

**SOME GENERAL LIE THEORY
NOTES FOR MATH 261, SPRING 2002**

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1. THE EXPONENTIAL MAP

Theorem. *Let G be a Lie group, $\vec{v} \in \mathfrak{g}$. Then there exists a unique homomorphism $e : \mathbb{R} \rightarrow G$ such that $(Te)|_0(\vec{1}) = \vec{v}$.*

Note that the image of e can be a nasty subgroup of G , in particular need not be closed. The standard example is irrational flow on a torus. (For K compact, this is really as bad as it gets.)

Proof. For each $r \in \mathbb{R}$, let $\vec{1}_r$ denote the usual tangent vector to r , with the property that left multiplication by r takes $\vec{1}_0$ to $\vec{1}_r$. So $\vec{1}_r = r \cdot \vec{1}_0$.

If e is a homomorphism as asserted by the theorem, then applying e to this equation gives $(Te)_r(\vec{1}_r) = e(r) \cdot \vec{v}$. In particular, e is an integral curve of the ODE defined by the vector field $X|_{\mathfrak{g}} = \mathfrak{g} \cdot \vec{v}$. So uniqueness of ODEs shows that e is unique.

As for existence, let e be the solution to this ODE (with initial condition being $e(0) = 1_G$). Then time $t_1 + t_2$ flow is time t_1 flow followed by time t_2 flow; this is the group homomorphism statement. \square

This defines the **exponential map** $\exp : \mathfrak{g} \rightarrow G$, which is *not* a group homomorphism (unless G abelian).

Exercise. Show that the exponential map intertwines the adjoint action on \mathfrak{g} with the conjugation action of G on G .

Exercise. Show that the exponential map is natural in the sense that given a group homomorphism $G \rightarrow H$, the diagram

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & \mathfrak{h} \\ \exp \downarrow & & \downarrow \exp \\ G & \longrightarrow & H \end{array}$$

commutes.

Exercise. What is the relation of these last two exercises?

Theorem. *If G is connected, then the image of \exp generates G .*

Proof. Since the derivative of \exp at 0 is the identity, the inverse function theorem says that it hits an open neighborhood of the identity, call it E . Replace E with $E \cap E^{-1}$ (still open) to be sure that it has all its inverses. Then $F := \bigcup_{n \in \mathbb{N}} E^n$ is an open subgroup of G . If we think of $G \setminus F$ as the union of the nonidentity cosets of F , we see that $G \setminus F$ is open too. Since G is connected, not both F and $G \setminus F$ can be open and nonempty, therefore $F = G$. \square

Corollary. Let G act on a manifold M , and therefore each $X \in \mathfrak{g}$ induces a vector field on M . If each X vanishes on a submanifold N of M , then N is pointwise invariant under G .

Proof. Let $\phi : G \rightarrow \text{Diff}(M)$ be the action. Then $\exp(\phi(\mathfrak{g}))$ generates $\phi(G)$, but each element of $\exp(\phi(\mathfrak{g}))$ fixes N . \square

Exercise. Show that the exponential map intertwines the adjoint action and the action of G on itself by conjugation. Use this to show that the center of G is discrete if and only if the adjoint action has no invariant vectors (except $\vec{0}$).

Exercise. Show that the exponential map from $\mathfrak{u}(\mathfrak{n}) \rightarrow U(\mathfrak{n})$ is onto.

Exercise. Show that the exponential map from $\mathfrak{sl}_2(\mathbb{R}) \rightarrow SL_2(\mathbb{R})$ is not onto.

What's more of a pain is showing that for every Lie subalgebra \mathfrak{s} (not just lines) in \mathfrak{g} , the group generated by $\exp(\mathfrak{s})$ has Lie algebra \mathfrak{s} . These subgroups tend not to be closed, of course.

2. COVERING GROUPS

Let H be a connected group, G a group of the same dimension, and $\phi : G \rightarrow H$ be a map with discrete fiber. Then ϕ is a covering space (the rank of its derivative can't change). The kernel is a discrete normal subgroup, and is quite restricted:

Theorem. Let G be a connected group, and N a discrete normal subgroup. Then N is central.

This doesn't actually use Lie-ness, if you want to work with algebraic groups over a characteristic p field or whatever.

Proof. Fix $n \in N$, and let $f : G \rightarrow N$ take $g \mapsto gng^{-1}$. Then the image is connected yet discrete, so just a point, n . Therefore each n is central. \square

So groups can be covering spaces: how about the reverse?

Proposition. Let H be a connected Lie group. Then its universal cover \tilde{H} has a canonical Lie group structure.

Proof. Recall that the universal cover is defined as paths in H starting at the identity, rel boundary; we define a product on them by concatenation. \square

Corollary. If G is a Lie group, then $\pi_1(G)$ is abelian.

Proof. We can see $\pi_1(G)$ as the kernel of the map $\tilde{G} \rightarrow G$. \square

Proposition. If the center Z of a connected Lie group G is discrete, then G/Z has no center.

Proof. G and G/Z have the same Lie algebra, and Z discrete implies that the Lie algebra \mathfrak{g} has no center. Therefore G/Z also has discrete center, call it Y .

The preimage of Y in G is therefore a discrete normal subgroup, so central. It obviously contains Z , so it's equal to Z . Therefore Y is trivial. \square

The easy way to kill the center of a group G is to look at its image under the adjoint representation $G \rightarrow \text{End}(\mathfrak{g})$. So we call a Lie group *adjoint* if it has no center.

Theorem. Let \mathfrak{g} be a centerless Lie algebra, and G a connected group with that algebra. Then the Lie groups with algebra \mathfrak{g} are exactly the covers of G/Z (and correspond to the subgroups of $\pi_1(G/Z)$).

Proof. Let $G_{\text{ad}} \cong G/Z$ be the image of G in $\text{End}(\mathfrak{g})$. Then any H with algebra \mathfrak{g} maps to G_{ad} under H 's adjoint representation, so H is a covering group of G_{ad} .

To see that every cover has a natural group structure, see it as a quotient of $\widetilde{G}_{\text{ad}}$ by a subgroup of $\pi_1(G_{\text{ad}}) \cong Z(\widetilde{G}_{\text{ad}})$. These are central, therefore normal in the whole group $\widetilde{G}_{\text{ad}}$. \square

Something's missing here: if \mathfrak{g} is a Lie algebra, is there a group G with which to get started? It's not so hard to see that \mathfrak{g} sits inside the Lie algebra of $\text{Aut}(\mathfrak{g})$, and so one can exponentiate it to get a subgroup (which we didn't show).

If \mathfrak{g} isn't centerless, then things are a bit worse, because there isn't a smallest group G_{ad} . For example, if G is a torus, then there are infinitely many covering spaces and discrete quotients. It's then a much harder theorem ("Lie's theorem") that \mathfrak{g} is indeed the Lie algebra of some Lie group (which can even be made a group of matrices).

Lie's original "proof" of this was to look at the adjoint representation! His student Engel pointed out that this wasn't enough, so several years later he published a real proof.

(We haven't talked about these enough to even set up the definitions for real, but it's worth mentioning that infinite-dimensional Lie algebras are *not* always the Lie algebras of any Lie groups!)