ANSWERS TO VARIOUS HOMEWORK PROBLEMS IN MATH 3210, FALL 2015

HOMEWORK #11 (DUE FRIDAY DEC 4)

Let M be a manifold. Define the **Poincaré polynomial** $p_M(t) := \sum_{i=0}^{\dim M} (\dim H^i(M)) t^i$ and the **Euler characteristic** as $\chi(M) := p_M(-1)$.

Q. Use Mayer-Vietoris to show that if $M = U \cup V$ (open submanifolds), then $\chi(M) = \chi(U) + \chi(V) - \chi(U \cap V)$.

A. Let $0 \to A_1 \to \ldots A_n \to 0$ be an exact sequence (soon to be M-V). We claim that $\sum_i (-1)^i \dim A_i = 0$. Why? (1) This is true if the sequence is just

$$0 \to \ldots \to 0 \to \mathbb{R} \to \mathbb{R} \to 0 \to \ldots \to 0,$$

(2) every other exact sequence is a direct sum of these, and (3) the condition stated is linear in the n-vector $(\dim A_i)$.

Now apply that to Mayer-Vietoris. Since 3 is an odd number, the signs on any column alternate, and we get $\chi(M) - (\chi(U) + \chi(V)) + \chi(U \cap V) = 0$.

Q. Compute the Euler characteristic of a genus g surface M using this.

A. Let χ_g be the answer. We already did genus $g = 0, 1$ and got $\chi_0 = 2, \chi_1 = 0$. So consider $g > 1$ and cut off a torus at the end of M i.e. make U a punctured genus $g-1$ surface, and V a punctured torus, glued along a cylinder.

If we glued discs V', U' to U, V to fix the punctures, we could use M-V to compute $\chi(U), \chi(V)$:

$$\chi_{g-1} = \chi(U) + \chi(V') - \chi(U \cap V') = \chi(U) + 1 - 0$$
$$\chi_1 = \chi(U') + \chi(V) - \chi(U' \cap V) = 1 + \chi(V) - 0$$

so $\chi(U) = \chi_{g-1} - 1, \chi(V) = -1$. Now we can do the real calculation:

$$\chi_g = \chi(U) + \chi(V) - \chi(U \cap V) = (\chi_{g-1} - 1) + (-1) - 0$$

i.e. it decreases by 2 each time. So $\chi_g = 2 - 2g$, to match our initial conditions.

(In fact $p_M = 1 - 2gt + t^2$.)

Q. What's the best statement you can make like that formula, relating the Poincaré polynomials?

A. Break the M-V LES into a sum of 2-steps like we did in the first question. These 2-steps come in three types:

- a part of $H^i(M) \to H^i(U) \oplus H^i(V)$,
- a part of $H^i(U) \oplus H^i(V) \to H^i(U \cap V)$, and
- a part of $H^i(U \cap V) \to H^{i+1}(M)$.

If all the snake maps $H^i(U \cap V) \to H^{i+1}(M)$ were zero, instead of a LES we'd have a SES for each i, and we'd get $\dim H^i(M) - (\dim H^i(U) + \dim H^i(V)) + \dim H^i(U \cap V) = 0$. Putting these together, $p_M - (p_U + p_V) - p_{U \cap V} = 0$.

How do the third type spoil that? They add a t^i to p_M and a t^{i+1} to $p_{U\cap V}$. So if we let $c = \sum_i rank(H^i(U \cap V) \to H^{i+1}(M))t^i$, which in effect measures how many snakes are involved, then $p_M - (p_U + p_V) - p_{U\cap V} = (1+t)c$.

HOMEWORK #10

Q. Let $T : A \to B$, $U : C \to B$. Break up A, B, C as direct sums in ways that T, U respect, such that on each piece T, U are either 0 or isomorphisms.

A. Let's write this as $A \xrightarrow{T} B \xleftarrow{U} C$, in general. The first pieces to go are

$$\ker T \to 0 \leftarrow 0$$

$$0 \to 0 \leftarrow \ker U$$

and we pick linear complements $A' \leq A$, $C' \leq C$ to $\ker T, \ker U$.

(Note that the definition of "complementary linear subspace" X' to $X \leq Y$, meaning that every vector in Y is uniquely of the form $\vec{x} + \vec{x}'$, is **not** the same as "complementary subset" $Y \setminus X$. That's not a subspace! It doesn't even have 0 in it! A very big difference is that there's only one complementary subset, vs. lots of potential choices for X'.)

Inside C, we have two subspaces $image(T), image(U)$ from which we can build $M := image(T) \cap image(U)$ and $image(T) + image(U)$. These are coming from the following pieces:

- $M = image(T) \cap image(U)$ from $1 \to 1 \leftarrow 1$ only
- $image(T)$ also comes from $1 \to 1 \leftarrow 0$
- $image(U)$ also comes from $0 \to 1 \leftarrow 1$.

Let $I_T \leq image(T)$, and $I_U \leq image(U)$, each be complements to M but inside $image(T), image(U)$ respectively. Then we claim

$$image(T) + image(U) = M \oplus I_T \oplus I_U.$$

Proof: certainly

$$image(T) + image(U) = (M + I_T) + (M + I_U) = M + I_T + I_U$$

and the dimension is right.

Then we can split $A' \to C \leftarrow B'$ into

$$A' \cap T^{-1}(M) \xrightarrow{\sim} M \xleftarrow{\sim} B' \cap U^{-1}(M)$$

$$A' \cap T^{-1}(I_T) \xrightarrow{\sim} I_T \leftarrow 0 \qquad 0 \to I_U \xleftarrow{\sim} B' \cap U^{-1}(I_U)$$

$$0 \to C' \leftarrow 0$$

where $C' \leq C$ is a complement to $image(T) + image(U)$.

Q. Put in another space $V : D \to B$, and assume B is 2-dimensional while A, C, D are 1-dimensional. Describe all the possibilities, up to change of basis in A, B, C, D.

[For laziness in typesetting I'm going to write these as $A, C \implies B \leftarrow D$ rather than drawing pictures.]

A. If any of the maps into B are 0, then (up to permuting A, C, D) we can split off a $0, 0 \implies 0 \leftarrow 1$, and use the previous problem; the remaining $1 \to 1 \leftarrow 1$ could come from

$$111, \quad 110 \oplus 001, \quad 100 \oplus 011, \quad \text{or} \quad 100 \oplus 010 \oplus 001.$$

So now assume that all the maps into B are $1:1$. Then the image is (up to) three lines in the plane.

If there's just one line, we can change basis in B to make it be the x-axis, and change bases in A, C, D to take their basis element to $\vec{e}_1$. Then we're looking at $(1, 1 \implies 1 \leftarrow 1) \oplus (0, 0 \implies 1 \leftarrow 0)$.

If there are two lines (which, up to change of basis in B, are the axes), i.e. $image(T) = image(U)$, then we're looking at $(1, 1 \implies 1 \leftarrow 0) \oplus (0, 0 \implies 1 \leftarrow 1)$.

If there's three, though, then change of basis can make them be slope 0, ∞, and 1. We then can't split up the representation as a direct sum. In this last case, we have a new phenomenon we didn't see when the graph of vector spaces was a line; $\dim B = 2$ occurs in one of the fundamental building blocks.

HOMEWORK #9: BOOK PROBLEMS 9.1, 9.2, 9.5

9.1. Let $U \subseteq \mathbb{R}^n$ open, $f, g : U \to \mathbb{R}$, $f < g$ pointwise, $M = \{(x, y) : f(x) \leq y \leq g(x)\}$.

(i) Show M is a manifold with boundary.

A. Let $(x, y) \in M$. Since U is an open set, it contains a ball $B \ni x$. Let $V = \{(x, y) \in M : x \in B\}$, which we correspond with $B \times [0, 1]$ by

$$\begin{aligned} c : B \times [0,1] &\xrightarrow{\sim} V \\ (x, r) &\mapsto (x, f(x) + (g(x) - f(x))r) \end{aligned}$$

with smooth inverse $c^{-1}(x, y) = (x, \frac{y - f(x)}{g(x) - f(x)})$.

The LHS has three open sets $B \times [0, 1)$, $B \times (0, 1)$, $B \times (0, 1]$, each obviously diffeomorphic to $\mathbb{R}^{n+1}$ or $\mathbb{R}^n \times [0, \infty)$. Composing c with those diffeomorphisms, we get open sets covering V showing that V is a manifold-with-boundary.

More specifically, if $y = f(x)$ or $y = g(x)$, then $(x, y) \in \partial M$, otherwise not.

(ii) Draw this when $f(x) = -\sqrt{1 - x^2 - y^2}$, $g(x) = 2 - x^2 - y^2$.

A. Hard to do here; let me say that below $y = 0$ we see half a ball, and above we see a parabolic mirror. (Those are really cool btw – if you point two at each other and have two people stand at their foci, you can talk clearly over quite long distances.)

(iii) Give an example with $f(x) \not< g(x)$ and M not a manifold.

A. E.g. $U = \mathbb{R}$, $f(x) = -x$, $g(x) = x$, and M is then the first quadrant but rotated $45°$ down. So it's got this nasty corner at $(0, 0)$ where it's not a manifold-with-boundary. (What happens to the c above?)

9.2. Let $\alpha, \beta \in \Omega^\bullet(M)$. Show $supp(\alpha \wedge \beta) \subseteq supp(\alpha) \cap supp(\beta)$. Given an example where it's strict.

A. If $\alpha(x) = 0$ or $\beta(x) = 0$, then $(\alpha \wedge \beta)(x) = 0$; this is a reformulation of the first statement.

For strictness, let $M = \mathbb{R}$, and $\alpha, \beta = dx$. Then their support is M, but $\alpha \wedge \beta = 0$ with empty support.

(Note this couldn't happen with 0-forms!)

9.5. (i) Let $\alpha = x\ dy - y\ dx$ and $M \subseteq \mathbb{R}^2$ be a compact 2-manifold-with-boundary. Show $\int_{\partial M} \alpha = 2area(M)$.

A. Recall $x\ dy$ means $x \wedge dy$, regarding x as a 0-form. So,

$$d\alpha = d(x \wedge dy) - d(y \wedge dx) = dx \wedge dy - dy \wedge dx = 2dx \wedge dy$$

hence $\int_{\partial M} \alpha = \int_M d\alpha = \int_M 2dx \wedge dy = 2area(M)$.

Btw there are cool devices vaguely like compasses with which to measure areas of plane figures, that work on this principle: `https://en.wikipedia.org/wiki/Planimeter` `https://www.youtube.com/watch?v=kdxPEZnv-U0`

(ii) Compute the area within $x = \cos^3 t, y = sin^3 t$.

A.

$$\begin{aligned}
\int_{t=0}^{2\pi} x\ dy - y\ dx &= \int_{t=0}^{2\pi} (\cos^3 t\ d(\sin^3 t) - sin^3 t\ d(\cos^3 t)) \\
&= \int_{t=0}^{2\pi} (\cos^4 t 3 \sin^2 t\ dt + sin^4 3 \cos^2 t\ dt) \\
&= 3 \int_{t=0}^{2\pi} (\sin^2 t \cos^2 t (\sin^2 t + \cos^2 t)\ dt) \\
&= 3 \int_{t=0}^{2\pi} \left(\frac{\sin 2t}{2} \right)^2\ dt \qquad = \frac{3}{4} \int_{t=0}^{2\pi} (\sin 2t)^2\ dt \qquad = \frac{3}{4}\pi
\end{aligned}$$

so the area is $\frac{3}{8}\pi$.

(iii) Let $\alpha = x \wedge *dx$ on $\mathbb{R}^n$.

A. Oops. I didn't mean to assign a problem on this, and there won't be any Hodge-stars on the final exam.

(Though the answer is: each of the n terms in the wedge product has the same d, namely the standard volume n-form; hence $d\alpha = n$ times volume.)

Homework #8

Q1. Let $M = \mathbb{R}^2 \setminus 0$, the punctured plane. We computed its ordinary cohomology $H^*(M)$ a few weeks ago, with dimensions $1, 1, 0$.

Compute its compactly supported cohomology $H^*_c(M)$, i.e. using the d operator on compactly supported forms. (Careful: the set of vectors $\vec{v}$ in $\mathbb{R}^2$ with $|\vec{v}| \leq r$ is compact, but once you remove $\vec{0}$ it's not compact.)

Hint: if you look at Q2 first, you know the dimensions are going to go $0, 1, 1$.

A. $H^0_c = \{$compactly supported locally constant functions$\}/0$, so as before, this is 0. More generally, $\dim H^0_c =$ the number of compact components of M.

$H^1_c = \{$compactly supported closed 1-forms$\}/d\Omega^0$. Since we want to show this is 1-dim, we should find a map $\{$compactly supported closed 1-forms$\} \to \mathbb{R}$ that kills $d\Omega^0$, show it's nonzero, and show its kernel is exactly $d\Omega^0$.

In the ordinary (not compactly supported) cohomology case, we used an integral around a circle containing the origin. This time, use the radial 1-chain $\gamma : \mathbb{R}_+ \to \mathbb{R}^2$, $t \mapsto (t, 0)$.

We need to check that $\alpha \mapsto \int_\gamma \alpha$

(1) kills $d\Omega^0$
(2) and only $d\Omega^0$
(3) the value is not always zero.

For part (3), let $b : \mathbb{R}^2 \to \mathbb{R}$ be a function that is 0 inside the unit disc, 3 outside radius 1, and smooth in between. (It's easy to cook up one of these using bump functions, e.g. make f only a function of the distance from $\vec{0}$.) This b isn't compactly supported. But db is supported inside the annulus, and $\int_\gamma df = 3$ by the FTC. We'll use this db like the $d\theta$ we did before.

For part (1), let g be a compactly supported function on M, hence there exist radii r, R such that $g(x) = 0$ for $|x| < r$ or $|x| > R$. Then $\int_\gamma dg = \int_{t=r}^R dg = g(R) - g(r) = 0 - 0$.

Now part (2). Let α be a closed compactly supported 1-form on M with $\int_\gamma \alpha = 0$. We want to show $\alpha = dg$ for some compactly supported function g. First observe that α extends over $\mathbb{R}^2$ by letting it be 0 at $\vec{0}$. We already computed $H^1(\mathbb{R}^2) = 0$, so since $d\alpha = 0$, we know $\alpha = dg$ for some function g, not yet necessarily compactly supported. By subtracting $g(0)$ we can ensure this g vanishes at 0. (And since $dg = 0$ in a neighborhood of 0, g is constant in that neighborhood, i.e. 0 there.)

Now let x be a point with $|x| > R$, and let γ' be a path from $(0, 0)$ straight to $(r, 0)$ straight to $(R, 0)$ (along γ), then over to x while staying outside radius R. Then

$$g(x) = g(0) + \int_{\gamma'} \alpha = 0 + \int_{\gamma'} \alpha = 0$$

i.e. g vanishes outside radius R. This completes (2).

Finally, $H^2_c(M)$, i.e. {compactly supported 2-forms}$/d\Omega^1_c$. Again, we'll show this space is nonzero by mapping it to $\mathbb{R}$, and check out (1)-(3). This time, we take $\alpha \in \Omega^2_c(M)$ to $\int_M \alpha$.

For part (1), if $\alpha = d\beta$ where β is a 1-form supported inside the open annulus A of radii r, R, then

$$\int_M \alpha = \int_M d\beta = \int_{\bar{A}} d\beta = \int_{\partial\bar{A}} \beta = 0$$

much as before.

For part (3), let $\alpha(x, y) = \mathrm{bump}(x^2 + y^2) dx \wedge dy$, where bump is a nonnegative bump function supported inside $[\sqrt{r}, \sqrt{R}]$.

Q2. Show that each pairing $H^i_c(M) \times H^{2-i}(M) \to \mathbb{R}$, by wedging then integrating, is perfect.

A. This is trivial for $i = 2$. For $i = 0, 1$ we have to check that the map is nonzero. I.e. for $i = 0$ let $\beta = 1 \in \Omega^0(M)$ and α be the bump form just above. Then $\int_M \alpha \wedge \beta > 0$.

Q. Book problem 8.6: Let F_n be the set of orthonormal bases of $\mathbb{R}^n$ and F_n^+ the positive ones. (a) Identify these with $O(n)$ and $SO(n)$. (b) is mostly about reading.

A. The correspondence takes a basis to the matrix with those columns.

Q. Given orientations on a pair $V < W$ of vector spaces, show how to define an orientation on the quotient space W/V. Your rule should have the property that if you flip V or W the rule flips W/V.

A. Pick a basis $b_1, \ldots, b_k$ of V and extend to one of W (of dim n). Then the images $\bar{b}_{k+1}, \ldots, \bar{b}_n$ form a basis of W/V, and we define

$$O_{W/V}(\bar{b}_{k+1}, \ldots, \bar{b}_n) = O_W(b_1, \ldots, b_n)/O_V(b_1, \ldots, b_k)$$

(obviously we could use multiplication instead of division, since these are ± 1, but this fits better with the volume-form idea). Plainly this does the flips required.

We still need to know it's well-defined, i.e. didn't depend on the choices. If we change V's basis but not $b_{k+1}, \ldots, b_n$, then numerator and denominator change by the same sign. If we change the remaining part of the basis, then both sides change by the same sign.