MATH 2240 FINAL, SPRING 2015

Name, **written slowly and legibly:** ________________________________

In each answer, write as much (on front and back) as it takes to convey your thought process; full English sentences are much easier to give credit to than bare, unmotivated scribbled formulæ. (They won't do any good if they can't be read, so *do* put effort into making them legible.)

Feel free to ask me questions during the test, especially if you need a little reminder about a definition. Worst case is I don't answer. (It's very sad to afterward hear "I didn't realize I could ask you that" — find out!)

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1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function, and $g : \mathbb{R} \to \mathbb{R}$ its restriction to an axis, i.e. $g(x) := f(x, 0)$.

a [5]. Give an example of such an f where f is Riemann integrable, but g isn't.

Answer. $f(x, y) = 1$ only where $x = 0$ and $y \in \mathbb{Q} \cap [0, 1]$.

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b [5]. Give an example of such an f where g is Riemann integrable, but f isn't.

Answer. $f(x, y) = 1$ where $x \in \mathbb{Q} \cap [0, 1]$ and $y \in [0, 1]$.

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2. Call two manifolds M, N "diffeomorphic" if there exists a smooth function $f : M \to N$ with smooth inverse.

a [5]. Give a complete list of compact 0-manifolds M, i.e. any compact 0-manifold should be diffeomorphic to exactly one element of your list.

(I.e. how would you minimally describe a particular M over the phone?)

Answer. M is a finite set of points; two compact 0-manifolds are diffeomorphic if they have the same number of points. If M were an infinite set of separate points, it wouldn't be compact.

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b [5]. For which M from (a) is $M = \partial N$, where N is a compact 1-manifold-with-boundary?

Answer. M has to have an even number of points; each closed interval in N contributes two. (Each circle contributes none.) We can't have clopen intervals (with one) because they're not compact.

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c [5]. Now, the oriented version.
If M, N are oriented, call them "oriented-diffeomorphic" if f can be taken to be orientation-preserving.

Give a complete list of the compact, oriented 0-manifolds M in $\mathbb{R}^3$ (say).

Answer. M is a finite number of points, some positively oriented, some negatively.

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d [5]. For which M from (c) is $M = \partial N$, where N is a compact oriented 1-manifold-with-boundary?

Answer. M has to have the same number of positive as negative points; each closed interval contributes one positive, one negative.

Incidentally, every compact 1-manifold is a union of circles, and is the boundary of something, a bunch of discs. Every compact oriented 2-manifold is a doughnut with g holes, and is the boundary of something, a solid doughnut with g holes. It's harder to show that every compact oriented 3-manifold is the boundary of something, but it's true. Finally, at $d = 4$ one again finds a compact oriented d-manifold that isn't a boundary:

$$\mathbb{CP}^2 = \{\text{unit vectors } \vec{v} \in \mathbb{C}^3\} / \left(\vec{v} \sim e^{i\theta}\vec{v}\right)$$

but this is the "only" example; for any compact oriented 4-manifold M, there is a unique integer s (called the signature of M) such that $M \cup \bigcup^s \mathbb{CP}^2$ is a boundary.

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3. Let $M = \mathbb{R}^3 \setminus \{(0, 0, z) \ : \ z \in \mathbb{R}\}$, i.e. rip out a line.

a [5]. Write down a 1-form α on M such that $d\alpha = 0$, but $\alpha \neq d\beta$ for any function β.

Answer. Let α be our old friend "$d\theta$" = $\frac{x\ dy - y\ dx}{\sqrt{x^2+y^2}}$.

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b [5]. Same thing, but $M = \mathbb{R}^3 \setminus \{(f(z), g(z), z)\}$, where f, g are two smooth functions $\mathbb{R} \to \mathbb{R}$.

Answer. Let $M_{0,0}$ be the M from part (a), and $M_{f,g}$ the M from part (b). Then there's a diffeomorphism

$$F : \ (x, y, z) \mapsto (x - f(z), y - f(z), z)$$

from $M_{f,g}$ to $M_{0,0}$, and $F^*(\alpha)$ works now.

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4. Let $\omega = \sum_{i=1}^{n} (dx_{2i-1} \wedge dx_{2i})$, so a 2-form; e.g. $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6$ for $n = 3$.

a [5]. Compute $\omega^{\wedge n}$, at least for $n \leq 3$ if you can't do the general case.

Answer.

$$\begin{aligned}
\omega^{\wedge n} &= \sum_{i=1}^{n} (dx_{2i-1} \wedge dx_{2i})^{\wedge n} \\
&= \sum_{i_1,\ldots,i_n} (dx_{2i_1-1} \wedge dx_{2i_1}) \wedge \cdots (dx_{2i_n-1} \wedge dx_{2i_n}) \\
&= \sum_{i_1,\ldots,i_n \text{ distinct}} (dx_{2i_1-1} \wedge dx_{2i_1}) \wedge \cdots (dx_{2i_n-1} \wedge dx_{2i_n})
\end{aligned}$$

where the last equality holds (i.e. we get to drop some terms) because $dx_i \wedge dx_i = 0$ kills any term with a repeated i_j. Then, when we rearrange the $i_1 < \ldots < i_n$ in increasing order, every one of these $n!$ terms becomes $+(dx_1 \wedge \cdots \wedge dx_{2n})$, since these 2-forms commute with each other.

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b [5]. Compute $d(\omega^k)$, for each $k \in [1, n]$.

Answer. $d\omega = 0$ handles $k = 1$, and the Leibniz rule kills all the other $d(\omega^k)$, too.

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5 [10]. Say you knew Stokes' theorem $\int_S d\alpha = \int_{\partial S} \alpha$, but only for unit squares $S = [a, a+1] \times [b, b+1]$ in the plane. (Which has corners, but you can still guess what $\int_{\partial S} \alpha$ should mean.)

How could you, from that, derive Stokes' theorem for the "square annulus"

$$T = ([0,3] \times [0,3]) \setminus ([1,2] \times [1,2]) \quad \subset \mathbb{R}^2$$

of area $3^2 - 1^2 = 8$?

Answer. T is the union of 8 unit squares. For each one, Stokes' theorem says that the integral of $d\alpha$ over the square is the sum of the edges along the four edges (always clockwise). If you add these 8 Stokes' theorems together, the sum of 32 integrals along edges becomes 12 around the outside loop, 4 around the inside loop, and $2*8$ more that come in opposite pairs.

A couple of people used change-of-variable to derive the size-3 square Stokes' theorem from the unit squares, then subtracted off the unit square in the middle.

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6 [15]. Let $\tau : \{(r, \theta) \ : \ r \in [0,1], \theta \in [0, \frac{3}{4}\pi]\} \to \mathbb{R}^2$ take $(r,\theta) \mapsto (r\cos\theta, r\sin\theta)$, and T be its image, 3/8 of the unit disc.

For $f : \mathbb{R}^2 \to \mathbb{R}$ a function, write down $\int_T f\ dx \wedge dy$ as an iterated integral three different ways: over x then y, over y then x, and over r then θ.

Answer.

$$\int_{x=-1/\sqrt{2}}^{1} \int_{y=\max(-x,0)}^{+\sqrt{1-x^2}} f(x,y) dx \wedge dy$$

$$\int_{y=0}^{1} \int_{x=\max(-y,-\sqrt{1-x^2})}^{+\sqrt{1-x^2}} f(x,y) dx \wedge dy$$

$$\int_{r=0}^{1} \int \theta = 0^{3\pi/4} f(r\cos\theta, r\sin\theta) r\ dr \wedge^d \theta$$

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7 [10]. Let M be the set of 2×3 matrices of rank 1. We've looked at this nonorientable 4-manifold before.

Set up the Lagrange multiplier problem that would find the closest point $\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ to the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. *Actually* finding it would take a while, so don't do that.

Answer. M is defined by three equations, the three 2×2 determinants, even though its codimension is only two. You need all three! Their differentials are

$$D(ae - bd) = [e - d0 - ba0]$$

$$D(af - cd) = [f0 - d - c0a]$$

$$D(bf - ce) = [0f - e0 - cb]$$

The function to minimize is $(a-1)^2 + b^2 + c^2 + d^2 + (e-1)^2 + f^2$, whose D is

$$[2(a-1)\ 2b\ 2c\ 2d\ 2(e-1)\ 2f]$$

You could minimize the square root of that, which is unnecessarily yucky.

One person pointed out that the minimum p should have last column is zero. (Why? If you zero out the last column of p, that shrinks its ditance to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, and doesn't increase its rank.) At that point the Lagrange multiplier question is about the single condition $ae - bd = 0$ and function $(a-1)^2 + b^2 + d^2 + (e-1)^2$, so

$$[e \ -d \ -b \ a] = \lambda[2(a-1) \ 2b \ 2d \ 2(e-1)].$$

8 [15]. Let

$$0 \to \mathbb{R}^2 \xrightarrow{a} \mathbb{R}^4 \xrightarrow{b} \mathbb{R}^m \xrightarrow{c} \mathbb{R}^3 \xrightarrow{d} \mathbb{R}^n \xrightarrow{e} \mathbb{R}^3 \to 0$$

be an exact sequence, i.e. each kernel = the image of the previous map.

What are the possible triples $(m, n, \mathrm{rank}(c))$? At least find some conditions they must satisfy.

Answer. The alternating sum of the dimensions must be 0,

$$2 - 4 + m - 3 + n - 3 = m + n - 8 = 0$$

Also

$$\mathrm{rank}(c) = m - \mathrm{rank}(b) = m - (4 - \mathrm{rank}(a)) = m - (4 - 2) = m - 2$$

so everything's determined by $\mathrm{rank}(c) \in [0, 3]$.

To show that each choice of $\mathrm{rank}(c)$ actually works, one can write down tuples of maps, most easily indicated by drawing arrows from one basis element to another (with no arrow out if the element is in the kernel). Then an exact sequence is one such that any basis element has one arrow in or one arrow out, and not both.