MATH 3360 PRELIM #1, SPRING 2018

Name, **written slowly and legibly:** ________________________________

In each answer, write as much (on front and back) as it takes to convey your thought process; full English sentences are much easier to give credit to than bare, unmotivated scribbled formulæ. (They won't do any good if they can't be read, so *do* put effort into making them legible.)

If two questions have the same number, they concern the same object. For example, the relation in 4b is the one defined in 4. [5π] indicates that a problem is worth 5π points.

Feel free to ask me questions during the test, especially if you need a little reminder about a definition. Worst case is I don't answer. (It's very sad to afterward hear "I didn't realize I could ask you that" — find out!)

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1. Let A, B be two finite sets of sizes k, n respectively.

1a [5]. How many relations Q are there from A to B?

Answer. These are just subsets of $A \times B$, so there are 2^{kn}.

1b [5]. Of those, how many are functions?

Answer. For each $a \in A$, we need to choose $f(a) \in B$, so n^k.

1c [10]. Of those, how many are one-to-one and onto?

Answer. If $k = n$ there are $k!$, and otherwise 0.

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2 [15]. Let $V = \mathbb{R}^3$. Define a relation R from V to V as follows: $\vec{v}\, R\, \vec{w}$ if $\exists r \in \mathbb{R}$ such that $\vec{v} = r\vec{w}$.

Is R an equivalence relation? Prove it is or prove it isn't.

Answer. It's not, since it's not symmetric:

Let $\vec{v} \neq \vec{0}$. Then $\vec{v}\ R\ \vec{0}$ but not $\vec{0}\ R\ \vec{v}$.

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3 [15]. Find two natural numbers x, y such that

(1) the Euclidean algorithm requires three "reductions", i.e. replacement of some $a = mb + r$ with the remainder r
(2) in each of those reductions, the multiple m is 2
(3) the resulting GCD is 5.

Answer. Working backwards from $(0, 5)$ three times, we get $(10, 5)$, $(10, 25)$, $(60, 25)$.

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4. Let $X = \mathbb{N}_+$ be the positive integers.
Define a relation

$$x \sim y \qquad \text{if} \qquad \exists n > 0 \text{ such that } x | y^n \text{ and } y | x^n.$$

4a [15]. Prove this is an equivalence relation.

Answer. Reflexive: $x \sim x$ since $n = 1$ does the job.

Symmetric: tautological from the definition.

Transitive: if $x \sim y$ using n and $y \sim z$ using n', then $x | y^n | z^{nn'}$ and $z | y^{n'} | x^{nn'}$, so we can check that $x \sim z$ using nn'.

4b [15]. Give a concrete rule for recognizing the minimum element in an equivalence class. (For example, is 5500 the minimum element in its equivalence class? If not, then what is?)

Answer. Two numbers are $\sim$ if they're divisible by the same set of primes. To be smallest, we shouldn't repeat any primes, i.e. not be divisible by any p^2. (For $5500 = 2^2 5^3 11$, the smallest thing it's $\sim$ to is $2 * 5 * 11 = 110$.)

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5. Define the *Chifobani numbers* by $C_0 = 3$, $C_1 = 5$, $C_{n+2} = 2C_{n+1} - C_n$.

5a [5]. Guess a formula for them by experiment.

Answer. Computing the first few we get $3, 5, 7, 9, 11, 13, \ldots$ so $C_n = 2n + 3$ looks like a good guess.

5b [15]. Prove your formula.

Answer. Proof by (strong) induction on n. For $n = 0, 1$ we're good. Then for $n \geq 2$,

$$\begin{aligned} C_n &= 2C_{n-1} - C_{n-2} \\ &= 2(2(n-1)+3) - (2(n-2) - 3) \qquad \text{by induction} \\ &= 2n + 3 \qquad \checkmark \end{aligned}$$