle on any curve has a constant section so $k = l = 0$. On an elliptic curve, the trivial line bundle has H^0 and H^1 each one-dimensional. But there are other line bundles with H^0 and H^1 both zero (topologically trivial, but algebraically nontrivial).

2. THE ATIYAH-BOTT WOODS HOLE FORMULA

Let M be a compact complex T-manifold, with a holomorphic T-equivariant vector bundle $\mathcal{V}$. (We of course mainly care about the case that $\mathcal{V}$ is a line bundle.) Then all the spaces at hand are T-spaces, so we can do better than summing the alternating dimensions: we can alternate-sum the T-characters.

Theorem (Atiyah-Bott, '65). *Assume M^T is finite. Over each point $m \in M^T$, there is a T-action on the fiber $\mathcal{V}_m$. Then*

$$\sum_i (-1)^i \operatorname{Tr}(t|_{H^i(M;\mathcal{V})}) = \sum_{m \in M^T} \frac{\operatorname{Tr}(t|_{\mathcal{V}_m})}{\det(1 - t|_{T^*_m M})}$$

(Note that this denominator is not zero, by the assumption that m is an isolated fixed point on M.)

We won't prove this, but we'll attempt to motivate it. Consider the case of T acting on a 1-d vector space M (which is bogus, of course, since we need M compact), and say V is the trivial line bundle. Then $H^0 = \mathrm{Fun}(M) \cong \mathbb{C}[x]$, and there is no higher cohomology.

The action of t on x is the dual to the action on $M = \operatorname{Spec} \mathbb{C}[x]$ itself, so the LHS is $1 + t^{-1} + t^{-2} + \ldots$ Meanwhile the numerator on the RHS is 1 (the action on the line over the fixed point $\vec{0}$), and the denominator is $1 - t^{-1}$ (because of the cotangent bundle). Ta-da!

Exercise. Apply this to the case M a flag manifold $GL_n(\mathbb{C})/B$, and $\mathcal{V}$ the line bundle associated to a dominant weight λ. Assume that all higher cohomology vanishes. Show that we recover the Weyl character formula.

3. THE DEMAZURE CHARACTER FORMULA

The hard theorems, that we won't prove, are the following:

Theorem (Demazure et al.). *Let λ be a dominant weight for G, and let $\mathcal{L}_\lambda$ be the corresponding equivariant line bundle on G/B. Let X^w be a Schubert variety in G/B.*

Then the line bundle $\mathcal{L}_\lambda$, restricted to X^w, has no higher cohomology. Also, the restriction map on sections is onto.

Let BS^I be a Bott-Samelson manifold mapping onto $X^w \subseteq G/B$. Then the line bundle $\mathcal{L}_\lambda$, pulled back to BS^I, has no higher cohomology, and the pullback map on sections from X^w to BS^I is an isomorphism.

Nowadays they can all be proved quickly via "Frobenius splitting", which uses special properties of these varieties to prove these results in all characteristics p, and thereby derive them in characteristic 0.

Define a **Demazure module** $D_{w,\lambda}$ as the space of sections of the λ line bundle on G/B over the Schubert variety X^w. It is a B-representation, and therefore a T-representation, so we can ask about its T-multiplicities. One of them is very easy, and one is very interesting:

$$D_{1,\lambda} \cong \mathbb{C}_{w_0 \lambda}$$
$$D_{w_0,\lambda} \cong V_\lambda$$

The first is the space of sections over a point, the second over G/B (and is therefore V_λ, by Borel-Weil). Therefore, if we had a way of inductively calculating the characters of Demazure modules, we could base our induction at $D_{1,\lambda}$ and arrive at $D_{w_0,\lambda}$.

Given a simple root α, define the **Demazure operator** d_α on the ring of functions $\mathrm{Fun}(T)$ by

$$d_\alpha p = \frac{p}{1-\alpha} + r_\alpha \cdot \frac{p}{1-\alpha} = \frac{p - \alpha r_\alpha \cdot p}{1-\alpha}$$

Here $\alpha : T \to \mathbb{C}^\times$ is considered an element of $\mathrm{Fun}(T)$, and the reflection $r_\alpha : T \to T$ is made to act on $\mathrm{Fun}(T)$.

There's something to check about this definition: why doesn't this rational function blow up as $1 - \alpha \to 0$?

$$\frac{p - \alpha r_\alpha \cdot p}{1-\alpha} = p + \alpha \frac{p - r_\alpha \cdot p}{1-\alpha}$$

This numerator negates when we hit it with r_α, and therefore vanishes along $\ker \alpha$. So it is a multiple of the linear Laurent polynomial $1 - \alpha$ which vanishes to first order along $\ker \alpha$, and the ratio is indeed a Laurent polynomial.

Theorem (The Demazure character formula)**.** *Let α be a simple root, and w a Weyl group element, such that $r_\alpha w > w$. Let λ be a dominant weight. Then*

$$\mathrm{char}\, D_{r_\alpha w, \lambda} = d_\alpha \cdot \mathrm{char}\, D_{w,\lambda}.$$

In particular, if I is a reduced expression for the long element w_0, then

$$\mathrm{char}\, V_\lambda = \Big(\prod_I d_{\alpha_i}\Big) \cdot (w_0 \cdot \lambda).$$

Proof. By the big theorems, we can calculate these characters by applying the Atiyah-Bott formula to relevant Bott-Samelsons. Let I be a reduced word for w; then the Atiyah-Bott formula applied to BS^I gives $\mathrm{char}\, D_{w,\lambda}$.

By assumption, αI is also a reduced word, giving us a bigger Bott-Samelson $BS^{\alpha I} = \partial_\alpha BS^I$. This is a bundle over $P_\alpha/B \cong \mathbb{CP}^1$ with fiber BS^I. In particular, the T-fixed points on $BS^{\alpha I}$ separate according to whether they lie over $B \in P_\alpha/B$ or over $r_\alpha B$. It's easy to understand the tangent space to such a fixed point: it's the sum of the tangent space to the fiber, plus the pullback of the tangent space to the $\mathbb{CP}^1$.

So if $M = BS^{\alpha I}$, and $\mathcal{V}$ is the λ line bundle on G/B pulled back to M, then

$$\sum_{m \in M^T} \frac{\mathrm{Tr}\,(t|_{\mathcal{V}_m})}{\det(1 - t|_{T^*_m M})} = \sum_{J \subseteq \alpha I} \frac{t^{\prod J}}{\det(1 - t|_{T^*_J BS^{\alpha I}})}$$

$$= \sum_{J \subseteq I} \frac{t^{\prod J}}{(1-\alpha)\det(1 - t|_{T^*_J BS^I})} + \sum_{\alpha J \subseteq \alpha I} r_\alpha \cdot \frac{t^{\prod J}}{(1-\alpha)\det(1 - t|_{T^*_J BS^I})} = d_\alpha \sum_{J \subseteq I} \frac{t^{\prod J}}{(1-\alpha)\det(1 - t|_{T^*_J BS^I})}$$

but then this inner sum is the Atiyah-Bott formula applied to BS^I. □

Exercise. If we have the same notation as in the theorem, but $r_\alpha w < w$, show that

$$d_\alpha \cdot \mathrm{char}\, D_{w,\lambda} = \mathrm{char}\, D_{w,\lambda}.$$

Exercise. Show that the formula for V_λ is Weyl-invariant. (Hint: it's enough to show it's invariant under simple reflections.)

4. Some examples of multiplicity diagrams for Demazure modules

As with the Weyl character formula, it may be more interesting to think about the Fourier transform, which concerns multiplicity diagrams rather than characters. In this context the formula

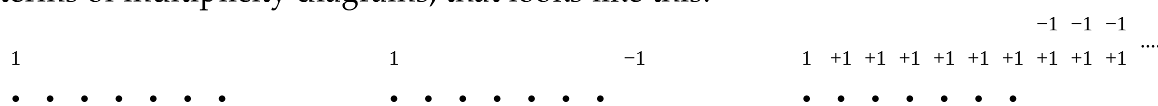
$$d_\alpha p = \frac{p - \alpha\, r_\alpha \cdot p}{1-\alpha}$$

is "flip, shift, subtract from the original, sum in the α direction".

4.0.1. *The* SL_2 *case.* In this case the torus T is already isomorphic to $\mathbb{C}^\times$, and the only simple root α is $t \mapsto t^2$. Then the formula is

$$\mathrm{Tr}\,(t|_{V_\lambda}) = \left(d_\alpha(u \mapsto u^{-\lambda})\right)(t) = \frac{t^{-\lambda} - t^2 t^\lambda}{1-t^2} = t^{-\lambda} + t^{2-\lambda} + \ldots + t^\lambda$$

In terms of multiplicity diagrams, that looks like this:



4.0.2. *The* SL_3 *case.* We use the reduced word $(12)(23)(12)$ for the long element 321, corresponding to the simple roots $x_2 - x_1, x_3 - x_2, x_2 - x_1$. Here are the diagrams, corresponding to the case $\lambda = 2\omega_1 + 5\omega_2$. In each row, we start with the multiplicity diagram of a Demazure module; we flip, shift, and subtract it from the original; and then we sum in the direction of the simple root. The simple roots here point east, northwest, east.