CLASSIFYING COMPACT LIE GROUPS
NOTES FOR MATH 261, SPRING 2002

ALLEN KNUTSON

1. Decomposing the Lie algebra

Let K be a compact connected Lie group, $\mathfrak{k}$ its Lie algebra, which we decompose under the adjoint action into irreps $\mathfrak{k} = \bigoplus_i V_i$. Some of these are the trivial representation, some not. The trivial isotypic component is the Lie algebra of the center of K (which may not be connected).

Proposition. *Let $\mathfrak{k}$ be the Lie algebra of a compact group K. The following subspaces of $\mathfrak{k}$ are equal:*

1. *The linear span of the Lie brackets* $[A, B], A, B \in \mathfrak{k}$
2. *The linear span of the set* $\{kAk^{-1} - A\}, A \in \mathfrak{k}, k \in K$
3. *The sum of the nontrivial irreps in* $\mathfrak{k}$*.*

Proof. If I is an irreducible subrep of $\mathfrak{k}$, then

1. the linear span of the Lie brackets $[A, B], A \in I, B \in \mathfrak{k}$
2. the linear span of the set $\{kAk^{-1} - A\}, A \in I, k \in K$

are each obviously subreps of I. Therefore they're I or 0, and they're only 0 if I is a trivial rep. □

Define the **commutator subalgebra** $\mathfrak{k}'$ of an arbitrary Lie algebra $\mathfrak{k}$ using the first or second definition above. For bad Lie algebras (like that of the following exercise) $\mathfrak{k}$ is not a direct sum, and so the third definition doesn't make sense.

Exercise. Let $\mathfrak{h}$ be the Lie algebra of strictly upper $n \times n$ complex matrices. Show that the center is contained within the commutator subalgebra (in particular, $\mathfrak{h}$ is not the direct sum of the two).

2. A little Lie algebra cohomology

Let K be a compact group. We already know that we're interested in $\pi_1(K)$, since each covering space of K is a Lie group too. Since it's necessarily abelian, By Hurewicz's theorem it's the same as $H_1(K)$. Then $H_1(K) \otimes \mathbb{R} \cong H^1(K, \mathbb{R})$, which we can compute with de Rham cohomology.

Proposition. *Let $\Lambda^\bullet \mathfrak{k}^*$ denote the space of differential forms on K invariant under left multiplication by K. Then $\Lambda^\bullet \mathfrak{k}^*$ is a subcomplex of the de Rham complex of all differential forms, and gives the same cohomology.*

Proof. First, that it's closed under d is pretty obvious, since d is also left-invariant. We have maps between the two complexes, inclusion ι and averaging α (using K compact). The composition $\alpha \circ \iota$ is the identity; since left-translation by a single group element is homotopic to the identity, averaging over all of them induces an isomorphism on cohomology. □

In general, one defines the **Lie algebra cohomology** of a Lie algebra $\mathfrak{g}$ as the cohomology of the complex $\Lambda^\bullet \mathfrak{g}^*$, with the differential $d : \Lambda^k \mathfrak{g}^* \to \Lambda^{k+1}\mathfrak{g}^*$ defined by

$$d\omega(\xi_1 \wedge \ldots \wedge \xi_{k+1}) = \sum_{i=1}^{k} (-1)^i \omega(\ldots \wedge [\xi_i, \xi_{i+1}] \wedge \ldots).$$

In the case $\mathfrak{g}$ is the Lie algebra of a group G, one can show this is the differential induced on left-invariant forms from d on G. We'll check this for H^1 now, as part of the following theorem.

Theorem. *Let K be a compact group, $\mathfrak{k}$ its Lie algebra, and $\mathfrak{k}'$ the commutator subalgebra. Then there is a natural identification of $H^1(K;\mathbb{R}) \cong (\mathfrak{k}')^\perp$, thought of as a subspace of $\mathfrak{k}^*$.*

Proof. We use the formula for Lie derivative of a form ω w.r.t. a vector field X:

$$D_X\omega = \iota_X \cdot d\omega + d(\iota_X \cdot \omega)$$

where $\iota_X \cdot$ denotes interior product with the vector field.

In our case, ω is a left-invariant form, and X a left-invariant vector field. If ω is a 1-form, then $\iota_X \cdot \omega$ is a left-invariant *function*, so constant, and $d(\iota_X \cdot \omega) = 0$.

Now we apply Leibniz to $D_Y(\iota_Y \cdot \omega)$. Ordinarily this gives

$$D_X(\iota_Y \cdot \omega) = \iota_Y \cdot (D_X\omega) + \omega(D_X\iota_Y).$$

We hit this with three equations:

1. The LHS is D_X of a left-invariant function, so zero.
2. The first term of the RHS is $\iota_Y \cdot \iota_X d\omega$ by the previous, or put another way it's $d\omega(X \wedge Y)$.
3. The second term of the RHS is $\omega([X, Y])$.

All put together, $d\omega(X \wedge Y) = -\omega([X, Y])$. (From there, one can do an induction to prove the general d formula above.)

Now let's compute $H^1(K)$ = closed left-invariant 1-forms modulo the image of d applied to left-invariant 0-forms. The latter is zero, so we just want the subspace of $\Lambda^1\mathfrak{k}^* = \mathfrak{k}^*$ that's killed by d. By the formula just proven, that's equivalent to it vanishing on any commutator. We're done. □

Corollary. *Let $\mathfrak{k}$ be the Lie algebra of a compact group. If $\mathfrak{k}$ is centerless, then it is the Lie algebra of only finitely many compact Lie groups.*

Proof. Let K be the adjoint group (recall, we can construct this as $\bar{K}/Z$ for any other compact group $\bar{K}$ with this Lie algebra). Then since $\mathfrak{k} = \mathfrak{k}'$, the first cohomology $H^1(K;\mathbb{R}) = 0$. By the universal coefficient theorem, $H_1(K;\mathbb{R}) = 0$. Therefore $H_1(K;\mathbb{Z})$ is a finite abelian group. By the Hurewicz theorem, it's equal to $\pi_1(K)$. And we already saw that the groups with Lie algebra $\mathfrak{k}$ correspond to the subgroups of $\pi_1(K)$, of which we now know there to be a finite number. □

Lie algebra cohomology can be defined for any Lie algebra. Note that it is not always the topological cohomology of a group; for example the group $\mathbb{R}^n$ has the same Lie algebra, hence the same Lie algebra cohomology, as T^n. Whereas the manifold $\mathbb{R}^n$ has no cohomology.

3. Adjoint groups

A connected Lie group K is called **simple** if its only normal subgroups are discrete (therefore central), and if it's not a circle. This doesn't fit with the definition used in finite group theory, that there are no nontrivial normal subgroups at all. Oh well. It's equivalent to the condition that the adjoint representation be irreducible (and not the trivial rep, to rule out the circle case), for any subrep would exponentiate to a normal subgroup.

Theorem. *Let K be a compact connected group with discrete center. Then K is a product of simple groups, modulo a discrete central subgroup.*

Proof. Let $\mathfrak{k} = \sum_i V_i$ be a decomposition into irreps. Then for each i, look at the differential of the action map $K \to GL(V_i)$; the kernel contains $\sum_{j \neq i} V_j$. So the image of $K \to GL(V_i)$ has dimension at most that of V_i. Call this group K_i; it is also compact connected, and its Lie algebra is the image of V_i in $\mathfrak{gl}(V_i)$.

So the adjoint action gives a map $K/Z \hookrightarrow \prod K_i$. We've already computed the dimension of $\prod K_i$ as bounded above by $\dim K$, so now we now that the dimensions are equal, and in particular the Lie algebra of K_i is V_i. So the map induces an isomorphism of Lie algebras, making it a degree 1 map between compact connected oriented manifolds, and therefore onto. In particular the adjoint group K/Z of K is a product.

Therefore its universal cover $\prod \widetilde{K}_i$ is a covering group of K. ☐

Exercise. If N is a connected normal subgroup of K, show that K has another normal subgroup M such that $K \cong M \times N$ modulo a finite central subgroup.

The same line of argument gives

Proposition. *If K is compact, and $\mathfrak{k}$ has no center, and H is another group with the same Lie algebra, then H is compact.*

Proof. H is a finite cover of the compact group K/Z. ☐

Example. The group $SO(4)$ has adjoint group $SO(3) \times SO(3)$, whose universal cover is $S^3 \times S^3$ (think of S^3 as a group because it's the unit quaternions). To see that $SO(4)$ is a quotient of $S^3 \times S^3$, consider the action of $S^3 \times S^3$ acting on $\mathbb{H}$ by left and right multiplication. That's a map to $SO(\mathbb{H})$, and the kernel is $\langle(-1,-1)\rangle$, the diagonal Z_2 central subgroup. By dimension count it's onto.

4. The commutator subgroup

If K's center is not discrete, we can still consider its image under the adjoint representation; this K/Z has Lie algebra $\mathfrak{k}'$ (considered as a quotient of $\mathfrak{k}$ by $\mathfrak{z}$).

Now do the odd thing of considering the *subgroup* generated by $\exp(\mathfrak{k}')$ (considered as a subspace of $\mathfrak{k}$). A priori it's some horrible irrational-flow non-closed subgroup of K. But its Lie algebra is centerless, and that of a compact group K/Z, so this subgroup is actually compact. It's easily seen to be the commutator subgroup K' of K.

Theorem. *Any compact connected Lie group is the product of a torus and a list of simple subgroups, modulo a finite central subgroup.*

Proof. The map $Z(K)_0 \times K' \to K$ induces an isomorphism of Lie algebras, therefore is a covering map modding out a finite central subgroup, and we've already analyzed groups like K'. □

So to classify arbitrary compact groups, we need to classify the simple Lie algebras of compact groups. We've already seen that each one leads to a Dynkin diagram; the theorem yet to be proven is that each Dynkin diagram corresponds to a unique compact Lie algebra.