SYMPLECTIC GEOMETRY OF K/T
NOTES FOR MATH 261, SPRING 2002

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1. The Lie-Kirillov-Kostant-Souriau form on coadjoint orbits

Let K be a compact connected Lie group, and T a chosen maximal torus. We'll analyze the space K/T and use it to show, among other things, that all tori in K are conjugate (so the choice of T is unimportant) and are self-centralizing.

There is a slight technical annoyance, that will go away once we prove that maximal tori are self-centralizing, but unfortunately this comes at the end. Until then, let T_0 be a maximal torus, and T its centralizer. The group T/T_0 is discrete (otherwise a maximal torus of it lifts to a torus in K larger than T_0, contradiction), so T_0 is the identity component of T, justifying the notation. (Eventually we will show $T = T_0$.)

There is a natural way to get hold of K/T as a homogeneous space for an action: pick a generator $\vec{v} \in \mathfrak{k}$ for a dense one-parameter subgroup of T_0 (an irrational-flow subgroup), and take the **adjoint orbit** $K \cdot \vec{v} \subseteq \mathfrak{k}$. The stabilizer of $\vec{v}$ must centralize T_0, so is T.

Since K is compact, any real representation (such as the adjoint representation $\mathfrak{k}$) can be given a positive definite invariant form, making it isomorphic to the dual representation (in this case the coadjoint representation $\mathfrak{k}^*$). It turns out that coadjoint orbits have some geometry that adjoint orbits generally don't:

Proposition. *Let G be a Lie group (not necessarily compact). Let $\mathcal{O}$ be a* **coadjoint orbit** *of G, an orbit in $\mathfrak{g}^*$. Then there is a natural G-invariant 2-form ω on $\mathcal{O}$, which defines a symplectic form (a nondegenerate skew form) on each tangent space.*

Proof. We first compute the action, so as to compute the stabilizer. The adjoint action is given by $X \cdot Y := [X, Y]$, $X, Y \in \mathfrak{g}$. The coadjoint action of $X \in \mathfrak{g}$ on $\mu \in \mathfrak{g}^*$ is then defined by

$$\langle (X \cdot \mu), Y \rangle = \langle \mu, [X, Y] \rangle.$$

Let $\mu \in \mathcal{O}$. Then from the submersion $K \to \mathcal{O}$, we can model μ's tangent space as $\mathfrak{k}/\mathrm{stab}(\mu)$. So it's enough to define a 2-form on $\mathfrak{k}$ whose kernel is exactly $\mathrm{stab}(\mu)$. The above formula gives us one,

$$\omega_\mu(X, Y) := \langle \omega, [X, Y] \rangle,$$

whose antisymmetry comes from the Lie bracket.

Then if $X \in \mathrm{stab}(\mu)$, this is $\langle (X \cdot \mu), Y \rangle = 0$. Conversely, if $X \notin \mathrm{stab}(\mu)$, then $X \cdot \mu \neq 0$, so there exists a $Y \in \mathfrak{g}$ pairing nontrivially with it. Hence the form descends to a well-defined, nondegenerate, form on $T_\mu\mathcal{O}$.

Finally, the formula is defined using only the (G-invariant) pairing on $\mathfrak{g}$ and $\mathfrak{g}^*$, and the (G-invariant) Lie bracket, so it too is G-invariant. ∎

(It is also true that $d\omega = 0$, but we won't need this.) It's easy to show that a symplectic vector space is necessarily even-dimensional, so this shows that coadjoint orbits are even-dimensional.

Exercise. Let G be the group of 3×3 upper triangular real matrices with 1s on the diagonal. Compute the adjoint orbits of G and show that some are odd-dimensional. Explain how to compute the coadjoint orbits using lower triangular matrices.

2. Stabilizers and critical points

Let $\Phi : \mathcal{O} \to \mathfrak{g}^*$ denote the inclusion of a coadjoint orbit. Given $X \in \mathfrak{g}$, denote by $\Phi_X : \mathcal{O} \to \mathbb{R}$ the X-component of Φ, defined by

$$\Phi_X(\mu) = \langle X, \mu \rangle.$$

Proposition. *Let $\vec{X}$ denote the vector field induced on a coadjoint orbit $\mathcal{O}$ by a Lie algebra element $X \in \mathfrak{g}$. Then the following equation holds:*

$$\iota_{\vec{X}}\omega = -d\Phi_X.$$

Consequently, the critical points of Φ_X are exactly the fixed points of the flow of $\vec{X}$.

Proof. We want to show that if we feed each side the same tangent vector at a point μ, we get the same answer. Since G submerses onto $\mathcal{O}$, it is enough to check using vector fields $\vec{Y}$ from some other Lie algebra element $Y \in \mathfrak{g}$. Then the statement reads

$$\omega_\mu(\vec{X}, \vec{Y}) = -(d\Phi_X)_\mu(\vec{Y})$$

$$\langle \mu, [X, Y] \rangle = -\langle d\Phi_\mu(\vec{Y}), \vec{X} \rangle = -\langle \vec{Y} \cdot \mu, \vec{X} \rangle.$$

For the second statement, we use ω's nondegeneracy to say that $(\vec{X})_\mu$ vanishes if and only if $(d\Phi_X)_\mu$ vanishes. □

Consider the case that $X_T \in \mathfrak{k}$ generates a dense subgroup of a maximal torus T_0 of a compact group K. Then we can figure out the fixed points of X_{T_0}'s flow on K/T, because being X_{T_0}-invariant is the same as being T_0-invariant:

$$T_0 k T_0 = k T_0 \iff k^{-1} T_0 k T_0 = T_0 \iff k^{-1} T_0 k \in T_0 \iff k \in N(T_0).$$

Since K/T_0 is a covering space of K/T, and the projection map is K-equivariant, it takes the vector field induced by X_T on K/T_0 to that induced on K/T. In particular the critical points on K/T are covered by the critical points on K/T_0, which we now know to be indexed by $N(T_0)/T_0$. We already showed that $N(T)/T$ is finite. (See the footnote on the last page.)

So we have a function Φ_{X_T} on K/T with finitely many critical points, and once we show $T = T_0$, we'll have them indexed by the Weyl group $W := N(T)/T$. [1] It remains to show that it's Morse. (Also to compute the indices of the critical points, but we'll do that later.)

[1] Only once we pick an isomorphism of $\mathcal{O}$ and K/T are they indexed by W. Before picking a basepoint, they're only relatively indexed by W.

Proposition. *Let $\mathcal{O} \cong K/T$ be a coadjoint orbit in $\mathfrak{k}$ (not some small coadjoint orbit, like $\{\vec{0}\}$). Let $\mu \in \mathcal{O}^{T_0}$ be a T_0-fixed point. Then as a T_0-rep, $T_\mu \mathcal{O}$ is a sum of nontrivial 2-dimensional real representations, corresponding to the positive roots of K.*

As a consequence, the function Φ_{X_T} is Morse, with all even indices.

Proof. The map $\mathfrak{k} \to T_\mu \mathcal{O}$ has kernel $\mathrm{stab}(\mu) = \mathfrak{t}$, so descends to $\mathfrak{k}/\mathfrak{t} \cong T_\mu \mathcal{O}$. We've more often thought about its complexification, which split into 1-dimensional spaces indexed by the roots. The original real rep is only half as big, and so it's enough to use the positive roots.

By picking a T_0-invariant (or even K-invariant) metric, we can identify a neighborhood of μ T_0-equivariantly with $T_\mu \mathcal{O}$. On $\mathfrak{k}/\mathfrak{t}$, the vector field induced by X_T only vanishes to first order at the origin. Therefore the same is true on $\mathcal{O}$ at μ.

By the equation $\iota_{\vec{X}_T} \omega = -d\Phi_{X_T}$, and ω's nondegeneracy, we see that Φ_{X_T} only vanishes to second order at μ. So it is Morse.

The function Φ_{X_T} is T_0-invariant, so on each of the 2-d irreps it contributes two to either the positive index or the negative index. Hence the indices are even. □

3. Topological and group-theoretic consequences

Using Morse theory, we get a cell complex where all the cells are even-dimensional. Pierre Deligne has described this as "paradise": the (co)homology is also even-dimensional, has no torsion, and has a basis corresponding to the cells.

We begin by abandoning the group T_0:

Corollary. *Let T_0 be a maximal torus of a compact connected group K. Then T_0 is its own centralizer.*

Proof. Let T be the centralizer. Consider the connected covering space map $K/T_0 \to K/T$, with (finite) fiber T/T_0. Since the latter space has a cell decomposition with no 1-cells, it is simply connected, so the map is a diffeomorphism, and $T = T_0$. □

In particular, maximal tori are maximal abelian. (Alas, the converse is not true, though it is in simply-connected groups.)

We will mainly use this

Corollary. *The space K/T has Euler characteristic $|W|$.*

Theorem. *Let K be a compact connected Lie group, and T a maximal torus. Then every element $k \in K$ can be conjugated into T. Also, any other torus S can be conjugated into T. In particular all the maximal tori are conjugate.*

Proof. Let k act on K/T by left multiplication; call this map $k\cdot$. Recall that the *Lefschetz number* of a self-map $k\cdot$ of a compact manifold K/T is defined as

$$L(k\cdot) := \sum_{i=0}^{\dim K/T} (-1)^i \mathrm{Tr}\,(k\cdot)|_{H^i(K/T)},$$

the alternating sum of the traces of the induced action on cohomology. It depends only on the homotopy class of $k\cdot$.

In the case that $k\cdot$ has isolated fixed points (such as none), the Lefschetz number can be computed by as a sum of local contributions around the fixed points (the index of the self-map of a small sphere).

In our case, since K is connected, $k\cdot$ is homotopic to the identity, and the Lefschetz number reduces to the Euler characteristic. Hence there must be a fixed point gT:

$$kgT = gT.$$

As we saw before, this is equivalent to $g^{-1}kg \in T$.

For the second statement, let k be an element of S whose powers are dense in S. Then conjugating k into T puts S in there too. □

4. Homotopical consequences

Theorem. *The space* $K/N(T)$ *has trivial rational cohomology (just* $H^0 = \mathbb{Q}$*).*

Proof. Consider the covering space $K/T \to K/N(T)$ with fiber $W := N(T)/T$. By pullback, we can embed $K/N(T)$'s cochain complex in K/T's. By pushforward and dividing by $|W|$ (requiring rational cohomology), we can go in reverse. This embeds $K/N(T)$'s rational cohomology in K/T's. In particular $K/N(T)$ has only even-dimensional rational cohomology.

On the other hand, the Euler characteristic of $K/N(T)$ is the ratio of that of K/T and the size of the fiber (proof: pull back a triangulation). So it's $|W|/|W| = 1$. There can't be any cancelation in computing it from Betti numbers, since they're all in even degree, so except for H^0 they have to be zero. □

Exercise. What is this space for $K = SU(2)$? What is its integer cohomology?

The next result is not usually considered to be within the scope of this class, so don't be too afraid of it if you haven't seen the concepts before. We won't prove anything else with it.

Corollary. *Let* BK, BT *be classifying spaces for* K, T*. Then there is a natural action of* W *on* $H^*(BT)$*, and a natural map* $BT \twoheadrightarrow BK$*, such that the pullback gives an isomorphism* $H^*(BK;\mathbb{Q}) \to H^*(BT;\mathbb{Q})^W$.

Proof. Let EK be a contractible space on which K acts freely. Then T acts freely too, and we can use EK/T as a BT. (They're unique up to homotopy, so this one will do.)

Then we have a map from $EK/T \twoheadrightarrow EK/K$, the latter being our BK. This map factors as $EK/T \twoheadrightarrow EK/N(T) \twoheadrightarrow EK/K$. Hereafter we call these by their B names.

First consider the map $H^*(BN(T)) \to H^*(BT)$. Since the map $BT \twoheadrightarrow BN(T)$ is W-invariant, the map on cohomology must land inside $H^*(BT)^W$. By averaging as before, we can set up a homotopy equivalence of complexes, showing that $H^*(BN(T);\mathbb{Q}) \cong H^*(BT;\mathbb{Q})^W$.

Now consider the map $BN(T) \twoheadrightarrow BK$, with fibers $K/N(T)$. Ordinarily the Leray-Hirsch theorem would give us the module structure of the cohomology of the total space $H^*(BN(T)$, as a module over that of the base $H^*(BK)$, with generators lifted from $H^*(K/N(T))$. In this case, since $H^*(K/N(T);\mathbb{Q})$ is trivial, we have $H^*(BN(T);\mathbb{Q}) \cong H^*(BK;\mathbb{Q})$. □

The space $BS^1 \cong \mathbb{CP}^\infty$, so $BT \cong (\mathbb{CP}^\infty)^n$, and $H^*(BT;\mathbb{Q}) \cong \mathbb{Q}[x_1, \ldots, x_n]$. In the case $K = U(n)$, then, the ring we're computing is the ring of S_n-symmetric polynomials in n variables. It is then a famous result that this is a polynomial ring itself, in the elementary symmetric polynomials. In fact the same is true for all other K.