NOTES FOR MATH 7670, ALGEBRAIC GEOMETRY

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CONTENTS

I plan to spend a week or two on some very classical constructions, largely to have examples at hand to motivate the generalizations required by schemes. Then I'll jump to [Vakil, chapter 3], delving back into chapter 2 from time to time. If there's a common request that I discuss something in chapter 1, we can do that.

All rings are commutative with 1, and homomorphisms preserve 1.

1. ALGEBRAIC SUBSETS OF $\mathbb{C}^n$

In this section $\mathbb{F}$ will be a field, typically $\mathbb{R}$ or $\mathbb{C}$.

Given a list of polynomial equations in n variables, namely $f_i = 0$ for $f_i \in R = \mathbb{F}[z_1, \ldots, z_n]$, we can associate its "solution set" or "zero set" or **algebraic subset**

$$V(I) := \left\{\vec{z} = (z_1, \ldots, z_n) \in \mathbb{F}^n : f_i(\vec{z}) = 0 \ \forall i\right\}.$$

Here $I = \langle f_1, \ldots, \rangle$ as it's pretty obvious that the set only depends on the ideal generated.

To what extent can we reverse this? For an arbitrary subset $X \subseteq \mathbb{F}^n$, we can define

$$I(X) := \{f \in R : \forall x \in X, f(x) = 0\}.$$

Exercise 1.1. (1) *Show* $I(V(I)) \geq I$ *and* $V(I(X)) \supseteq X$.

(2) *Given an ideal* I, *show that the* **radical**

$$\sqrt{I} := \{f \in R : \exists n \geq 1, f^n \in I\}$$

is again an ideal.

(3) *Show* $I(X)$ *is a radical ideal.*

(4) *Show* $I(X) = I(\overline{X})$, *where* $\overline{X}$ *is the closure in the usual topology on* $\mathbb{F}^n$, *for* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$.

This map $V(\bullet)$ from ideals to subsets of $\mathbb{C}^n$ is very far from onto (and there's no really independent way to characterize its image, so we just take the definition above as our only definition of "algebraic subset"). There are two reasons that $V(\bullet)$ is not 1:1:

- $V(I) = V(\sqrt{I})$.
- If $\mathbb{F}$ is not algebraically closed, there may not be enough solutions to characterize the ideal (e.g. $V(\langle x^2+1\rangle) = V(\langle 1\rangle)$ for $\mathbb{F} = \mathbb{R}$).

Theorem (Hilbert's Nullstellensatz)**.** *If $\mathbb{F}$ is algebraically closed, then $I(V(I)) = \sqrt{I}$. In particular $V(\bullet), I(\bullet)$ are inverse bijections between radical ideals and algebraic subsets.*

While one can pound away at the algebra and prove this, the proof is much more comprehensible (or, less magical) once one has absorbed some concepts we'll introduce later.

One usually meets ideals as kernels of homomorphisms. So one might wonder how to relate an algebraic subset X to R/I_X, instead of to I_X directly. One of the many good reasons to do that is that $V(\bullet)$ is order-*reversing* on ideals, $I \geq J \implies V(I) \subseteq V(J)$, but more order-preserving on quotients, $V(J) \supseteq V(I) \implies R/J \twoheadrightarrow R/I$.

For $\mathbb{E}$ a field extension of $\mathbb{F}$, define the **$\mathbb{E}$-points of a ring** S as the set of $\mathbb{F}$-algebra homomorphisms $S \to \mathbb{E}$ such that the fraction field of the image is all of $\mathbb{E}$.

Exercise 1.2. (1) *Give a correspondence between the $\mathbb{F}$-points of $R = \mathbb{F}[z_1, \ldots, z_n]$ and $\mathbb{F}^n$, for $\mathbb{F}$ algebraically closed.*
(2) *Find the $\mathbb{R}$-points and $\mathbb{C}$-points of $\mathbb{R}[x]$.*
(3) *If $\phi : S_1 \to S_2$ is an $\mathbb{F}$-algebra homomorphism, show there's a natural map ϕ^** backwards *on sets of $\mathbb{E}$-points.*
(4) *In the case $\phi : R \twoheadrightarrow R/I$, show that ϕ^* is injective, and the image is $V(I)$.*
(5) *If R is finitely generated over $\mathbb{F}$ algebraically closed, show that each maximal ideal I of R gives an $\mathbb{F}$-point. Find counterexamples if either hypothesis is omitted.*
(6) *The ideal $I \leq R$ is radical iff the quotient ring R/I is ...?*

If we were just interested in the set X, not its embedding in affine space, then the above lets construct it from the ring R/I_X. One of our many tasks is to do the same for arbitrary commutative rings, not just those that are finitely generated over an algebraically closed field.

Even before that, we can look at some of the maps on $\mathbb{F}$-point sets induced by maps of rings. If $g_1, \ldots, g_k \in \mathbb{F}[x_1, \ldots, x_n]$ is a list of polynomials, we can define a function

$$m : \mathbb{F}^n \to \mathbb{F}^k, \qquad (x_1, \ldots, x_n) \mapsto (g_1(\vec{x}), \ldots, g_k(\vec{x}))$$

and a ring homomorphism

$$\phi : \mathbb{F}[x_1, \ldots, x_k] \mapsto \mathbb{F}[y_1, \ldots, y_n], \qquad p(x_1, \ldots, x_k) \mapsto p(g_1(\vec{y}), \ldots, g_k(\vec{y})).$$

Exercise 1.3. (1) *Check that $m = \phi^*$.*
(2) *Consider the map $m(x) = (x^2, x^3)$ from $\mathbb{F}^1 \to \mathbb{F}^2$. Its image is an algebraic set $X = V(I)$; what is the I defining it?*
(3) *Show that the map $m : \mathbb{F}^1 \to X$ is bijective.*
(4) *Find a ring homomorphism $\phi : \mathbb{F}[s,t]/I \to \mathbb{F}[x]$ inducing this map (as $m = \phi^*$). Show it's unique, and not invertible.*
(5) *Show that the image of $m(x, y) = (x, xy)$ is not an algebraic set, nor even locally closed in the usual topology.*

1.1. **Operations on ideals.** An arbitrary union of algebraic subsets is not usually an algebraic subset. But a *finite* union is:

Proposition 1.4. (1) $V(\cap_{I\in\Gamma} I) \supseteq \bigcup_{I\in\Gamma} V(I)$. (2) $V(I\cap J) = V(I)\cup V(J)$.

Proof. (1) Since $\cap_{I\in\Gamma} I$ is in each I, $V(\cap_{I\in\Gamma} I)$ contains each $V(I)$, hence their union.
(2) We need the opposite inclusion, $V(I\cap J) \subseteq V(I)\cup V(J)$. Let IJ be the ideal generated by $\{ij : i \in I, j \in J\}$. Then $V(I\cap J) \subseteq V(IJ) = \{\vec{z} : i(\vec{z})j(\vec{z}) = 0\ \forall i, j\}$. If $\vec{z} \notin V(I)$, then $\exists i \in I$ such that $i(\vec{z}) \neq 0$, and similarly $\exists j \in J$ such that $j(\vec{z}) \neq 0$, with which to show $\vec{z}$ is not in the right-hand side, forcing that to be $V(I)\cap V(J)$.

□

This also shows $V(I\cap J) = V(IJ)$. We won't have much other use for IJ.

Exercise 1.5. *If* I, J *are radical, show* $I\cap J = \sqrt{IJ}$. *Give an example where* IJ *is not radical.*

By contrast, an arbitrary intersection is algebraic:

$$\cap_{I\in\Gamma} V(I) = V(\sum_{I\in\Gamma} I) \qquad \text{tautological}$$

and these two facts put together give as good a reason as any to define a **Zariski topology** in which the algebraic subsets are the closed sets. (Though properly speaking the Zariski topology is on a set containing $\mathbb{F}^n$ that we'll define later.)

Note that a sum of radical ideals is *not* always radical, e.g. $\langle y\rangle + \langle y - x^2\rangle$.

The **colon ideal** $I : J$ is defined as $\{r \in R : rJ \leq I\}$, and has a surprisingly useful interpretation (for I radical):

Proposition 1.6. *If* I *is radical,* $I : J = I\big(V(I) \setminus V(J)\big)$.

Proof.

$$\begin{aligned} r \in I\big(V(I)\setminus V(J)\big) &\iff r(\vec{z}) = 0 \quad \forall \vec{z} \in V(I)\setminus V(J) \\ &\iff (rj)(\vec{z}) = 0 \quad \forall \vec{z} \in V(I), j \in J \\ &\iff rj \in I, j \in J \qquad \text{where “} \Longrightarrow \text{” uses I radical} \\ &\iff r \in I : J. \end{aligned}$$

□

For example, figure out the geometric interpretation of $\langle xy\rangle : \langle x\rangle = \langle y\rangle$.

If $\phi : S_1 \to S_2$ is a ring homomorphism, for example an inclusion $\mathbb{F}[x_1, \ldots, x_k] \hookrightarrow \mathbb{F}[x_1, \ldots, x_n]$, we can push forward and pull back ideals. The pullback of $J \leq S_2$ is easy to understand in terms of the composite $S_1 \to S_2 \twoheadrightarrow S_2/J$; it corresponds to taking the subset of $\mathbb{E}(S_2)$ and applying ϕ^* to put it in $\mathbb{E}(S_1)$, *then taking the Zariski closure*. (Recall the $(x, y) \mapsto (x, xy)$ example above.)

The pushforward of $I \leq S_1$ is a little bit stranger, just because $\phi(I)$ is not itself an ideal in S_2, usually. Rather, we have to talk about the smallest ideal in S_2 mapped into by $\phi(I)$, and dually, the largest subset in $\mathbb{E}(S_2)$ mapping into $\mathbb{E}(S_1/I)$; this is exactly the preimage.

2. ALGEBRAIC SUBSETS OF $\mathbb{CP}^n$

The only compact algebraic subsets of $\mathbb{C}^n$ are finite, so we're missing out on a lot of nice spaces (and proper maps) if we stick with algebraic subsets of $\mathbb{C}^n$. But we needn't rush into general schemes just to scratch that itch.

Let $\mathbb{F}^\times$ act on $\mathbb{F}^{n+1}$ by dilation, and for now, let

$$\mathbb{FP}^n := (\mathbb{F}^{n+1} \setminus \vec{0})/\mathbb{F}^\times.$$

One obvious way to define a family of "algebraic subsets" of $\mathbb{FP}^n$ is to look for those algebraic subsets of $\mathbb{F}^{n+1}$ that are invariant under dilation.

Exercise 2.1. *Let $R = \mathbb{F}[z_0, z_1, \ldots, z_n]$, where $\mathbb{F}$ is an infinite field. Show that an ideal $I \leq R$ is invariant under the action of $\mathbb{F}^\times$ on R (where $c \cdot z_i := cz_i$), and called a* **homogeneous ideal**, *iff I is generated by homogeneous polynomials.*

We can then define $\mathbb{P}V(I)$ for I homogeneous, and $I(X)$ for $X \subseteq \mathbb{FP}^n$, in obvious ways. The analogue of the Nullstellensatz is almost true, but not quite; the **irrelevant ideal** $\mathfrak{m} := \langle z_0, \ldots, z_n \rangle$ and the unit ideal R are radical ideals that both have $\mathbb{P}V = \emptyset$.

(Later when we define sub*schemes* of $\mathbb{FP}^n$, we'll see that the map $\mathbb{P}V(\bullet)$ is actually very far from 1:1, but in an easily controlled way.)

Inside $\mathbb{FP}^n$ are $n+1$ copies of affine space, coming from the maps

$$(z_0, \ldots, \hat{z_i}, \ldots, z_n) \mapsto [z_0, \ldots, 1, \ldots, z_n]$$

whose images cover $\mathbb{FP}^n$.

Exercise 2.2. *Let $\mathbb{F} = \bar{\mathbb{F}}$.*

1. *Let $I \leq \mathbb{F}[z_0, \ldots, z_n]$ be a radical homogeneous ideal, defining a subset of $\mathbb{FP}^n$. Intersect it with the image of the $i = 0$ map given above. What is the ideal in $\mathbb{F}[z_1, \ldots, z_n]$ defining that?*
2. *Let $J \leq \mathbb{F}[z_1, \ldots, z_n]$ be a radical ideal, defining a subset of $\mathbb{F}^n$. Take its image under the $i = 0$ map given above. What is the homogeneous ideal in $\mathbb{F}[z_0, \ldots, z_n]$ defining that?*
3. *Call these two operations $I \mapsto I-$ and $J \mapsto J+$. Show that $J+- = J$, and find the ideal-theoretic condition on I that governs whether $I-+ = I$.*

The **Hilbert function** h_S of an $\mathbb{N}$-graded ring S with $R_0 = \mathbb{F}$ is $h_S(n) = \dim_{\mathbb{F}} R_n$.

Theorem 2.3 (Hilbert).

1. *If S is finitely generated, i.e. a quotient of a polynomial ring R in finitely many generators, then there is a finite resolution of S as a graded R-module. (We don't need to spell out what this means at the moment.)*
2. *(Corollary) $\exists D$ such that for each $i = 1, \ldots, D$, the function $m \mapsto h_S(Dm + i)$ is a polynomial for large m.*
   Also, if R is generated in degree 1, then $D = 1$ works.

In this $D = 1$ case, we speak of the **Hilbert polynomial** of S, and call its degree the **Hilbert dimension** of $\mathbb{PF}(S)$. The prototype is $S = \mathbb{F}[z_0, \ldots, z_n]$, whose Hilbert dimension is n.

2.1. **The projective analogue of $\mathbb{E}$-points.** Instead of looking at $\mathbb{F}$-algebra morphisms to $\mathbb{E}$, we should look at graded $\mathbb{F}$-algebra morphisms. But this doesn't give the right answer in the case $S = \mathbb{F}[x]$. So we look at graded $\mathbb{F}$-algebra morphisms to $\mathbb{E}[x]$, not contained in $\mathbb{E}$ or any $\mathbb{E}'[x]$, modulo rescaling x. Call this $\mathbb{P}\mathbb{E}(S)$ for the moment.

In the following graded algebras, the parenthesized exponents indicate the degree of the generator.

$\mathbb{F}[z_0^{(1)}, \ldots, z_n^{(1)}]$, *all degree* 1.

$S[y^{(1)}]$, *where* S *is degree* 0. Up to scaling, $y \mapsto x$, and $\mathbb{P}(\mathbb{E}(S[y])) \cong \mathbb{E}(S)$.

$\mathbb{F}[x_1^{(0)}, \ldots, x_n^{(0)}, y_0^{(1)}, \ldots, y_m^{(1)}]$

...rational maps and basepoints...

...the patches above as examples of such maps (without basepoints)...

...the blowup of $\mathbb{F}^n$...

3. Irreducible algebraic sets and prime ideals

For $r \in S$, let $\mathrm{ann}(r) := \{s \in S : sr = 0\}$ (which can more generally be defined for r in an S-module), and extend it to subsets, so we can define $\mathrm{ann}(\mathrm{ann}(r))$. Put another way, $\mathrm{ann}(r) = \langle 0\rangle : \langle r\rangle$, and $\mathrm{ann}(\mathrm{ann}(r)) = \langle 0\rangle : \mathrm{ann}(r)$.

Call a ring **reduced** if it has no nilpotents, i.e. $s^n = 0 \implies s = 0$.

Proposition 3.1. *If* S *is a reduced ring, then for* $r \in S$, *the natural map* $\phi : S \to S/\langle r\rangle \oplus S/\mathrm{ann}(r)$ *is* 1:1. *On* $\mathbb{E}$*-points it becomes* $\mathbb{E}(S) = \mathbb{E}(S/\langle r\rangle) \cup \mathbb{E}(S/\mathrm{ann}(r))$ *(not usually a disjoint union).*

Proof. If $a \in \ker\phi$, $a = rb$ and $ar = 0$. Hence $a^2 = arb = 0$. Since S has no nilpotents, $a = 0$.

The $\mathbb{E}$-points of a finite direct sum are the disjoint union of the $\mathbb{E}$-points (check!). The $\mathbb{E}$-points of a quotient S/I naturally include into those of S. Let $p : S \to \mathbb{E}$ be an $\mathbb{E}$-point. If $p(r) = 0$, then p factors through $S/\langle r\rangle$. Otherwise, for any $a \in \mathrm{ann}(r)$, $p(a)p(r) = p(ar) = p(0) = 0$, so $p(a) = 0$, hence p factors through $S/\langle \mathrm{ann}(r)\rangle$. □

This is not very impressive if $\mathrm{ann}(r) = 0$, i.e. if r is not a "zero divisor". An **(integral) domain** S is one with $0 \neq 1$ and no zero divisors, i.e. $ab = 0$ implies $a = 0$ or $b = 0$. Non-example: any ring with nilpotents, or $S = \mathbb{F}[a, b]/\langle ab\rangle$. A **prime ideal** $I \leq S$ is one such that S/I is an integral domain, i.e. $I \neq S$ and $ab \in I \implies a \in I$ or $b \in I$.

Exercise 3.2. *Let* S *be finitely generated over* $\mathbb{F} = \overline{\mathbb{F}}$, *and put the Zariski topology on* $\mathbb{F}(S)$. *Assume* S *has no nilpotents. Then* S *is a domain iff* $\mathbb{F}(S)$ *is not the union of two closed proper subsets.*

This is obviously very unlike the usual topology on $\mathbb{C}^n$! If $I \leq R$ is a prime ideal, we call $V(I)$ an **irreducible** algebraic subset. To my mind, that one can make this definition is one of the most interesting aspects of *algebraic* geometry.

Exercise 3.3. *Let* $f \in \mathbb{F}[x, y]$ *be a quadratic such that* $V(f)$ *is reducible. What does* $V(f)$ *look like?*

...Example: if we take 2×3 matrices, and kill two of the 2×2 minors, the result is reducible. What are the components? One of them is not a complete intersection...

4. Three classic morphisms

4.1. **The Segre embedding.** Let V, W be vector spaces, and define the map $\sigma : V \times W \to V \otimes W$ by $(\vec{v}, \vec{w}) \mapsto \vec{v} \otimes \vec{w}$. This is not quite 1:1, but becomes so if we descend to $\mathbb{P}V \times \mathbb{P}W \to \mathbb{P}(V \otimes W)$, where it is called the **Segre embedding**.

On the algebra level, we have $\phi : \mathbb{F}[(z_{ij})] \to \mathbb{F}[(x_i), (y_j)]$ by $z_{ij} \mapsto x_i y_j$. Plainly this kills $\langle z_{ij} z_{kl} - z_{il} z_{kj} \rangle$, and indeed these generate the kernel (which we probably won't prove).

Exercise 4.1. *Show that the kernel of* ϕ *is* $\sqrt{\langle z_{ij} z_{kl} - z_{il} z_{kj} \rangle}$. *(In fact the* $\sqrt{}$ *is unnecessary.)*

4.2. **The Veronese embedding.** Fix $k \in \mathbb{N}_+$. Let $\nu_k : V \to Sym^k V \leq V^{\otimes k}$ take $\vec{v} \mapsto \vec{v} \otimes \cdots \otimes \vec{v}$. Again, this descends to an embedding $\mathbb{P}V \to \mathbb{P}(Sym^k V)$.

On the algebra level, we have $\phi : \mathbb{F}[(x_S)] \to \mathbb{F}[z_1, \ldots, z_n]$, where S ranges over functions $\{1, \ldots, n\} \to \mathbb{N}$ with $\sum_i S(i) = k$, and the map is $x_S \mapsto \prod_i z_i^{S(i)}$. Again, there are some obvious quadratic relations $x_S x_{S'} - x_T x_{T'} \mapsto 0$ when $S + S' = T + T'$, and the nontrivial theorem is that they generate the kernel, i.e. cut out the image.

Ring-theoretically, the image of ϕ is exactly the graded subring of $R = \mathbb{F}[z_1, \ldots, z_n]$ using the degree $0, k, 2k, 3k, \ldots$ pieces.

Exercise 4.2. *Let* $I \leq R$ *be a homogeneous ideal, and let* $R_{k\mathbb{N}}$ *be the image of the* k*th Veronese map. Show that for some* k, $I \cap R_{k\mathbb{N}}$ *is generated in degrees* k *and* $2k$.

4.3. **The Plücker embedding.** Fix k, n, and let $M_{k \times n}$ be the space of $k \times n$ matrices. Let $\pi : M_{k \times n} \to \mathbb{F}^{\binom{n}{k}}$ take a matrix to the list of the determinants of all its $k \times k$ submatrices.

If we projectivize the target, ripping out its $\vec{0}$, we have to rip out from $M_{k \times n}$ all the matrices with rank $< k$. (So k had better be $\leq n$.) Call the remaining (Zariski open) set $M_{k \times n}^{\text{rank } k}$. Then the injective map is

$$GL(k) \backslash M_{k \times n}^{\text{rank } k} \to \mathbb{F}\mathbb{P}^{\binom{n}{k} - 1}.$$

What is this space on the left? When we do row operations on a full rank matrix, the only invariant is its row span, a k-plane. So as a set this is the **Grassmannian** of all k-planes in n-space. But now we can regard it as a projective algebraic subset.

Note that the $k = 1$ case just recovers projective space.

Once again, there are quadratic relations that define the image, but this time they are tricky to state, and trickier yet to prove that they give the prime ideal.

Exercise 4.3. *Find these relations in the* 2×5 *case (but don't prove that they generate the prime ideal).*

5. Localization

If $A \subseteq S$ is closed under multiplication, we can form the **ring of fractions** $A^{-1}S = \{\text{“}s/a\text{”}\}$ where $s/a \sim s'/a'$ if $sa' = s'a$. There is a natural map $S \to A^{-1}S$, which need not be injective; it is only if A contains no zero divisors. (The worst case is $A \ni 0$, so A an ideal is not a good choice.) Therefore there's a map backwards on $\mathbb{F}$-points that we should consider.

Example:

(1) Let $S = \mathbb{F}[x, y]$, and $A = \langle x \rangle$. Then $A^{-1}S \cong \mathbb{F}[x, x^{-1}, y]$.
(2) Let $S = \mathbb{F}[x, y]/\langle xy \rangle$, and $A = \langle x \rangle$. Then $A^{-1}S \cong \mathbb{F}[x, x^{-1}]$.
(3) Let P be an ideal. Then $A = S \setminus P$ is closed under multiplication iff P is prime.

In the first two examples, the image of $(S \to A^{-1}S)^*$ is an open set. The third is trickier, and shows the inadequacy of our current approach to the geometry.

Proposition 5.1. (1) *The map $(S \to A^{-1}S)^*$ is injective. Perhaps because of this, $A^{-1}S$ is called a* **localization** *of S.*
(2) *If $A = S \setminus P$, its image is empty unless $S/P \cong \mathbb{F}$, in which case its image is the point given by the maximal ideal P.*

Proof. (1) An $\mathbb{F}$-point of $A^{-1}S$ is determined by its values on S.
(2) An $\mathbb{F}$-point $\phi : S \to \mathbb{F}$ factors through $A^{-1}S$ iff $a \in A \implies \phi(a) \neq 0$, iff (contrapositively) $\phi(a) = 0 \implies a \in P$, iff $\ker \phi \subseteq P$. Then since P contains the maximal ideal $\ker \phi$ and $P \neq S$, $P = \ker \phi$.

□

So the ring $A^{-1}S$ has at most one $\mathbb{F}$-point.

Proposition 5.2. *If $A = S \setminus P$ for P a prime ideal, then $A^{-1}S$ has a unique maximal ideal $A^{-1}P$.*

Proof. Let I be an ideal; we want to show $I \leq A^{-1}P$ or $I = A^{-1}S$. Let $r/a \in I \setminus A^{-1}P$, with $r \in S$, $a \in S \setminus P$. Then $r \in I \setminus P$. So $r \in A$, i.e. it's invertible, hence $I = S$. □

A **local ring** S is one with a unique maximal ideal $\mathfrak{m}$. (Note that a localization of a ring need not be a local ring!) Under the right definition of the geometry associated to a ring, that should define a "point" even if it's not an $\mathbb{F}$-point.

Example. Let $S = \mathbb{F}[x, y]$, $P = \langle x \rangle$, so $A^{-1}S$ is rational functions whose denominator is not a multiple of x. Then $A^{-1}S/A^{-1}P \cong \mathbb{F}(y)$, rational functions in one variable.

Local rings have an extra structure that helps one prove theorems about them, which is the valuation $q : S \to \mathbb{N} \cup \infty$, $q(s) = \max\{n : s \in \mathfrak{m}^n\}$ (here $\mathfrak{m}$ is the maximal ideal), and s is invertible $\iff q(s) = 0$.

The **residue field** of a local ring $(S, \mathfrak{m})$ is $S/\mathfrak{m}$. It's the first term in $\operatorname{gr} S := \oplus_i \mathfrak{m}^i/\mathfrak{m}^{i+1}$, which is the sort of ring we can define the Hilbert dimension of! Define the **height of a prime** P as *one plus* the Hilbert dimension of $\operatorname{gr} A^{-1}S$, where $A = S \setminus P$.

Exercise 5.3. *Let $P = \langle x_k, \ldots, x_n \rangle \leq S = \mathbb{F}[x_1, \ldots, x_n]$. Show P is prime. What is its height?*

A local ring has a sequence

$$\ldots \twoheadrightarrow S/\mathfrak{m}^3 \twoheadrightarrow S/\mathfrak{m}^2 \twoheadrightarrow S/\mathfrak{m}$$

and S maps into the inverse limit of this sequence, the **completion** of S. Call S a **complete local ring** if that map is an isomorphism.

Example. Let $S = \mathbb{F}[x]$, $P = \langle x \rangle$, so $S/\mathfrak{m}^k \cong \mathbb{F}[x]/\langle x^k \rangle$. Then this completion is $\mathbb{F}[[x]]$.

Example. Let $S = \mathbb{F}[x, y]/\langle y^2 - x^2(x+1) \rangle$, and $P = \langle x, y \rangle$. Then both S and $A^{-1}S$ are domains, but the completion is $\cong \mathbb{F}[x, y']/\langle y'^2 - x^2 \rangle$, with $y' = y\sqrt{1+x}$. (That nonalgebraic function is a well-defined power series living in the completion.)

Obviously these rings are not usually finitely generated over a field. So we instead consider the more robust class of **Noetherian** rings, in which any increasing chain of ideals must terminate. Here is a reason to study complete Noetherian local rings:

Theorem 5.4 (Cohen structure theorem). *Let S be a complete local Noetherian commutative ring with maximal ideal $\mathfrak{m}$ and* **residue field** *$S/\mathfrak{m} = K$. If S contains a field, then $S \cong K[[x_1, x_2, \ldots, x_n]]/I$ for some n and some ideal I.*

6. Stunt lecture by Matthias

7. Nilpotents

First we argue that in passing from I to $V(I)$ we lose important information in $I \neq \sqrt{I}$, then describe some extra geometry on $V(I)$ to at least partially remember I.

Whenever we have a map $\operatorname{Spec} S \to \operatorname{Spec} R$, we can consider $\operatorname{Spec} R$ as parametrizing a family of schemes. Especially if we have $\operatorname{Spec} S \hookrightarrow \mathbb{A}^n_R := \operatorname{Spec} R[x_1, \ldots, x_n]$, in which case we're parametrizing a family of subschemes of affine space.

For example, let $R = \mathbb{F}[t]$, and $S = R[x, y]/\langle y, y - x^2 + t\rangle$. Then $\operatorname{Spec}(S/\langle t - f\rangle)$ is two points $(\pm\sqrt{t}, 0)$ for $t \neq 0$, but only one for $t = 0$. So for $t = 0$ we would like to call it a "double point".

The utility lies in being able to say "a line and parabola always intersect in two points" rather than "usually two, and occasionally one", exactly analogous to the way that two distinct lines in the projective plane always intersect in one point, not "usually one, and occasionally zero". Really, we want this to be true:

Theorem 7.1 (Bezout). *Let p, q be homogeneous polynomials in x, y, z of degrees a, b. Assume that $\gcd(p, q) = 1$. Then counted correctly, there are ab points in $\mathbb{P}^2$ satisfying $p = q = 0$.*

On the algebra level, this will be easy. Recall that a graded quotient of a polynomial ring has a Hilbert polynomial, whose leading term's exponent gives the projective "Hilbert" dimension d. Now we look at the leading term's coefficient, multiply it by $d!$ to get an integer (as it turns out), and call that the **degree** of R/I.

Exercise 7.2.
1. *Show that the degree of the nth Veronese of $\mathbb{P}^1$ is n.*
2. *Figure out the degree of the nth Veronese of $\mathbb{P}^k$.*
3. *Figure out the degree of the Segre variety $\mathbb{P}^1 \times \mathbb{P}^2$.*
4. *Figure out the degree of the multi-Segre variety $(\mathbb{P}^1)^d$.*

Lemma 7.3. *Let $S = R/I$ be a graded quotient of a polynomial ring. If $p \in S_k$ is not a zero divisor, then $h\dim S/\langle p\rangle = h\dim S - 1$, and $\deg S/\langle p\rangle = k \deg S$.*

Proof. The graded short exact sequence

$$0 \to S[k] \xrightarrow{\cdot p} S \twoheadrightarrow S/\langle p\rangle \to 0$$

gives a formula on Hilbert functions,

$$h_{S/\langle p\rangle}(m) = h_S(m) - h_S(m - k)$$

which on leading terms looks like

$$\begin{aligned}
\deg(S/\langle p\rangle)\frac{m^d}{d!}+\ldots &= \deg S\frac{m^d}{d!}+\ldots-\deg S\frac{(m-k)^d}{d!}-\ldots \\
&= \frac{\deg S}{d!}(m^d-(m-k)^d)-\ldots \\
&= \frac{\deg S}{d!}(m^d-(m^d-kdm^{d-1}+\ldots))-\ldots \\
&= \frac{\deg S}{d!}kdm^{d-1}-\ldots \\
&= k\deg S\frac{m^{d-1}}{(d-1)!}-\ldots
\end{aligned}$$

□

(This proof also suggests why the degree is an integer, and what it counts. If we can show that there always exists a linear function p such that $p \not| 0$, a "general hyperplane", we can slice with it and not change the degree. Repeat d times. In general, this requires an infinite field so as to have enough hyperplanes, and is the first of a family of "Bertini theorems".)

Now we can define the "number of points" in a 0-dimensional projective scheme: take the degree. It's even easier in an affine scheme: take the vector space dimension of the underlying ring. Of course we'd like these to agree:

Exercise 7.4. *Let* $R = \mathbb{F}[z_0,\ldots,z_n]$. *Let* $S = R/I$ *be graded, of Hilbert dimension zero, and degree* m. *If* $f \in R$ *is a linear polynomial that is not a zero divisor on* S, *show that* $\dim_{\mathbb{F}} S[f^{-1}] = \deg S$.

Proof of Bezout's theorem. Start with $R = \mathbb{F}[z_0, z_1, z_2]$, and check that $\deg R = 1$, $\dim R = 2$. Then let $S = R/\langle p\rangle$, so $\deg S = a$, and $\dim S = 1$.

Is q a zero divisor in S? Say $qr = 0$, so $qr \in \langle p\rangle$, hence $qr = ps$. By unique factorization and $\gcd(q,p) = 1$, we have $p|r$, so $r \equiv 0$ in S.

Hence q is not a zero divisor, so $\deg S/\langle q\rangle = ab$, $\dim S/\langle q\rangle = 0$, and we have ab points in the plane. □

7.1. **A bestiary.** In the above, when two points were atop one another all we cared about was "that's 2 not 1". But $\langle x^2, y\rangle \neq \langle x, y^2\rangle$, even though they both describe two points at the origin.

Exercise 7.5.
(1) *Find all ideals* $I \subset \mathbb{C}[x,y]$ *of* $\mathbb{C}$*-codimension* 2 *such that* $\sqrt{I} = \langle x,y\rangle$.
(2) *Show the* monomial *ideals* $I \subseteq \mathbb{C}[x,y]$ *of codimension* n, *and such that* $\sqrt{I} = \langle x,y\rangle$, *correspond to partitions of* n.

We can have nilpotents with higher-dimensional objects, too; $I = \langle x^2y\rangle$ describes a double vertical axis union the reduced horizontal axis.

Exercise 7.6. *(Hard) If* $I \leq R$ *is a homogeneous ideal, show* $h\dim R/I = h\dim R/\sqrt{I}$.

(I'm not sure we quite have the tools for this yet.)

If $\deg R/I = \deg R/\sqrt{I}$, we can say I is **reduced in top dimension**. But that doesn't imply it's reduced; consider $\langle xy, y^2\rangle$.

Exercise 7.7. (1) *Let* $S = R/I$ *be graded, and* p *be a nonzero divisor. If* $S/\langle p\rangle$ *is reduced in top dimension, show* S *was too.*

(2) *Let* $R = \mathbb{F}[(x_{ij}, y_{ij})]$ *be the polynomial ring in the matrix entries of* X, Y*. Let* I *be generated by the matrix entries of* $XY - YX$*. Assume that the open set in which* X, Y *have no repeated eigenvalues is dense (for more general Lie algebras this is Richardson's theorem). Show that* R/I *is reduced in top dimension.*

(3) *Show that* R/I *is reduced. Then collect your PhD, and at the very least, multiple tenure-track offers.*

Consider two lines approaching one another in space; the $z = x = 0$ line fixed, and the $z = t, y = 0$ line falling onto it. With Macaulay 2 we can confirm $\langle z, x\rangle \cap \langle z - t, y\rangle \cap = \langle z(z - t), x(z - t), zy, xy\rangle$. Now set $t = 0$ to get $\langle z^2, xz, zy, xy\rangle$, which contains the ideal $\langle z, xy\rangle$ in codimension 1. That is, there is one embedded point.

7.2. **One kind of Gröbner degeneration.** Let $R = \mathbb{F}[z_0, \ldots, z_n]$, and R_m be spanned by monomials with z_n-degree at most m. Then $R_m R_k \subseteq R_{m+k}$, so we can take the **associated graded ring** $\mathrm{gr}\, R := \oplus_m R_m/R_{m-1}$. Of course this is stupid as $\mathrm{gr}\, R \cong R$.

However, it's not stupid if we consider an ideal $I \leq R$, and its associated graded $\mathrm{gr}\, I := \oplus_m (I \cap R_m)/(I \cap R_{m-1})$, an ideal in $\mathrm{gr}\, R$. Traditionally this is pulled back to R along the isomorphism above, even though it's a rather confusing point of view; when it is it's called $\mathrm{init}\, I$, the "initial ideal".

Exercise 7.8. (1) *Let* $g_1', \ldots, g_k'$ *be homogeneous generators of* $\mathrm{init}\, I$*. Show that they can be lifted to generators of* I*.*

(2) *If* I *is homogeneous in the usual sense, show* $h_I = h_{\mathrm{init}\, I}$*, so they have the same Hilbert dimension and degree.*

Repeating this $I \mapsto \mathrm{init}\, I$ with other variables, we can produce a monomial ideal. A combinatorial argument shows that monomial ideals are finitely generated, and then by the exercise, all ideals are finitely generated. This is a case where we prove something about radical ideals by passing to very very nonradical ideals.

8. More nilpotents, bases, and base change

I have to integrate in the notes I made for Wednesday Sep 14.

9. Sheaves

Friday Sep 16: [Vakil, 3.1, 3.2]

References

[Vakil] R. Vakil, Foundations of Algebraic Geometry, downloadable at `http://math.stanford.edu/~vakil/216blog/`