SCHUBERT CALCULUS AND QUIVER VARIETIES

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Introduction

Notational conventions

Part 1. Equivariant Schubert calculus

1. Flag and Schubert varieties

Let V denote an n-dimensional vector space over $\mathbb{C}$, and $0 \le n_1 \le n_2 \le \ldots \le n_d \le n$ a sequence of d integral "steps". Our principal object of study in this book is the **d-step flag variety**

$$Fl(n_1, n_2, \ldots, n_d;\ V) := \left\{ (V_1, V_2, \ldots, V_d) \ :\ V_i \le V_{i+1} \le V,\ \dim V_i = n_i,\ \forall i = 1, \ldots, d \right\}$$

So far this is a set, of "d-step flags of subspaces in V"; our first endeavor is to give it the structure of complex manifold (or more specifically, smooth complex variety).

When $V = \mathbb{C}^n$, there is a natural map

$$\begin{array}{rcl} rowspan: \ GL_n(\mathbb{C}) & \to & Fl(n_1, n_2, \ldots, n_d;\ \mathbb{C}^n) \\ M & \mapsto & (\ldots,\ \text{span of top } n_i \text{ rows},\ \ldots)_{i=1\ldots d} \end{array}$$

which is easily seen to be onto (pick a basis of V_1, extend to a basis of $V_2, \ldots$, extend to a basis of $\mathbb{C}^n$, take that basis as rows of M). This map $rowspan$ is equivariant with respect to the *right*[1] action of $GL_n(\mathbb{C})$, transitive on both spaces.

The basic case $d = 1$ gets a special name and notation; it is the **Grassmannian** $Gr(n_1; V) := Fl(n_1; V)$ of n_1-planes in V (and we will usually write k instead of n_1). We consider this case first, to later build on it through the obvious inclusion

$$\begin{array}{rcl} Fl(n_1, n_2, \ldots, n_d; V) & \hookrightarrow & \prod_{i=1}^{d} Gr(n_i; V) \\ (V_1, V_2, \ldots, V_d) & \mapsto & (V_1, V_2, \ldots, V_d) \end{array}$$

whose image is the *nested* tuples, satisfying $V_1 \leq V_2 \leq \ldots \leq V_d$.

The **full flag variety** $Fl(1, 2, \ldots, n; \mathbb{C}^n)$ also gets special notation, $Fl(\mathbb{C}^n)$.

1.1. **Atlases on flag manifolds.** To make $Gr(k; V)$ into a complex manifold, not just a set, we give it an atlas of charts whose transition maps are holomorphic (in fact, algebraic). Begin by defining the **Stiefel manifold**

$$Stiefel(k; \mathbb{C}^n) := \{k \times n \text{ complex matrices of rank } k\}$$

and consider the surjective map

$$rowspan : Stiefel(k; \mathbb{C}^n) \twoheadrightarrow Gr(k; \mathbb{C}^n)$$

The Stiefel manifold is a Zariski open set inside $M_{k\times n}$, the union of $\binom{n}{k}$ principal open sets

$$U_\lambda := \{M \in M_{k\times n} \;:\; \det(\text{columns } \lambda \text{ in } M) \neq 0\} \qquad \lambda \in \binom{[n]}{k}$$

The fibers of the map $rowspan$ are exactly the left $GL_k(\mathbb{C})$-orbits on $Stiefel(k; \mathbb{C}^n)$, and this action preserves each of the open sets U_λ. Conveniently, this action on U_λ admits a continuous system of orbit representatives:

$$U'_\lambda := \{M \in M_{k\times n} \;:\; \text{columns } \lambda \text{ in } M \text{ form an identity matrix}\}$$

and consequently

- the restriction $rowspan|_{U'_\lambda} : U'_\lambda \to Gr(k; \mathbb{C}^n)$ is injective, and
- the images $rowspan(U'_\lambda)$, $\lambda \in \binom{[n]}{k}$ cover $Gr(k; \mathbb{C}^n)$
- if we give $Gr(k; \mathbb{C}^n)$ the quotient topology, i.e. the coarsest topology such that $rowspan$ is continuous (w.r.t. the Zariski topology on $Stiefel(k; \mathbb{C}^n)$), then the images $rowspan(U'_\lambda)$, $\lambda \in \binom{[n]}{k}$ are open.

Theorem 1. *The charts* $rowspan|_{U'_\lambda} : U'_\lambda \to Gr(k; \mathbb{C}^n)$ *form an algebraic atlas, making* $Gr(k; \mathbb{C}^n)$ *a smooth scheme over* $\mathbb{C}$ *of dimension* $k(n-k)$.

Proof. Let $G_{\lambda\mu} = rowspan(U'_\lambda) \cap rowspan(U'_\mu)$, so $G_{\lambda\mu} = rowspan(U_\lambda) \cap rowspan(U_\mu) = rowspan(U_\lambda \cap U_\mu)$. Its preimages in U'_λ and U'_μ are respectively

$$U'_\lambda \cap U_\mu = \{M \in M_{k\times n} \;:\; \text{columns } \lambda \text{ in } M \text{ form an identity matrix}, \det(\text{columns } \mu) \neq 0\}$$

and the very similar $U'_\mu \cap U_\lambda$. The overlap map $U'_\lambda \cap U_\mu \xrightarrow{\sim} G_{\lambda\mu} \xrightarrow{\sim} U'_\mu \cap U_\lambda$ is

$$M \mapsto (\text{columns } \mu \text{ in } M)^{-1} M$$

which is algebraic.

[1]For a number of reasons we choose $\mathbb{C}^n$ to, as a rule, denote *row* vectors. One important one is that wide, shallow matrices fit better on the page.

As to the dimension, the conditions on each U'_λ specify k^2 of the kn matrix entries, leaving $k(n-k)$ free. $\square$

For the most part it is easier to work with the kn Stiefel coördinates on the Grassmannian, rather than the $\binom{n}{k}$ Plücker coördinates that we won't[2] bother introducing.

As we explain below, much the same construction gives an atlas on $Fl(n_1, n_2, \ldots, n_d; \mathbb{C}^n)$, restricting the surjective $rowspan$ map to slices U'_λ that together cover the target. The difference is that the group we're dividing by, instead of $GL_k(\mathbb{C})$, is the group P of block lower triangular matrices in $GL_{n_d}(\mathbb{C})$ with diagonal blocks of size $n_1, n_2 - n_1, \ldots, n_d - n_{d-1}$.

We will need a finer analogue of the covering $Stiefel(k; \mathbb{C}^n) = \bigcup_{\lambda \in \binom{[n]}{k}} U_\lambda$. Let $M \in Stiefel(n_d; \mathbb{C}^n)$ be a matrix we're making sure not to exclude. Start by picking $\lambda_d \in \binom{[n]}{n_d}$ such that M's submatrix with columns λ_d (and all n_d rows) is invertible, exactly as we did before. But now we also pick a subset $\lambda_{d-1} \in \binom{\lambda_d}{n_{d-1}}$ such that M's submatrix with columns λ_{d-1} (and first n_{d-1} rows, a **top submatrix**) is invertible, then similarly $\lambda_{d-2} \in \binom{\lambda_{d-1}}{n_{d-2}}$, etc. Encode this sequence $(\lambda_1 \subseteq \lambda_2 \subseteq \ldots \subseteq \lambda_d \subseteq [n])$ of subsets as a **string** λ, whose ith letter is $\min\{j : j \notin \lambda_i\}$ (considering $\lambda_{d+1} = [n]$). To such a string we associate the manifolds

$$U_\lambda := \{N \in Stiefel(n_d; \mathbb{C}^n) : \det(N\text{'s top submatrix with columns } \lambda_i) \neq 0, \forall i = 1 \ldots d\}$$

$$U'_\lambda := \left\{ N \in Stiefel(n_d; \mathbb{C}^n) : \begin{array}{c} N\text{'s top submatrix with columns } \lambda_i \\ \text{is a permutation matrix,} \\ \text{whose 1s run NW/SE in rows } [1+n_{i-1}, n_i] \end{array} \quad \forall i = 1 \ldots d \right\}$$

As in the Grassmannian case,

- these (U_λ) cover $Stiefel(n_d; \mathbb{C}^n)$ (the construction of λ above made sure to hit M),
- they're invariant under the left action of P (the block lower triangular group defined above),
- the submanifold U'_λ serves as a system of P-orbit representatives in U_λ, so
- the maps $rowspan|_{U'_\lambda} : U'_\lambda \to Fl(n_1, n_2, \ldots, n_d; \mathbb{C}^n)$ form an atlas, again easily checked to be algebraic.

1.2. **The Bruhat decomposition of** $Gr(k; \mathbb{C}^n)$**.** The left $GL_k(\mathbb{C})$ action on $M_{k\times n}$ is a familiar topic in first linear algebra classes, under the name "row reduction" or "Gaussian elimination". The principal result there concerning it is the construction of a *discontinuous* system of orbit representatives, the **reduced row-echelon forms** (or RREF) such as

$$\begin{bmatrix} 0&0&1&*&*&0&0&*&0&0&* \\ 0&0&0&0&0&1&0&*&0&0&* \\ 0&0&0&0&0&0&1&*&0&0&* \\ 0&0&0&0&0&0&0&0&1&0&* \\ 0&0&0&0&0&0&0&0&0&1&* \\ 0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}$$

characterized by the properties

- The zero rows are at the bottom
- The nonzero rows start with a 1, called a **pivot**
- The pivotal 1s run NW/SE

[2]They are coördinates on the smallest $GL(V)$-equivariant embedding of $Gr(k; V)$ into projective space, via the map $V_1 \mapsto Alt^k V_1 \in \mathbb{P}(Alt^k V)$. In particular, they let one study the Grassmannian as a *sub*variety of a simpler space, rather than as a quotient.

- Above each pivotal 1 (and below, automatically) is all 0s.

If we work not with $M_{k\times n}$ but $\mathtt{Stiefel}(k;\mathbb{C}^n)$, then there can be no zero rows. *We assume this hereafter.* Let C_λ denote the space of RREFs with pivots in columns $\lambda \in \binom{[n]}{k}$.

Exercise 1. Consider the $0, 1, *$ matrix, as above, describing the elements of a fixed C_λ. If we remove the pivotal columns, show that the 0s in the resulting diagram form a "French partition" in the SW corner of the resulting $k \times (n-k)$ matrix. Find a recipe that, starting from such a partition, says where to reinsert the k pivot columns.

Hereafter we will make reference to the corresponding "English partition", obtained by flipping the French partition upside down, and by abuse of notation speak of "the partition λ" to refer to this English partition.

Exercise 2. Show that the matrix cell $(a, b) \in [k] \times [n-k]$ lies in the partition λ iff for each $M \in C_\lambda$, the left $(k-a)+b$ columns of M have rank $k-a$.

In this notation, the principal result about RREFs is that $\mathtt{rowspan} : \bigcup_{\lambda\in\binom{[n]}{k}} C_\lambda \to \mathtt{Gr}(k;\mathbb{C}^n)$ is bijective. This map is continuous, but with discontinuous inverse. Define the **Bruhat cell** $X^\circ_\lambda \subseteq \mathtt{Gr}(k;\mathbb{C}^n)$ as the image of C_λ, and its closure X_λ as the **Schubert variety** associated to λ.

Theorem 2. *The homology classes associated to the cycles X_λ give a $\mathbb{Z}$-basis for the homology of $\mathtt{Gr}(k;\mathbb{C}^n)$. Let $[X_\lambda]$ denote the Poincaré dual basis element, in* co*homology; hence $\{[X_\lambda]\}$ form a $\mathbb{Z}$-basis for $H^*(\mathtt{Gr}(k;\mathbb{C}^n))$.*

Proof. Each C_λ is a complex vector space, so even-real-dimensional; hence in the cellular homology complex (freely spanned by the cycles $\{X_\lambda\}$) the differentials are zero. □

Now that we have a ring-with-basis, the natural question is of how to expand a product of two basis elements in the basis. Studying such questions is the principal aim of the book. Right now we don't have enough handle on that basis to be able to do much in the way of such computation. The following gives a start.

Theorem 3. (1) *The Bruhat cells $\{X^\circ_\lambda\}$ are exactly the orbits of the right action of $B \le GL_n(\mathbb{C})$, the upper triangular matrices; this is called the* **Bruhat decomposition** *of $\mathtt{Gr}(k;\mathbb{C}^n)$.*

(2) *The subset $GL_k(\mathbb{C})C_\lambda \subseteq M_{k\times n}$ is defined as a set by the rank* equalities $\mathrm{rank}(\textit{left } i \textit{ columns}) = \#(\lambda \cap [i]),\ \forall i = 1\ldots n$.

(3) *Given $V \in \mathtt{Gr}(k;\mathbb{C}^n)$, we can determine the unique cell $X^\circ_\lambda \ni V$ as follows:*
$$\lambda = \{i \in [n] \ :\ (V \cap \mathbb{C}^{i,\ldots,n}) > (V \cap \mathbb{C}^{i+1,\ldots,n})\},$$
where $\mathbb{C}^{j,\ldots,n} \le \mathbb{C}^n$ is the coördinate subspace using coördinates $j, \ldots, n$.

(4) $\overline{GL_k(\mathbb{C})C_\lambda} \subseteq M_{k\times n} = \{N \in M_{k\times n}\ :\ \mathrm{rank}(\textit{left } i \textit{ columns of } N) \le \#(\lambda \cap [i]),\ \forall i = 1\ldots n\}$. *Call this subset $Y_\lambda \subseteq M_{k\times n}$.*

(5) $Y_\lambda \supset C_\mu$ *iff μ's partition defined in exercise 1 contains λ's, iff there is a chain $Y_\lambda \subset Y_{\lambda_1} \subset Y_{\lambda_2} \subset \cdots \subset Y_{\lambda_k} = Y_\mu$ where the associated partitions differ in size by one square. This containment on partitions is called the* **Bruhat order** *on the Bruhat cells in the Grassmannian. ... this probably isn't quite what I want yet...*

(6) $X_\lambda = \{V \in \mathtt{Gr}(k;\mathbb{C}^n)\ :\ \dim(V \cap \mathbb{C}^{i,\ldots,n}) \ge \#(\lambda \cap \{i, \ldots, n\})\}$

Proof. First, we need argue that each X°_λ is B-invariant, or equivalently upstairs in the Stiefel manifold, that $C_\lambda B \subseteq GL_k(\mathbb{C})C_\lambda$. The right action of B consists of scaling the columns and adding columns to further-rightward columns. If we scale a pivot column (spoiling the 1), we can use $GL_k(\mathbb{C})$ to unscale the corresponding row (restoring the pivot entry to 1). If we add a multiple of a column to a later non-pivotal column, the result is still in RREF. If we add a multiple of a column to a later pivotal column, spoiling some 0s above the pivotal 1, we can use $GL_k(\mathbb{C})$ to add multiples of the pivotal row to cancel those entries back to 0. This settles the B-invariance.

It's more obvious that B acts transitively on X_λ°: already a subgroup of it acts transitively on C_λ, in which we use rightward column operations to cancel all the $*$ entries (but don't add pivot columns to other pivot columns) thereby reducing any element of C_λ to the partial permutation matrix of the pivots. This proves (1).

The action of $GL_k(\mathbb{C}^n) \times B$ (on left and right) leaves the rank measurements of (2) invariant. On the partial permutation RREF with pivot columns λ, the rank equalities obviously hold; hence they hold on the rest of the orbit, $GL_k(\mathbb{C})C_\lambda$. This proves (2).

That rank is the rank of the composite $V \hookrightarrow \mathbb{C}^n \twoheadrightarrow \mathbb{C}^n/\mathbb{C}^{i+1,\ldots,n}$, whose corresponding nullity is used to compute λ in (3).

Since Y_λ is $GL(k) \times B$-invariant, it contains C_μ iff it contains the partial permutation matrix with pivots in columns μ. That connects straightforwardly to the first partition statement in (5), which in turn proves the second.

The closed subset Y_λ defined in (4) obviously contains the closure $\overline{GL_k(\mathbb{C})C_\lambda} \subseteq M_{k\times n}$, and both sets are $GL_k(\mathbb{C}) \times B$-invariant. To check that $\overline{GL_k(\mathbb{C})C_\lambda} \subseteq M_{k\times n}$ contains Y_λ, we need .... □

Exercise 3. Let $X_\lambda \subseteq Gr(k;\mathbb{C}^n)$ be a Bruhat cell. Show there is a subgroup of B that acts *simply* transitively on X_λ.

Exercise 4. Show that the matrix cell $(a,b) \in [k] \times [n-k]$ lies in the partition λ iff for each $M \in rowspan^{-1}(X_\lambda)$, the left $(k-a)+b$ columns of M have rank at most $k-a$.

The T-fixed points in $Gr(k;\mathbb{C}^n)$ are exactly the coördinate k-planes and in natural correspondence with the set $\binom{[n]}{k}$ of strings with content $0^k 1^{n-k}$; as such we will denote those k-planes directly as λ, abusing notation.

1.3. **First examples of Schubert calculus.** Let $k=1$, so $Gr(k;\mathbb{C}^n)$ is projective space, and the Bruhat decomposition is the familiar one

$$\mathbb{CP}^{n-1} = \mathbb{C}^{n-1} \coprod \mathbb{C}^{n-2} \coprod \cdots \coprod \mathbb{C}^0$$

We'll index classes by their strings, $[X_{01\ldots1}], [X_{101\ldots1}], \ldots, [X_{1\ldots10}]$. Then $X_{1^k01\ldots1}$ is a codimension k subspace, and our first product calculation is

$$[X_{1^k01\ldots1}][X_{1^j01\ldots1}] = \begin{cases} [X_{1^{j+k}01\ldots1}] & \text{if } j+k<n \\ 0 & \text{if } j+k \geq n \end{cases}$$

Proof: $X_{1^k01\ldots1}$ is the set of (projectivized) vectors with first k coördinates vanishing, and if we move $X_{1^j01\ldots1}$ to have the *next* j coördinates vanishing, then the two intersect transversely in $X_{1^{j+k}01\ldots1}$. Meanwhile, for $j+k \geq n$ we know $H^{2(j+k)}(\mathbb{CP}^{n-1}) = 0$.

The product rule is equally simple for $k = n-1$ (by Grassmannian duality), so the next case to consider is $k=2, n=4$, in which

$$X_{0101} = \{V \in Gr(2;\mathbb{C}^4) \ : \ \dim(V \cap \mathbb{C}^{3,4}) \geq 1\}$$

which considered projectively is the set of lines in $\mathbb{CP}^3$ meeting the line $\mathbb{P}(\mathbb{C}^{3,4})$.

When two cycles $A, B \subseteq X$ are *transverse*, we can compute $[A][B]$ as $[A \cap B]$, but this isn't useful for computing $[X_{0101}]^2$. Consider the divisor

$$Y = \{V \in Gr(2; \mathbb{C}^4) \ :\ \dim(V \cap \mathbb{C}^{2,4}) \geq 1\}.$$

that's $GL_4(\mathbb{C})$-equivalent to X_{0101}, and defines the same cohomology class. Then

$$[X_{0101}]^2 = [X_{0101}][Y] = [X_{0101} \cap Y]$$

This intersection $X_{0101} \cap Y$ is reducible

$$X_{0101} \cap Y \quad = \quad X_{0110} \quad \cup \quad \{V \in Gr(2; \mathbb{C}^4) \ :\ V \leq \mathbb{C}^{1,2,4}\}$$

and its second component if $GL_4(\mathbb{C})$-equivalent to X_{1001}, resulting in the computation

$$[X_{0101}]^2 = [X_{0110}] + [X_{1001}]$$

It is not an accident that the coefficients we've run into so far have all been positive, though they can be > 1, the first example being

$$[X_{010101}]^2 = [X_{011100}] + [X_{110001}] + 2[X_{101010}].$$

In §2 we'll give a general rule for these coefficients, counting a set of "puzzles".

1.4. **The Bruhat decomposition of flag manifolds.** We already have the surjection of $\mathtt{Stiefel}(n_d; \mathbb{C}^n)$ onto $Fl(n_1, n_2, \ldots, n_d;\ \mathbb{C}^n)$, and need now to adapt the usual RREF theorem to reduce it to a bijection. For each string λ with content $0^{n_1} 1^{n_2 - n_1} \cdots d^{n-n_d}$, define $C_\lambda \subseteq \mathtt{Stiefel}(n_d; \mathbb{C}^n)$ as the space of matrices such that

- Each row starts with a "pivotal" 1 ($\nexists$ zero rows since we're in $\mathtt{Stiefel}(n_d; \mathbb{C}^n)$).
- The pivotal 1s in rows $[n_{i-1} + 1, n_i]$ run NW/SE, for each $i = 1, \ldots, d$, and are in the columns j such that λ's jth letter is i.
- Above each pivotal 1 (and below, automatically) is all 0s.

We leave the interested reader to adapt the usual proofs of RREF uniqueness to this more general context. As in the Grassmannian case, the images of the C_λ give an even-real-dimensional cell decomposition $\coprod_\lambda X^\circ_\lambda$ again called the **Bruhat decomposition**, hence $\mathbb{Z}$-bases of homology and cohomology.

> **WARNING. Our indexation of our basis using strings, forced on us by the product rule computations to come, is *inverse* to the standard indexation in the literature. Outside of the full flag manifold case, this should cause no confusion; for example, on Grassmannians we index with strings like 001010110 (or, permutations with only two unambiguous values) whereas the standard indexation is by permutations with a single descent (or, permutations with only two unambiguous positions).**

Exercise 5. Check that

$$[X_{132}][X_{213}] = [X_{231}] + [X_{312}]$$

using the fact that X_{132}, X_{213} already intersect transversely.

Exercise 6. Check that

$$[X_{213}]^2 = [X_{231}]$$

but as in the example in §1.3, you'll need to move one of the X_{213}s to make the intersection transverse.

Given a string λ, define an **inversion** of λ as a pair of positions $i < j$ such that the λ-values are inverted: λ at position $i > \lambda$ at position j. Then define

$$\ell(\lambda) := \#\{\text{inversions in } \lambda\}$$

Proposition 1. $\operatorname{codim} X_\lambda = \ell(\lambda)$, *so,* $[X_\lambda] \in H^{2\ell(\lambda)}(Fl(n_1, n_2, \ldots, n_d;\ \mathbb{C}^n))$.

Proof. □

Exercise 7. Recall that we associated a partition to a string with two values $0, 1$. How can one compute the number of inversions of the string, directly from the partition?

Exercise 8. Extend exercise 3 to the $Fl(n_1, n_2, \ldots, n_d;\ \mathbb{C}^n)$ situation.

1.5. **Poincaré polynomials of flag manifolds.** Recall that the **Poincaré polynomial** $p_M(t) := \sum_i t^i \dim H^i(M)$ of a topological space M is the generating function of its Betti numbers.

Theorem 4. *Define the polynomial in* q

$$n^{q}_{\bullet} := \prod_{i=1}^{n} \frac{1 - q^i}{1 - q},$$

the q**-factorial of** n *or "*n q**-torial**". *Then* $p_{Fl(1,\ldots,n;\mathbb{C}^n)}(t) = n^{q}_{\bullet}$ *when* $q = t^2$.

Proof. By proposition 1 we need to compute

$$\sum_{\pi \in S_n} q^{\ell(\pi)} = \sum_{i \in [n]} \sum_{\pi \in S_n,\ \pi(1)=i} q^{\ell(\pi)} = \sum_{i \in [n]} q^{i-1} \sum_{\rho \in S_{n-1}} q^{\ell(\rho)}$$

where the second follows from the bijection $S_n \to [n] \times S_{n-1}$, where $\rho \in S_{n-1}$ is determined by the order of $\pi(2), \ldots, \pi(n)$.

Then use $\sum_{i \in [n]} q^{i-1} = \frac{q^n - 1}{q - 1}$ and induction. □

Proposition 2. *Let* $\pi :\ Fl(\mathbb{C}^n) \twoheadrightarrow Fl(n_1, \ldots, n_d;\ \mathbb{C}^n)$ *be the projection forgetting all subspaces except those of dimensions* $(n_i)_{i=1\ldots d}$. *Let* w_0^P *denote the longest element in* $S_{n_1} \times S_{n_2 - n_1} \times \cdots \times S_{n - n_d}$, *reversing the individual blocks. Then* π *is a fiber bundle, with fiber* $X^{w_0^P} \cong \prod_{i=1}^{d+1} Fl(\mathbb{C}^{n_i - n_{i-1}})$ *over the basepoint (taking* $n_0 = 0$ *and* $n_{d+1} = n$*).*

Proof. The source and target of π are compact (in the analytic topology). The set of critical points of π is closed, hence compact, hence its image (the critical values) is closed. By Sard's theorem, there exist regular values, hence they form an open set in the target, over which π is a bundle. Since π is $GL(\mathbb{C}^n)$-equivariant, and the target is homogeneous, this open set must be everything (there are no critical points).

The base fiber consists of flags whose n_i-plane is the standard one, which turns directly into the conditions defining $X^{w_0^P}$. Such a flag induces flags on $\mathbb{C}^{n_1}, \mathbb{C}^{n_2}/\mathbb{C}^{n_1}, \ldots$ which gives the isomorphism. □

Corollary 1.

$$p_{Fl(n_1,\ldots,n_d;\mathbb{C}^n)}(t) = \frac{n^{q}_{\bullet}}{\prod_{i=0}^{d} (n_{i+1} - n_i)^{q}_{\bullet}} = \frac{\prod_{i=1}^{n} (1 - q^i)}{\prod_{i=0}^{d} \prod_{j=1}^{n_{i+1} - n_i} (1 - q^j)}$$

where $q = t^2$ *and by convention* $n_0 := 0, n_{d+1} := n$.

Proof. If $F \to E \to B$ is a fiber bundle, and $H^*(E) \to H^*(F)$ is onto, then it is a consequence of the Leray-Hirsch theorem that $p_E = p_F p_B$.

Since $X^w \hookrightarrow Fl(\mathbb{C}^n)$ is a closed union of cells in the Bruhat decomposition, it induces a map on cellular cohomology, and since the differential is trivial (as in theorem 2), this map is onto.

Hence we may apply the Leray-Hirsch theorem to the fiber bundle in proposition 2, and apply theorem 4 to compute p_E and p_F.

The second equality follows from multiplying top and bottom by $(1-q)^n$. □

Exercise 9. Give a direct combinatorial proof of corollary 1.

1.6. **Self-duality of the Schubert basis.** The cohomology ring $H^*(Fl(n_1, n_2, \dots, n_d; \mathbb{C}^n)$ comes with a perfect pairing $\langle \alpha, \beta \rangle \mapsto \int \alpha\beta$, and now that we have a basis $\{[X_\lambda]\}$ for the ring, it is natural to seek (and will be handy to have) the dual basis. By convention, the integral over a compact oriented manifold M of a cohomology class is zero for classes of degree $< \dim M$, so this is really a series of perfect pairings, between $H^i(M)$ and $H^{\dim M - i}(M)$.

Define the **opposite Schubert variety** X^λ as $X_{\lambda\,\mathrm{reversed}} w_0$, where $w_0 \in GL_n(\mathbb{C})$ is an antidiagonal matrix (of all 1s, say, but since X_μ is T-invariant this doesn't matter). Since each X_λ is B-invariant, these X^λ are invariant under $w_0^{-1} B w_0 = B_-$.

...make sure to mention finite-dim vs. finite-codim of strata...

Proposition 3. *...lemma to give the equations determining X^λ...*

Lemma 1. *Let X_λ, X^μ be a Schubert and an opposite Schubert variety, respectively, in the Grassmannian $Gr(k; \mathbb{C}^n)$. Then there are several cases:*

(1) *If $\ell(\lambda) > \ell(\mu)$ then $X_\lambda \cap X^\mu = \emptyset$, so $\int [X_\lambda][X^\mu] = 0$.*
(2) *If $\ell(\lambda) < \ell(\mu)$ then $\dim(X_\lambda \cap X^\mu) > 0$, so $\int [X_\lambda][X^\mu] = 0$.*
(3) *If $\ell(\lambda) = \ell(\mu)$ but $\lambda \neq \mu$, then $X_\lambda \cap X^\mu = \emptyset$, so $\int [X_\lambda][X^\mu] = 0$.*
(4) *$X_\lambda \cap X^\lambda = \{\mathtt{rowspan}($the matrix in C_λ with all non-pivots $= 0)\}$, so $\int [X_\lambda][X^\lambda] = 1$.*

Proof. (1) Equivalently, the areas of the partitions λ, μ^c add up to more than $k(n-k)$. Consequently there must be some overlap in the partitions μ-rotated-180° and λ. Let (a, b) be a square of the overlap, so (a, b) is a square in the partition λ and $(k+1-a, n-k+1-b)$ a square in the partition μ. By exercise 4, each $M \in X_\lambda$ has its left $(k-a)+b$ columns of rank $\leq k-a$, whereas each $M \in X^\mu$ has its right $(k-(k+1-a)) + (n-k+1-b) = a+n-k-b$ columns of rank $\leq a-1$. Together, these left $(k-a)+b$ columns and right $n-k+a-b$ columns cover the entire M, but only have rank $\leq (k-a)+(a-1) < k$ together, so don't define any elements of $\mathtt{Stiefel}(k; \mathbb{C}^n)$.

(2) For any $A, B \subseteq C$ for C smooth, we have $\operatorname{codim}(A \cap B) \leq \operatorname{codim} A + \operatorname{codim} B$, which in this case gives us

$$\begin{aligned} \dim(X_\lambda \cap X^\mu) &= \dim Gr(k; \mathbb{C}^n) - \operatorname{codim}(X_\lambda \cap X^\mu) \\ &\geq \dim Gr(k; \mathbb{C}^n) - \operatorname{codim} X^\mu - \operatorname{codim} X_\lambda \\ &= \dim X^\mu - \operatorname{codim} X_\lambda = \ell(\mu) - \ell(\lambda) > 0 \end{aligned}$$

(3) Once one observes that $\lambda \cap \mu^\circ$ contains some square (a, b), then the argument from (1) applies.

(4) Consider $M \in C_\lambda$ and apply the rank conditions from X^λ, which say that the rank of the right i columns of M is bounded above by the number of λ's pivots in those columns. Consequently, each non-pivot column i of M must be in the span of the columns to the right, which (by downward induction on i) is only the span of those

pivot columns, hence (by the RREF shape of M) that column i must be 0. So M is unique.

□

Proposition 4. *Let X_λ, X^μ be a Schubert and an opposite Schubert variety, respectively, in the same flag manifold $Fl(n_1, n_2, \ldots, n_d; \mathbb{C}^n)$. Then there are several cases:*

1. *If $\ell(\lambda) > \ell(\mu)$ then $X_\lambda \cap X^\mu = \varnothing$, so $\int [X_\lambda][X^\mu] = 0$.*
2. *If $\ell(\lambda) < \ell(\mu)$ then $\dim(X_\lambda \cap X^\mu) > 0$, so $\int [X_\lambda][X^\mu] = 0$.*
3. *If $\ell(\lambda) = \ell(\mu)$ but $\lambda \neq \mu$, then $X_\lambda \cap X^\mu = \varnothing$, so $\int [X_\lambda][X^\mu] = 0$.*
4. *$X_\lambda \cap X^\lambda = \{\mathrm{rowspan}($the matrix in C_λ with all non-pivots $= 0)\}$, so $\int [X_\lambda][X^\lambda] = 1$.*

Proof. (1)

The dimension

To compute these intersections in $Fl(n_1, n_2, \ldots, n_d; \mathbb{C}^n)$, we can instead compute the intersection $\mathrm{rowspan}^{-1}(X_\lambda)$...

□

1.7. **Push and pull operations.** For each d-step flag manifold and $i \in [d]$, we get a projection map

$$\pi\colon\ Fl(n_1, \ldots, n_d; \mathbb{C}^n) \twoheadrightarrow Fl(n_1, \ldots, \widehat{n_i}, \ldots, n_d; \mathbb{C}^n)$$

to a $(d-1)$-step, forgetting the n_i-plane in the flag.

Theorem 5. *Let $\pi\colon\ Fl(n_1, \ldots, n_d; \mathbb{C}^n) \twoheadrightarrow Fl(n_1, \ldots, \widehat{n_i}, \ldots, n_d; \mathbb{C}^n)$ forget the n_i-plane.*

1. *$\pi(X^\lambda) = X^{\lambda'}$, where the string λ' is constructed from λ by turning all i into $i-1$, and more generally, subtracting 1 from every value $\geq i$.*
2. *The induced map on homology, in the Schubert basis, is*
$$\pi_*([X^\lambda]) = \begin{cases} [X^{\lambda'}] & \text{if every } i-1 \text{ in } \lambda \text{ is left of every } i \\ 0 & \text{otherwise.} \end{cases}$$
Alternately, one can describe this condition as $\ell(\lambda') = \ell(\lambda)$, the only other possibility being $\ell(\lambda') < \ell(\lambda)$.
3. *$\pi^*([X_\mu]) = [X_{\mu'}]$ in cohomology, where μ' is constructed from μ by adding 1 to the last $n_{i+1} - n_i$ many is (i.e. breaking ties), and to all values $> i$.*
4. *In particular, π_* is onto, and π^* is $1:1$.*

...examples...

Proof. (1) Since π is $GL(\mathbb{C}^n)$-equivariant and X^λ is B_--invariant, its image is B_--invariant. Since X^λ is closed in the analytic topology on $Fl(n_1, \ldots, n_d; \mathbb{C}^n)$, which is compact, X^λ is also compact, so its image is closed. Together, these and the Bruhat decomposition show that $\pi(X^\lambda)$ is a union of various $X^{\lambda'} \subseteq Fl(n_1, \ldots, \widehat{n_i}, \ldots, n_d; \mathbb{C}^n)$.

(Finally, since X^λ is irreducible, its image $\pi(X^\lambda)$ is also irreducible, hence it is a single $X^{\lambda'}$. We only include that argument due to its interest; in fact we'll show both containments.)

If we determine X^λ from proposition 3, but leave out the conditions constraining the n_i-plane, we get exactly the conditions defining $X^{\lambda'}$; hence $\pi(X^\lambda) \subseteq X^{\lambda'}$. Meanwhile, $\pi(\lambda) = \lambda'$, hence $\lambda' \in \pi(X^\lambda)$, so by B_--invariance $\lambda' B_- \subseteq \pi(X^\lambda)$, then since the latter is closed $X^{\lambda'} \subseteq \pi(X^\lambda)$.

(2) The composite $X^\lambda \twoheadrightarrow X^{\lambda'} \hookrightarrow Fl(n_1, \ldots, \widehat{n_i}, \ldots, n_d; \mathbb{C}^n)$ induces a composite map π_* on homology

$$[X^\lambda] \in H_{2\ell(\lambda)}(X^\lambda) \to\to H_{2\ell(\lambda)}(X^{\lambda'}) \to\to H_{2\ell(\lambda)}(Fl(n_1, \ldots, \widehat{n_i}, \ldots, n_d; \mathbb{C}^n)).$$

If $\ell(\lambda') < \ell(\lambda)$, then the middle group is 0, so $\pi_*([X^\lambda]) = 0$.

If $\ell(\lambda') = \ell(\lambda)$, then $\pi_*([X^\lambda]) = [X^{\lambda'}] \deg(\pi)$. To show that this degree is 1, we use exercise 8 to construct a group H such that in the maps $H \to X^\lambda_\circ \to X^{\lambda'}_\circ$, both the first and composite maps are isomorphisms, hence the second map is also an isomorphism.

(3) Effectively, in (2) we computed a $0,1$ matrix with rows and columns indexed by the homology Schubert bases $\{[X^\lambda]\}, \{[X^{\lambda'}]\}$ of the two flag manifolds. By proposition 4, the cohomology Schubert bases $\{[X_\lambda]\}, \{[X_{\lambda'}]\}$ are the dual bases under the natural pairings, so we can compute π^* as the transpose of this $0,1$ matrix. The claim follows.

(4) That matrix (and its transpose) visibly contains a full-rank identity matrix.

□

Hereafter let $Fl(\hat{i}; \mathbb{C}^n)$ denote $Fl(1, 2, \ldots, \hat{i}, \ldots, n; \mathbb{C}^n)$, and $\pi_i : Fl(\mathbb{C}^n) \twoheadrightarrow Fl(\hat{i}; \mathbb{C}^n)$ denote the natural projection forgetting the i-plane. Note that the fibers are 1-dimensional; knowing V_{i-1}, V_{i+1} the choice of V_i is equivalently that of a line in the plane V_{i+1}/V_{i-1}. In particular, if $A \subseteq Fl(\mathbb{C}^n)$ is an irreducible cycle, then $\dim \pi_i(A)$ is either $\dim A$ or $\dim A - 1$, the latter being true exactly if A consists of fibers of π_i.

Let ∂_i denote the degree -2 endomorphism of $H^*(Fl(\mathbb{C}^n))$ given by the composite

$$H^k(Fl(\mathbb{C}^n)) \cong H_{n(n-1)-k}(Fl(\mathbb{C}^n)) \xrightarrow{\pi_{i*}} H_{n(n-1)-k}(Fl(\hat{i}; \mathbb{C}^n)) \cong H^{k-2}(Fl(\hat{i}; \mathbb{C}^n)) \xrightarrow{\pi_i^*} H^{k-2}(Fl(\mathbb{C}^n))$$

Geometrically, if $A \subseteq Fl(\mathbb{C}^n)$ is an irreducible cycle, and d is the degree of $\pi_i|_A : A \to \pi_i(A)$ (or 0 if the map drops dimension as discussed above), then $\partial_i([A]) = d\,[\pi_i^{-1}(\pi_i(A))]$. In particular $\partial_i^2 = 0$, because π_i applied to $\pi_i^{-1}(\pi_i(A))$ definitely drops the dimension.

This map isn't a ring homomorphism (obviously – it takes $1 \mapsto 0$), but is a module homomorphism over the subring $H^*(Fl(\hat{i}; \mathbb{C}^n))$. We will compute this subring later in lemma 3.

Theorem 6. *Let $X_w \subseteq Fl(\mathbb{C}^n)$ be a Schubert variety. (Since this full-flag case is the situation where confusion is most likely to arise, we remind the reader here that our indexing $[X_w]$ of Schubert classes is inverse to the one standard in the literature.) Then*

(1)
$$\partial_i[X_w] = \begin{cases} [X_{(i\leftrightarrow i+1)\circ w}] & \text{if } i+1 \text{ left of } i \text{ in } w \\ 0 & \text{if } i \text{ left of } i+1 \text{ in } w \end{cases}$$

(2) *If $|i - j| > 1$, then*
$$\partial_i\partial_j[X_w] = \partial_j\partial_i[X_w] = \begin{cases} [X_{(i\leftrightarrow i+1)\circ(i\leftrightarrow i+1)\circ w}] & \text{if } i+1 \text{ left of } i \text{ and } j+1 \text{ left of } j \text{ in } w \\ 0 & \text{otherwise} \end{cases}$$

(3) *For $i < n$,*
$$\partial_i\partial_{i+1}\partial_i[X_w] = \partial_{i+1}\partial_i\partial_{i+1}[X_w] = \begin{cases} [X_{(i\leftrightarrow i+2)\circ w}] & \text{if } i+2 \text{ left of } i+1 \text{ left of } i \text{ in } w \\ 0 & \text{otherwise} \end{cases}$$

(4) *If $\ell(wv) = \ell(w) + \ell(v)$, then $[X_v] = \partial_{w^{-1}}[X_{wv}]$.*

In particular, $\partial_i\partial_j = \partial_j\partial_i$ *for* $|i-j|>1$, *and* $\partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1}$ *for* $i<n$, *since the Schubert classes* $[X_w]$ *span* $H^*(Fl(\mathbb{C}^n))$.

Proof. ...quote theorem 5 and proposition 4 appropriately...

Claims (2) and (3) follow directly from (1). Claim (4) follows from (1) and induction on $\ell(w)$. $\square$

Corollary 2. *Since any two minimal-length "***reduced***" words writing* w *as a product of simple reflections are related by commuting and braid moves (Tits' theorem), if* $w = \prod Q$ *is a reduced word then* $\partial_w := \prod_Q \partial_q$ *is well-defined (in either the polynomial or* $H^*(Fl(\mathbb{C}^n))$ *context).*

Exercise 10. Show

$$\partial_w\partial_v = \begin{cases} \partial_{wv} & \text{if } \ell(wv) = \ell(w)+\ell(v) \\ 0 & \text{otherwise, i.e. if } \ell(wv) < \ell(w)+\ell(v) \end{cases}$$

and in particular, that these are the only two possibilities.

Exercise 11. Interpret $\partial_i\partial_j$ and $\partial_i\partial_{i+1}\partial_i$ in terms of projections similar to π_i. (You'll know them when you've found them.)

Once we understand these $\{\partial_i\}$ better, theorem 6 will supply a computation of Schubert classes, starting from the class $[X_{w_0}]$ of a point (where $w_0(i) = n+1-i$ for all i):

Corollary 3. *For* $w \in S_n$, *we have* $\ell(ww_0) = \ell(w_0) - \ell(w)$, *and* $[X_w] = \partial_{ww_0}[X_{w_0}]$.

Proof. For each $i<j$ in $[1,n]$, exactly one is true of $w(i) > w(j)$ vs. $(ww_0)(n+1-i) > (ww_0)(n+1-j)$, so that's $\binom{n}{2} = \ell(w_0^n)$ inversions to share between $\ell(w)$ and $\ell(ww_0)$. Then apply theorem 6(4). $\square$

1.8. **Borel's presentation of ordinary cohomology: stable case.** In this subsection we give our first general results about the cohomology *ring* of flag manifolds. We begin with the easiest case:

Proposition 5. (1) *Let* $x_1 = [X_{101\ldots 1}]$ *be the class of the Schubert hyperplane in projective space* $\mathbb{CP}^{n-1} := Gr(1;\mathbb{C}^n)$. *Then* $H^*(\mathbb{CP}^{n-1}) \cong \mathbb{Z}[x_1^{(2)}]/\langle x_1^n\rangle$ *where the superscript on* x_1 *indicates the degree.*

(2) *Things are simpler still in infinite dimensions:* $H^*(\mathbb{CP}^\infty) \cong \mathbb{Z}[x_1^{(2)}]$, *where* $\mathbb{CP}^\infty$ *denotes the projectivization of* $\mathbb{C}^\infty := \bigoplus_{\mathbb{N}} \mathbb{C}$.

Proof. We already did the calculation in (1) in §1.3. As for (2), the inclusions

$$\mathbb{CP}^0 \hookrightarrow \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2 \hookrightarrow \mathbb{CP}^3 \hookrightarrow \cdots \hookrightarrow \mathbb{CP}^\infty$$

induce a map from $H^*(\mathbb{CP}^\infty)$ to the inverse limit $\mathbb{Z}[[x_1]]$ of the rings $\mathbb{Z}[x_1]/\langle x_1^n\rangle$. Since $H^*(\mathbb{CP}^\infty)$ is a graded ring, it maps to the graded part $\bigoplus_n \mathbb{Z}[[x_1]]_n = \mathbb{Z}[x_1]$. To map onto $H^2(\mathbb{CP}^1)$ or even just its element x_1, the map $H^*(\mathbb{CP}^\infty) \to \mathbb{Z}[x_1]$ must be onto. Finally, the cell decomposition of $\mathbb{CP}^\infty$ tells us that $H^*(\mathbb{CP}^\infty)$ is no bigger than $\mathbb{Z}[x_1]$ in each degree, so the map is also $1:1$. $\square$

(It is really a matter of taste whether $H^*(X)$ should be the direct sum or direct product of the individual groups $H^i(X)$, though "direct sum" is the industry standard. For many purposes, e.g. defining the Chern character map $K(X) \to H^*(X)$, the direct product is the more natural object.)

The manifold $Fl(n_1, \ldots, n_d; \mathbb{C}^n)$ comes with d tautological vector bundles $\mathcal{V}_i$, where $\mathcal{V}_i|_{(V_1,\ldots,V_d)} = V_i$. To construct them, we begin with the Grassmannian case, and do the "associated-bundle construction":

$$\mathcal{V}_k := \mathbb{C}^k \times^{GL_k(\mathbb{C})} Stiefel(k; \mathbb{C}^n) := (\mathbb{C}^k \times Stiefel(k; \mathbb{C}^n)) / GL_k(\mathbb{C}^n)_\Delta$$

Here "$\times^G$" means one divides by[3] the diagonal G-action on the two factors (right on $\mathbb{C}^k$, left on $Stiefel(k; \mathbb{C}^n)$). This essentially replaces the $GL_k(\mathbb{C}^n)$-fibers of the map $rowspan$ with $\mathbb{C}^k$-fibers. Now we pull back to $Fl(n_1, \ldots, n_d; \mathbb{C}^n)$ these bundles $\mathcal{V}_k, Alt^k\mathcal{V}_k$.

Exercise 12. Extend the definitions in §1.1 to put an algebraic atlas on the total space $\mathcal{V}_k$.

Consider now the first Chern classes / Euler class

$$c_1(\mathcal{V}_k^*) = c_1(Alt^k\mathcal{V}_k^*) = e(Alt^k\mathcal{V}_k^*)$$

The Euler $e(\mathcal{L})$ class of a line bundle $\mathcal{L}$ is, by definition, Poincaré dual to the zero set of a section $\sigma \in \Gamma(\mathcal{L})$ transverse to the zero section. Sections of $\mathcal{L}$ correspond to functions *on* the total space[4] of the dual line bundle $\mathcal{L}^*$, linear on each fiber. In the case at hand, that dual line bundle is associated to the $\det^{-1}$ representation R of $GL_k(\mathbb{C})$, so, a function on it is a $GL_k(\mathbb{C})$-invariant function on $R \times Stiefel(k; \mathbb{C}^n)$. Consider the following such function:

$$\sigma : (r, M) \mapsto r \det(\text{left } k \text{ columns of } M)$$

Being linear in r, this function σ of course vanishes on the zero section, but also vanishes on the entire fibers over the unique codimension 1 Schubert variety $X_{0^{k-1}10 1^{n-k-1}}$. Dually, as a section of $Alt^k\mathcal{V}_k^*$, σ vanishes exactly on $X_{0^{k-1}101^{n-k-1}}$, so gives us

$$c_1(\mathcal{V}_k^*) = [X_{0^{k-1}101^{n-k-1}}]$$

Such a class, a positive combination of classes of subvarieties, is called **effective**, and one could say that we work with the dual bundle $\mathcal{V}_k^*$ exactly to make this Chern class effective.

Since as we learned in proposition 5 that the cohomology rings are simpler (or more specifically, freer) in infinite dimensions, we next consider the case of n-step flags in $\mathbb{C}^\infty$.

Proposition 6. *The map*

$$\begin{array}{rcl} \beta : \mathbb{Z}[x_1, \ldots, x_n] & \to & H^*(Fl(1, 2, \ldots, n; \mathbb{C}^\infty)) \\ x_i & \mapsto & c_1(\mathcal{V}_{i-1}) - c_1(\mathcal{V}_i) \end{array}$$

is an isomorphism.

Proof. In the course of the proof, but not afterward, it will be mildly convenient to reverse the variables and instead take

$$x_{n+1-i} \mapsto c_1(\mathcal{V}_{i-1}) - c_1(\mathcal{V}_i).$$

The map $Fl(1, 2, \ldots, n; \mathbb{C}^\infty) \twoheadrightarrow Gr(1; \mathbb{C}^\infty)$ is a bundle with fiber $Fl(1, 2, \ldots, n-1; \mathbb{C}^\infty)$, inducing a restriction map to the fiber, which fits into a square

$$\begin{array}{ccc} H^*(Fl(1, 2, \ldots, n; \mathbb{C}^\infty)) & \to & H^*(Fl(1, 2, \ldots, n-1; \mathbb{C}^\infty)) \\ \uparrow & & \uparrow \\ \mathbb{Z}[x_1, \ldots, x_n] & \to & \mathbb{Z}[x_1, \ldots, x_{n-1}] \end{array}$$

[3]This is inspired by the $\times_S$ notation for fiber products, which results in a subset of the product rather than a quotient. One can unify the two concepts with enough talk about "stacks", in which case $\times^G = \times_{[pt/G]}$, but we will avoid such harsh language.

[4]Put another way, the dualization is here for the same reason one uses the *anti*tautological bundle $\mathcal{O}(1)$ on projective space.

where the bottom map is

$$x_i \mapsto \begin{cases} x_i & \text{if } i < n \\ 0 & \text{if } i = n. \end{cases}$$

We reversed the variables in order to make this square commute.

By induction this restriction map is onto, hence the Leray-Hirsch theorem applies, which says that

$$H^*(Fl(1,2,\ldots,n;\mathbb{C}^\infty) \cong H^*(\mathbb{CP}^\infty) \otimes_{\mathbb{Z}} H^*(Fl(1,2,\ldots,n-1;\mathbb{C}^\infty) = \mathbb{Z}[x_n] \otimes_{\mathbb{Z}} \mathbb{Z}[x_1,\ldots,x_{n-1}]$$

as graded $H^*(\mathbb{CP}^\infty)$-modules. In particular, $H^*(Fl(1,2,\ldots,n;\mathbb{C}^\infty)$ is linearly spanned by monomials, so our map $\mathbb{Z}[x_1,\ldots,x_n] \to H^*(Fl(1,2,\ldots,n;\mathbb{C}^\infty))$ is onto, but (as before) since the graded dimensions match the map must also be $1:1$. ☐

From here one can build the foundation of higher Chern classes:

Lemma 2. *Let* $\pi : Fl(1,\ldots,k;\mathbb{C}^\infty) \twoheadrightarrow Gr(k;\mathbb{C}^\infty)$ *be the projection forgetting all subspaces except the* k*-plane. Then* $\pi^* : H^*(Gr(k;\mathbb{C}^\infty)) \to H^*(Fl(1,\ldots,k;\mathbb{C}^\infty)) \cong \mathbb{Z}[x_1,\ldots,x_k]$ *is injective, and its image (at least* $\otimes_{\mathbb{Z}}\mathbb{Q}$*) is the subring of* S_k*-invariant polynomials.*

Later in theorem 10 we will show that the $\otimes_{\mathbb{Z}}\mathbb{Q}$ is unnecessary. (Note though, that our principal interest in this book is in the structure constants for multiplication of the Schubert basis, and none of that information is lost when extending scalars.) Part of the reason that the integrality result is so subtle is that the corresponding results for orthogonal and symplectic flag manifolds do *not* hold over $\mathbb{Z}$.

Proof. By theorem 5(2) (extended without incident to $n = \infty$), the map on homology is surjective, so the map on cohomology is injective. Let M denote the "frame bundle", the principal $GL(\mathbb{C}^k)$-bundle over $Gr(k;\mathbb{C}^\infty)$ whose fiber over V consists of the bases of V. Then M has three quotients:

$$\begin{array}{ccccc} Fl(1,\ldots,k;\mathbb{C}^\infty) & & & & Gr(k;\mathbb{C}^\infty) \\ \uparrow & & & & \uparrow \\ T\backslash M & \to & N(T)\backslash M & \to & GL(\mathbb{C}^k)\backslash M \end{array}$$

Here $N(T)$ is the normalizer of T, permutation matrices times diagonal matrices. The left vertical map is a homotopy equivalence, the right one an isomorphism. The map $\rho : T\backslash M \to N(T)\backslash M$ is a $k!$-sheeted covering space, dividing by the action of $S_k \cong N(T)\backslash T$, so the image of $H^*(N(T)\backslash M) \xrightarrow{\rho^*} H^*(T\backslash M)$ lies inside the invariant subring $H^*(T\backslash M)^{S_k} = \mathbb{Z}[e_1(x_1,\ldots,x_k),\ldots,e_k(x_1,\ldots,x_k)]$. A priori, the image of $GL(\mathbb{C}^k)\backslash M$ might be even smaller.

To show the image is indeed the entire invariant subring, we compute the Poincaré series (the q-adic[5] limit of corollary 1, as $n \to \infty$) of $H^*(Gr(k;\mathbb{C}^\infty))$.

$$\lim_{n\to\infty} p_{Gr(k;\mathbb{C}^n)}(t) = \lim_{n\to\infty} \frac{\prod_{i=1}^n (1-q^i)}{\prod_{i=1}^d (1-q^i)\prod_{i=1}^{n-d}(1-q^i)} = \lim_{n\to\infty} \frac{\prod_{i=n-d+1}^n (1-q^i)}{\prod_{i=1}^d (1-q^i)} = \frac{1}{\prod_{i=1}^d (1-q^i)}$$

which matches that of the polynomial ring $\mathbb{Q}[e_1(x_1,\ldots,x_k),\ldots,e_k(x_1,\ldots,x_k)]$ in generators of degrees $2,4,\ldots,2k$. Since we are now comparing two vector spaces of the same finite dimension (degree by degree), one containing the other, they must be equal. ☐

[5]This only means, coefficientwise as power series.

This $Gr(k;\mathbb{C}^\infty)$ is the "classifying space" for k-dimensional vector bundles over nice topological spaces M (e.g. finite CW-complexes): specifically, the map

$$\begin{array}{rcl} Map(M, Gr(k;\mathbb{C}^\infty)) & \to & \{\text{isomorphism classes of } k\text{-plane bundles over } M\} \\ \phi & \mapsto & \phi^*(\mathcal{V}^k) \end{array}$$

taking a map ϕ (considered up to homotopy) to the pullback along ϕ of the tautological k-plane bundle $\mathcal{V}^k$, is bijective. So from a k-plane bundle J on M, we infer a map ϕ to $Gr(k;\mathbb{C}^\infty)$, hence a pullback on cohomology, and from there construct $\phi^*(e_i) =: c_i(J) \in H^{2i}(M)$ and name these the Chern classes of J. We won't make much explicit use of them hereafter.[6]

Lemma 3. *Let* $\pi : Fl(1,\ldots,n;\mathbb{C}^\infty) \twoheadrightarrow Fl(n_1,\ldots,n_d,n;\mathbb{C}^\infty)$ *be the projection forgetting all subspaces except those of dimensions* $(n_i)_{i=1\ldots d}$. *Then the pullback* $\pi^* : H^*(Fl(n_1,\ldots,n_d,n;\mathbb{C}^\infty)) \to H^*(Fl(1,\ldots,n;\mathbb{C}^\infty))$ *is injective. Its image* $\otimes_{\mathbb{Z}}\mathbb{Q}$ *is equal* $\mathbb{Q}[x_1,\ldots,x_n]^{S_{n_1}\times S_{n_2-n_1}\times\cdots\times S_{n-n_d}}$ *i.e. polynomials symmetric in contiguous blocks of variables.*

As above, we'll be able to eliminate the "upon $\otimes_{\mathbb{Z}}\mathbb{Q}$" proviso later.

Proof. ...should be inductive on d using the previous... □

1.9. Borel's presentation of ordinary cohomology: unstable case.

Corollary 4. *Since the restriction map* $H^*(Fl(1,\ldots,n;\mathbb{C}^\infty)) \to H^*(Fl(1,\ldots,n;\mathbb{C}^n))$ *is onto, proposition 6 lets us write* $H^*(Fl(n))$ *as a quotient of* $\mathbb{Z}[x_1,\ldots,x_n]$.

To work out the kernel we need two properties of total Chern classes $c(\mathcal{V}) = \sum_i c_i(\mathcal{V})$:

- if $\mathcal{A}$ is a trivial vector bundle, then $c(\mathcal{A}) = 1$, and
- if $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{B}/\mathcal{A} \to 0$ is a short exact sequence of vector bundles, then $c(\mathcal{B}) = c(\mathcal{A})c(\mathcal{B}/\mathcal{A})$.

From these and the nested vector bundles $\mathcal{V}_1 < \mathcal{V}_2 < \ldots < \mathcal{V}_n$ on $Fl(\mathbb{C}^n)$, we learn

$$1 = c(\mathcal{V}_n) = \prod_{i=1}^n c(\mathcal{V}_i/\mathcal{V}_{i-1}) = \prod_{i=1}^n (1 + c_1(\mathcal{V}_i/\mathcal{V}_{i-1}))$$

In terms of our generators x_i, this says

$$1 = \prod_{i=1}^n (1 - x_i) = \sum_{i=0}^n e_i(-x_1,\ldots,-x_n) = \sum_{i=0}^n (-1)^i e_i(x_1,\ldots,x_n)$$

We explain one mystery now: why do the x_i appear symmetrically in this presentation? This can be explained from a homotopy equivalence or even a diffeomorphism with an S_n-space:

$$T\backslash GL_n(\mathbb{C}) \quad \sim \quad B_-\backslash GL_n(\mathbb{C}) \quad \cong \quad T_{\mathbb{R}}\backslash U(n)$$

The first (complex algebraic) space is the set of decompositions of $\mathbb{C}^n$ as a direct sum of n lines, and the third (compact real) space is the set of decompositions of $\mathbb{C}^n$ as a direct sum of n Hermitian-orthogonal lines. On each of those two spaces there is an S_n-action permuting the lines, their associated line bundles, and their Chern classes.

[6]Especially in work of Fulton, one finds cohomology of classifying spaces avoided in favor of "universal degeneracy loci" constructions, perhaps in order to avoid infinite-dimensional spaces that are commonplace in topology but less standard in algebraic geometry. The equivalence of the approaches follows from results like this.

Lemma 4. *The* **coinvariant ring** $\mathbb{Z}[x_1, \ldots, x_n] / \langle e_i(x_1, \ldots, x_n), \forall i = 1, \ldots, n \rangle$ *is spanned by the images of the bounded monomials* $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ *where* $a_i \in [0, n - i]$*, hence, has rank at most* $n!$ *over* $\mathbb{Z}$.

Proof. Let $p \in \mathbb{Z}[x_1, \ldots, x_n]$. We want to rewrite p as a $\mathbb{Z}$-combination of these "bounded monomials", plus elements of the ideal. It suffices to write x_i^{n-i+1} as a polynomial in $x_{j \le i}$ of degree $\le n - i$ in x_i, plus an element of the ideal, and then use induction.

This monomial x_i^{n-i+1} is the last monomial in a complete homogeneous symmetric polynomial $h_{n-i+1}(x_1, \ldots, x_i)$, the sum of *all* terms of degree $n - i + 1$ in the first i variables.

Consider the equation $\prod_{j=1}^n (1 - x_j) = \sum_{k=0}^n (-1)^k e_k(x_1, \ldots, x_n) = 1$ in the coinvariant ring, rewrite as $\prod_{j=i+1}^n (1 - x_j) = \prod_{j=1}^i (1 - x_j)^{-1}$, and then expand the right-hand side:

$$\prod_{j=1}^{i} (1 - x_j)^{-1} = \prod_{j=1}^{i} (1 + x_j + x_j^2 + \cdots) = \sum_k h_k(x_1, \ldots, x_i)$$

Now consider degree by degree: the LHS $\prod_{j=i+1}^n (1 - x_j)$ has terms only of degree $\le n - i$, so the RHS $\sum_k h_k(x_1, \ldots, x_i)$ has vanishing degree $n - i + 1$ term, namely $h_{n-i+1}(x_1, \ldots, x_i)$.

Now we use $h_{n-i+1}(x_1, \ldots, x_i) = 0$ to rewrite x_i^{n-i+1} as a polynomial in $x_{j \le i}$ of degree $\le n - i$ in x_i. □

Theorem 7 (The Borel presentation)**.** *The map*

$$\begin{array}{rcl} \beta : \mathbb{Z}[x_1, \ldots, x_n] / \langle e_i(x_1, \ldots, x_n),\ i = 1, \ldots, n \rangle & \to & H^*(Fl(\mathbb{C}^n)) \\ x_i & \mapsto & -c_1(\mathcal{V}_i / \mathcal{V}_{i-1}) \end{array}$$

is well-defined and an isomorphism. Moreover, β *takes* $x_1 + \ldots + x_k \mapsto [X_{1\ 2 \ldots (k-1)\ (k+1)\ k\ \ldots n}]$*, and hence the Schubert divisor classes* $\{[X_{1\ 2 \ldots (k-1)\ (k+1)\ k\ \ldots n}]\}$ *generate* $H^*(Fl(\mathbb{C}^n))$.

Proof. We already had the surjective map from the polynomial ring in corollary 4. The well-definedness, i.e. the requirement that each $e_i(x_1, \ldots, x_n) \mapsto 0$, follows from the total Chern class calculation after corollary 4. By lemma 4 the source is a quotient of $\mathbb{Z}^{n!}$, and by §1.4's Bruhat decomposition the target is free of rank $n!$, so the surjectivity shows that the map is an isomorphism. □

Proposition 7. *Embed* $\iota : Fl(\mathbb{C}^n) \hookrightarrow Fl(\mathbb{C}^{n+1})$ *by taking* $(V_1 < V_2 < \ldots < V_{n-1} < \mathbb{C}^n)$ *to* $(V_1 \oplus 0 < V_2 \oplus 0 < \ldots < V_{n-1} \oplus 0 < \mathbb{C}^n \oplus 0 < \mathbb{C}^n \oplus \mathbb{C} = \mathbb{C}^{n+1})$*. Then for* $w \in S_{n+1}$,

$$\iota^*([X_w]) = \begin{cases} [X_{w(1),\ldots,w(n)}] & \text{if } w(n+1) = n+1 \\ 0 & \text{otherwise} \end{cases}$$

and ι^* *fits into a commutative square with Borel presentations:*

$$\begin{array}{ccccc} \mathbb{Z}[x_1, \ldots, x_{n+1}] & \xrightarrow{/\langle x_{n+1} \rangle} & \mathbb{Z}[x_1, \ldots, x_n] & & \textit{i.e. } x_{n+1} \mapsto 0 \\ \downarrow \beta & & \downarrow \beta & & \\ H^*(Fl(\mathbb{C}^{n+1})) & \xrightarrow{\iota^*} & H^*(Fl(\mathbb{C}^n)) & & \end{array}$$

Proof. The line bundle $\mathcal{V}_i / \mathcal{V}_{i-1})$ on $Fl(\mathbb{C}^n)$ is the pullback of the line bundle $(\mathcal{V}_i \oplus 0)/(\mathcal{V}_{i-1} \oplus 0)$ on $Fl(\mathbb{C}^{n+1})$. The final line bundle $(\mathbb{C}^n \oplus \mathbb{C})/(\mathbb{C}^n \oplus 0)$ is trivial, with $c_1 = 0$. With these facts one can check that each $x_i, i \in [n+1]$ maps to the same value East then South, or South then East. □

This "stability" property of Schubert classes *does not hold* for inclusions of orthogonal or symplectic flag manifolds. The first case ($[X_w] \mapsto [X_{w'}]$) works, in an appropriate sense, but the "extra" classes at size $n+1$ don't necessarily restrict to 0; they may instead give a Schubert class, twice a Schubert class, or a sum of two.

Our next goal is to locate the Schubert classes in this presentation, i.e., to lift them to "Schubert polynomials" living in $\mathbb{Z}[x_1, \ldots, x_n]$ directly.

1.10. **Divided difference operators and Schubert polynomials.** It is time to unify the push-pull construction studied in §1.7 and the Borel presentation:

Theorem 8. *Define the* **divided difference operator** *on polynomials in* $\mathbb{Z}[x_1, x_2, \ldots]$*, again denoted* ∂_i*, by*

$$\partial_i p(x_1, x_2, \ldots) := \frac{p(\ldots, x_i, x_{i+1}, \ldots) - p(\ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}}$$

(While seemingly producing rational functions, the results are actually polynomials, with integer coefficients.) Then the square

$$\begin{array}{ccc} \mathbb{Z}[x_1, \ldots, x_n] & \xrightarrow{\partial_i} & \mathbb{Z}[x_1, \ldots, x_n] \\ \downarrow \beta & & \downarrow \beta \\ H^*(Fl(\mathbb{C}^n)) & \xrightarrow{\partial_i} & H^*(Fl(\mathbb{C}^n)) \end{array}$$

commutes, whose vertical arrows both are the Borel presentation of theorem 7 and whose lower ∂_i *is the push-pull map from §1.7.*

Proof. As to the polynomiality: the numerator obviously vanishes under the specialization $x_i = x_{i+1}$, so the long division algorithm (which incurs no rational numbers because the leading coefficient is 1) produces no remainder.

The modules in the square have no $\mathbb{Z}$-torsion, so to check commutativity it is safe to first tensor with $\mathbb{Q}$.

The upper ∂_i is linear over $\mathbb{Q}[x_1, \ldots, x_{i-1},\ x_i + x_{i+1}, x_i x_{i+1},\ x_{i+1}, \ldots, x_n]$, the subring of $(x_i \leftrightarrow x_{i+1})$-invariants. By lemma 3, that subring's image under β is $H^*(Fl(\hat{i}; \mathbb{C}^n))$. We already knew the bottom ∂_i to be linear over that subring, by ∂_i's construction.

So it remains to check the commutativity on $\mathbb{Q}[x_1, \ldots, x_{i-1},\ x_i + x_{i+1}, x_i x_{i+1},\ x_{i+1}, \ldots, x_n]$-module generators. We take $\{1, x_1 + x_2 + \ldots + x_i\}$ as generators.

The upper ∂_i takes $1 \mapsto \frac{1-1}{x_i - x_{i+1}} = 0$. The lower ∂_i takes $[Fl(\mathbb{C}^n)] \to 0$ because, for example, $H^{-2}(Fl(\mathbb{C}^n)) = 0$.

The upper ∂_i takes $x_1 + x_2 + \ldots + x_i \mapsto 1$. By theorem 7, the left vertical map takes $x_1 + x_2 + \ldots + x_i \mapsto [X_{1\ 2 \ldots (i+1)\ i\ \ldots n}]$, and then we apply theorem 6 to compute the effect of the bottom map. ☐

Exercise 13.
(1) Directly show that the polynomial operators ∂_i defined in theorem 8 satisfy $\partial_i^2 = 0$ and the relations in theorem 6, together called the **nil Hecke relations**.
(2) Give a ridiculously indirect proof by combining theorem 6 and proposition 7, and taking the inverse limit to get the ring $\mathbb{Z}[[x_1, x_2, \ldots]]$.

Let $p \in \mathbb{Z}[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree $\binom{n}{2}$. Then under the Borel presentation β, p maps to an element of $H^{n(n-1)}(Fl(\mathbb{C}^n)) = \mathbb{Z} \cdot [X_{w_0}]$, i.e. to some multiple of the point class. So our next goal is to find a p that maps to 1 times the point class. Now that we have theorem 8 and corollary 2, we can check whether $\beta(p) = [X_{w_0}]$ by computing $\partial_{w_0} \beta(p) = \beta(\partial_{w_0} p)$. In particular if one picks p randomly enough that $\partial_{w_0} p \neq 0$, and doesn't mind using rational coefficients, then $p / \partial_{w_0} p$ will map to $1[X_{w_0}]$.

These polynomial operators ∂_w were introduced in [?] "Schubert cells, and the cohomology of the spaces G/P" where they suggest a (rational-coefficient) formula for such a p, defined in a uniform way across all connected complex groups G (our $Fl(\mathbb{C}^n)$ case corresponding to $G = GL_n(\mathbb{C})$). We will use a different p discovered a few years later by Lascoux and Schützenberger[7], with some excellent properties:

Lemma 5. *Let* $w_0^k \in S_n$ *denote the element* $k\ k-1 \ldots 2\ 1\ k+1 \ldots n$. *Define* $\mathcal{S}_{w_0^k} := \prod_{i=1}^k x_i^{k-i}$. *Then*

(1) $\partial_{w_0^{k-1} w_0^k} \mathcal{S}_{w_0^k} = \mathcal{S}_{w_0^{k-1}}$
(2) $\partial_{w_0^k} \mathcal{S}_{w_0^k} = 1$

In particular $\beta(\mathcal{S}_{w_0^k})$ *is the point class* $[X_{w_0}]$.

Proof. There is a unique reduced word for $w_0^{k-1} w_0^k$, giving the operator $\partial_1 \partial_2 \cdots \partial_{k-1}$. Then (1) is a trivial calculation, and (2) follows from (1) by induction. $\square$

Theorem 9. *Let* S_∞ *denote permutations of* $\{1, 2, 3, \ldots\}$ *such that* $w(i) = i$ *for all sufficiently large* i. *Then there is an obvious inclusion* $S_n \hookrightarrow S_\infty$, *and* $S_\infty = \bigcup_{n \geq 1} S_n$.

Let $w \in S_n \leq S_\infty$, *and define*

$$\mathcal{S}_w := \partial_{w w_0^n} \mathcal{S}_{w_0^n}$$

Then this **(single) Schubert polynomial** *is independent of the* n *chosen, and represents the Schubert class in that* $\beta(\mathcal{S}_w) = [X_w] \in H^*(Fl(\mathbb{C}^n))$.

The fact that there exists a representative independent of n is *almost* automatic from proposition 7; that stability result guarantees a unique representative in the inverse limit $\mathbb{Z}[[x_1, x_2, \ldots]]$ of the system of rings $\ldots \leftarrow H^*(Fl(\mathbb{C}^n)) \leftarrow H^*(Fl(\mathbb{C}^{n+1})) \leftarrow \ldots$ but doesn't guarantee that that representative is *polynomial*.

Proof. First observe

$$\ell(w w_0^k) = \ell(w_0^k) - \ell(w) \qquad \forall k \geq n$$

hence

$$\ell(w w_0^{n+1}) = \ell(w_0^{n+1}) - \ell(w) = \ell(w_0^{n+1}) - \ell(w_0^n) \ + \ \ell(w_0^n) - \ell(w) = \ell(w_0^n w_0^{n+1}) + \ell(w w_0^n)$$

To see the independence, briefly denote the polynomial with choice of n indicated, i.e. $\mathcal{S}_w^n$. Then finally

$$\mathcal{S}_w^{n+1} \ = \ \partial_{w w_0^{n+1}} \mathcal{S}_{w_0^{n+1}} \ = \ \partial_{w w_0^n} \partial_{w_0^n w_0^{n+1}} \mathcal{S}_{w_0^{n+1}} \ = \ \partial_{w w_0^n} \mathcal{S}_{w_0^n} \ = \ \mathcal{S}_w^n$$

using exercise 10. $\square$

We compute the S_3 Schubert polynomials[8], starting with $\mathcal{S}_{321} = x_1^2 x_2$. Southwestern arrows are ∂_1s, Southeastern are ∂_2s.

$$\begin{tikzcd}
 & & & \mathcal{S}_{321} = x_1^2 x_2 \arrow[dll] \arrow[drr] & & & \\
 & \mathcal{S}_{312} = x_1 x_2 \arrow[dl] \arrow[dr] & & & & \mathcal{S}_{231} = x_1^2 \arrow[dl] \arrow[dr] & \\
0 & & \mathcal{S}_{213} = x_1 \arrow[dl] \arrow[dr] & & \mathcal{S}_{132} = x_1 + x_2 \arrow[dl] \arrow[dr] & & 0 \\
 & \mathcal{S}_{123} = 1 & & 0 & & \mathcal{S}_{123} = 1 &
\end{tikzcd}$$

[7] Pronounced Frenchly, Marcel SHÜT-sen-bear-ZHAY.

[8] And remind the reader, again, that our indexing is inverse to the standard one

It is intriguing that despite their divided *difference* origins, the coefficients have all come out positive! A formula for these coefficients was conjectured by Stanley, and proven independently by Billey-Jockusch and Fomin, so Stanley's name is on two independent papers. We'll happen upon this formula in §4.4, which will let us prove

Lemma 6. *[to be proven in §4.5]*

$$\mathcal{S}_{1\ 2\ldots(k-1)\ n\ k\ (k+1)\ldots(n-1)} = e_{n-k}(x_1,\ldots,x_k)$$

and from there the Borel presentation over $\mathbb{Z}$:

Theorem 10. *The image of* $\pi^* : H^*(Fl(n_1,\ldots,n_d;\mathbb{C}^n)) \to H^*(Fl(\mathbb{C}^n))$ *is the partially-symmetric polynomials* $\mathbb{Z}[x_1,\ldots,x_n]^{S_{n_1}\times S_{n_2-n_1}\times\cdots\times S_{n-n_d}}/\langle e_i(x_1,\ldots,x_n),\ i=1,\ldots,n\rangle$.

Proof. ... lemma 3, plus going from Gr to multistep □

Exercise 14. Let $\pi\in S_n$, and p a polynomial in finitely many (x_i) such that $p\mapsto[X_{\pi\oplus \mathrm{Id}_{N-n}}]\in H^*(Fl(\mathbb{C}^N))$ for every $N>n$ (where $\pi\oplus \mathrm{Id}_{N-n}$ is the image of π under the usual inclusion $S_n\hookrightarrow S_N$). Show that $p=\mathcal{S}_\pi$.

1.11. **Kleiman's theorem. Examples of Schubert calculus.** One of the principal differences in doing topology with complex manifolds, rather than real, is that every manifold comes with a canonical orientation, and the induced orientation on the transverse intersection $A\cap B$ of two complex submanifolds matches the intrinsic orientation of the submanifold $A\cap B$.

By the dual-basis property of proposition 4, one can compute the structure constants $c^\sigma_{\pi\rho}$ of the Schubert basis by integrating *triple* products:

$$c^\sigma_{\pi\rho} = \int_{Fl(\mathbb{C}^n)} S_\pi S_\rho S^\sigma$$

Theorem 11. [?] *Let* $C_1,\ldots,C_m$ *be subvarieties of a homogeneous space* G/H*, where* G *is an algebraic group in characteristic* 0*. Then for general choices of* $g_1,\ldots,g_m$*, the intersection* $\cap_{i=1}^m g_i\cdot C_i$ *is transverse.*

In particular, the cohomology product $[C_1]\cdots[C_m]$ *can be computed as* $[\bigcap_{i=1}^m(g_i\cdot C_i)]$.

The proof is not very difficult; the characteristic 0 assumption is used in the "generic freeness theorem", the algebraic analogue of Sard's theorem.

Corollary 5. *The structure constants* $c^\sigma_{\pi\rho}$ *in the Schubert basis are nonnegative.*

Proof. Since $H^*(Fl(\mathbb{C}^n))$ is a graded ring, we may assume $\ell(\sigma)=\ell(\pi)+\ell(\rho)$. In that case $\operatorname{codim} X_\lambda+\operatorname{codim} X_\rho+\operatorname{codim} X^\sigma=\dim Fl(\mathbb{C}^n)$, so (by Kleiman's theorem) p $\dim(g_1\cdot X_\lambda\cap g_2\cdot X_\mu\cap g_3\cdot X_\sigma)=0$. By the compatibility of the complex orientations, the intersection points come with positive orientations. □

It is a central problem in algebraic combinatorics to compute these structure constants in a manifestly positive way. Any single structure constant $c^\sigma_{\pi\rho}$ can be computed as $\partial_\sigma(\mathcal{S}_\pi\mathcal{S}_\rho)$ (assuming $\ell(\sigma)=\ell(\pi)+\ell(\rho)$), which is easy to implement on a computer. However, this method produces many positive and negative terms, with no obvious reason that more of them are positive.

To see how special homogeneous spaces are for this positive-multiplication property, consider the space $\widetilde{\mathbb{CP}^2}$, the plane blown up at a point, where $E\cong\mathbb{CP}^1$ is the curve lying over the blown-up point. By various means one may compute $[E]^2=-1[pt]$. A topologist would say, move E to a nearby S^2 called E' inside the real 4-manifold $\widetilde{\mathbb{CP}^2}$, then intersect

$E \cap E'$ transversely, and do tangent space calculations to determine the orientations on the resulting points – apparently the total is -1. If E' were a complex submanifold like E, then those intersections would be positive: *hence there can be no such* E'. This is why E is called an "exceptional" curve: it has no nearby neighbors.

Exercise 15. Use divided-difference operators to compute the entire ring structure on $H^*(Fl(\mathbb{C}^3))$.

Exercise 16. Use divided-difference operators to prove **Monk's rule**:

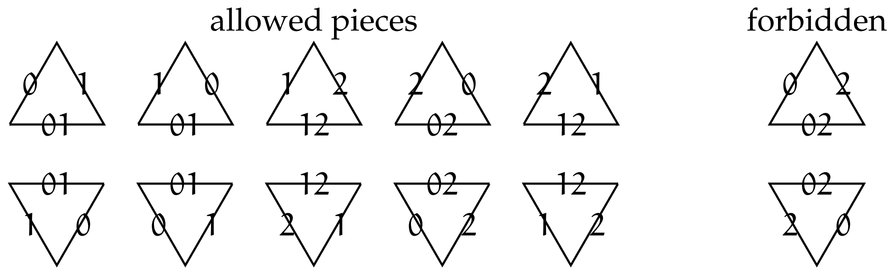
$$S_\pi S_{r_i} = \sum_{\substack{j \le i,\, i+1 \le k \\ (j\leftrightarrow k)\circ\pi \gtrdot \pi}} S_{(j\leftrightarrow k)\circ\pi}$$

Since the divisor classes S_{r_i} generate $H^*(Fl(\mathbb{C}^n))$ as a ring, this rule (rather implicitly) determines the entire ring structure.

2. A first look at puzzles

In this section we present our combinatorial formulæ for several families of Schubert calculus problems, all based on tiling problems we call "puzzles".

2.1. **Grassmannian puzzles.** The **Grassmannian $\triangle$ pieces** are unit triangles with NW and NE sides labeled 0, 1, or 2, and South side labeled 01, 02, or 12, with the South label the (unordered) disjoint union of the NW and NE labels, *except* that one of the six cases is forbidden. The **Grassmannian ∇ pieces** are the $180°$ rotations.



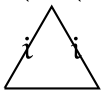
A **Grassmannian puzzle** of size n is a filling of $n\triangle$ with Grassmannian $\triangle$ and ∇ pieces, with a forbidden label on each side: the NE has no 2s, the NW has no 0s, and the South has no 02s.

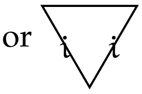
Theorem 12 (recasting of theorem 1 from **[?]**). *Let λ, μ, ν be three $0,1$-strings defining classes on $Gr(k; \mathbb{C}^n)$. Then the Schubert structure constant $c_{\lambda\mu}^{\nu}$ is the number of Grassmannian puzzles with*

- μ *on the NE side,*
- λ *on the NW side but $0, 1$ turned into $1, 2$ respectively,*
- ν *on the S side but $0, 1$ turned into $01, 12$ respectively,*

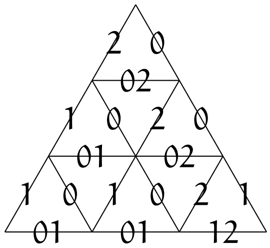
each read left-to-right.

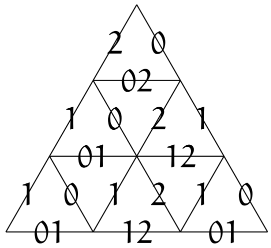
This theorem (whose proof will come later) solves the problem of Grassmannian Schubert calculus, which has received many rules, the first due to Littlewood and Richardson; for this reason these $\{c_{\lambda\mu}^{\nu}\}$ are also called **Littlewood-Richardson coefficients**.

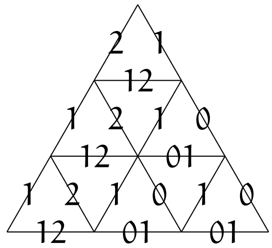
Example. We use this rule to compute all products in $H^*(\mathrm{Gr}(1;\,\mathbb{C}^3))$. The main trick, during creation, is to avoid any partial labeling in which 

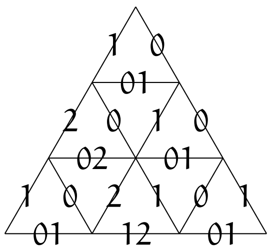
or 

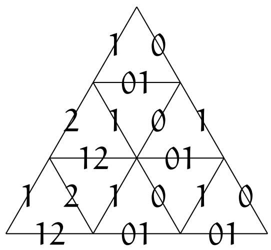
appears. The six possible puzzles are these:

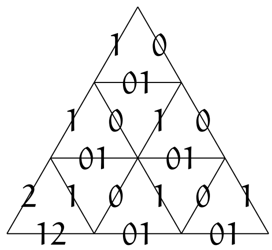








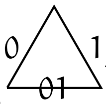


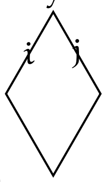


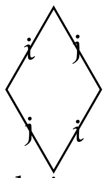
The first three show that S_{001} (written as 112 on the Northwest side) is the left multiplicative identity. The first, fourth, and sixth show that the same for right multiplication (now S_{001} is 001 on the Northeast). The fifth shows that two transverse lines in the projective plane intersect in one point.

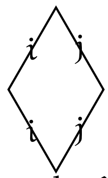
Observe that one can flip a puzzle left-right and replace each label i by $2-i$, obtaining another valid puzzle. Flipping the six puzzles above gives the corresponding six puzzles for $H^*(\mathrm{Gr}(2;\,\mathbb{C}^3))$. More generally, this **puzzle duality** corresponds the puzzles for $H^*(\mathrm{Gr}(k;\,\mathbb{C}^n))$ with those for $H^*(\mathrm{Gr}(n-k;\,\mathbb{C}^n))$, which is nice because the homeomorphism $\mathrm{Gr}(k;\,\mathbb{C}^n) \cong \mathrm{Gr}(n-k;\,\mathbb{C}^n), V \mapsto V^\perp$ takes Schubert classes to Schubert classes.

Some tips on puzzle generation/computation:

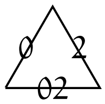
- Don't make the rookie mistake of drawing the initial labels on the outside of the puzzle, as in 

; this isn't sustainable. Draw them directly on the edges.
- Puzzles work pretty well on graph paper if you make them 45°-45°-90° triangles. Otherwise, draw in all the interior vertices before starting to add edges with labels.
- It's not too important to actually label the horizontal labels in the interior of the puzzle. You can just go from 

directly to 

or 

.
- You know that the 0s on the bottom must trace their way to the 0s on the NE side, similarly the 2s to the NW side.
- A good sanity check is that cohomology should be a graded ring, i.e. the number of inversions "some 12 West of some 01" on the South should be the sum of

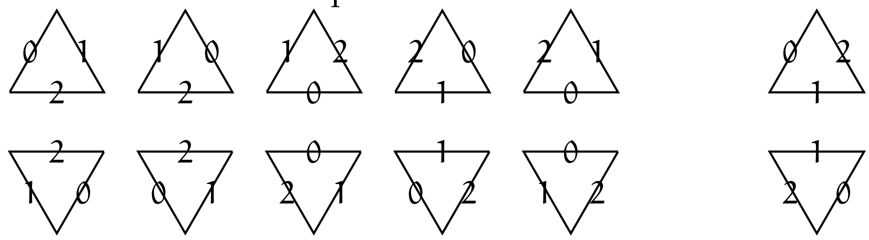
the numbers of inversions on the NW and NE sides. (This property of puzzles is tightly tied to having forbidden the  piece, as we will discuss in §???.)

Exercise 17. Use puzzles to compute all products in $H^*(Gr(2;\ \mathbb{C}^4))$. You can cut this particular calculation in about half with puzzle duality.

2.2. **Rotational symmetry of puzzles.** Recall the integral formula from §1.11 for structure constants (stated here on the Grassmannian), $c_{\lambda\mu}^{\nu} = \int_{Gr(k;\mathbb{C}^n)} S_\lambda S_\mu S^\nu = \int_{Gr(k;\mathbb{C}^n)} S_\lambda S_\mu S_{\nu \text{ reversed}}$. This suggests we seek a modification of Grassmannian puzzles that lets us see this $\mathbb{Z}/3$ rotational symmetry, where λ, μ, ν are read from the puzzle boundary *all clockwise* instead of all left-to-right.

We do this in two stages. The first is to replace each double-label ij by its complement, i.e. $01 \mapsto 2, 02 \mapsto 1, 12 \mapsto 0$. The resulting pieces are rather like hadrons assembled from red, green, and blue quarks.

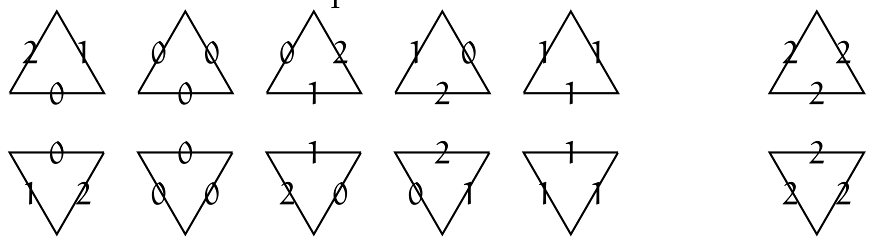
allowed pieces forbidden



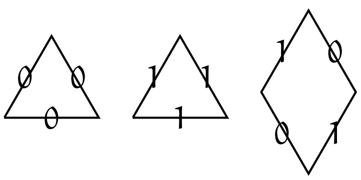
The boundary conditions on a puzzle are now that 2 does not appear on the NE side, nor 0 on the NW side, nor 1 on the South side. Without those conditions, and the forbidding of the one piece, we would have the $\mathbb{Z}/3$ symmetry.

The remaining step is to rotate the edge labels $0 \mapsto 1 \mapsto 2 \mapsto 0$, once on the South side (so the forbidden 1s become forbidden 2s), twice on the NW side (so the forbidden 0s become forbidden 2s). Now the pieces look as follows:

allowed pieces forbidden



Since there are no 2s allowed on the outside of the puzzle – the boundary conditions are determined by the 0, 1-strings λ, μ, ν directly rather than involving some label correspondence – one could also imagine gluing pieces together along 2-edges. Then the pieces become the original puzzle pieces from [**?**]:

 up to rotation.

Exercise 18. Download the Grassmannian triangle-and-rhombus pieces from ??? and cut them out on a laser cutter.

Exercise 19. Define a **region** in a triangle-and-rhombus puzzle to be a maximal set of identical (up to rotation) adjacent pieces. So there are 0-regions, 1-regions, and rhombus regions.

(1) Show that all the regions are convex.
(2) Show that the rhombus regions are parallelograms.
(3) Show that the 0-regions fit together, under translation but not rotation, into a size k triangle.
(4) Show that the 0-regions fit together, under translation but not rotation, into a size $n-k$ triangle.
(5) From these and area considerations, determine the number of rhombi as a function of k, n.

Exercise 20. Take the results of the last two exercises to a math circle for grade-school children, and challenge them to figure out how many 0-triangles, 1-triangles, and rhombi there are in a $Gr(k;\mathbb{C}^n)$ puzzle by experiment.

2.3. **Separated-codescent puzzles.** Let π,ρ be two strings of length n, not necessarily of the same content. Say that (π,ρ) have **codescents separated by** k if there exists a k such that

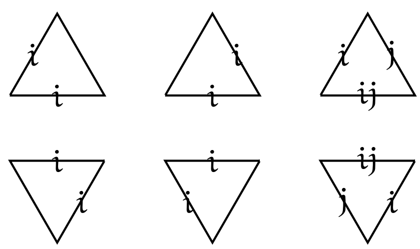
- for all $k \le j, j+1$, we have j left of $j+1$ in π, and
- for all $i, i+1 \le k$, we have i left of $i+1$ in ρ.

Equivalently, S_π is pulled back from $Fl(k,k+1,\ldots,n;\mathbb{C}^n)$ whereas S_ρ is pulled back from $Fl(1,\ldots,k;\mathbb{C}^n)$. To consider "multiplying" them, we have to pull them back to a common flag manifold, refining the content of each, e.g. along the inclusion

$$Fl(\mathbb{C}^n) \quad \hookrightarrow \quad Fl(k,k+1,\ldots,n;\mathbb{C}^n) \times Fl(1,\ldots,k;\mathbb{C}^n)$$

Obviously, if both are k-Grassmannian classes, then their codescents are separated by k.

Theorem 13. [?] *If (π,ρ) have codescents separated by k, then $c^{\sigma}_{\pi\rho}$ is the number of puzzles made of the puzzle pieces*



$i > j$

such that π is on the NW side, with $1\ldots k-1$ erased, ρ is on the NE side, with $k\ldots n$ erased, and σ is along the South side, all left-to-right.

k vs k − 1 confusion here

Exercise 21. Derive the Grassmannian puzzle rule from the separated-codescents puzzle rule.

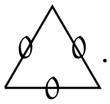
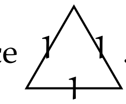
Exercise 22 (very difficult). Show that the separated-codescents puzzle rule defines a graded ring.

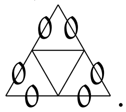
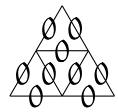
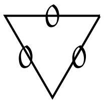
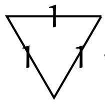
2.4. **Inventing 2-step puzzles.** There are also puzzle rules for Schubert calculus on 2- and 3-step flag manifolds, which were discovered and are most easily stated in the $\mathbb{Z}/3$-rotationally invariant terms.

Rather than stating them outright, we encourage the reader to rediscover them (as the author did) through a series of experiments. We take the following axioms:

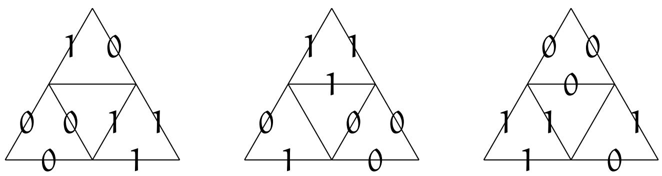
(0) All puzzle pieces are unit triangles (unlike the rhombus in §2.2).
(1) $c_{\pi\rho}^{\sigma} = \#\{$puzzles with π, ρ, σ on their NW, NE, South edges, all left-to-right$\}$.
(2) Two edge labels on a puzzle piece determine the third.

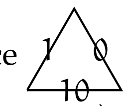
(We could add axioms like "The set of puzzle pieces for d-step flag varieties should be invariant under rotation (as in §2.2)" and "The set of puzzle pieces for d-step flag varieties should be invariant under reflection + taking labels $i \mapsto d - i$ (puzzle duality)" but we won't actually need them.)

We start small, with $Gr(0; \mathbb{C}^1)$, whose only relation is $S_0 S_0 = S_0$. This means we need a puzzle (and hence a puzzle piece!) . Similarly, $Gr(1; \mathbb{C}^1)$ gives us the piece .

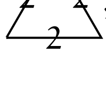
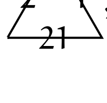
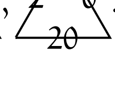
Now we try out $Gr(0; \mathbb{C}^2)$, whose only relation is $S_{00} S_{00} = S_{00}$. Axiom 1 says that there should be a way to fill in ; axiom 2 tells us how to do it , and in so doing, we discover the  piece in the center. The same process applied to $Gr(1; \mathbb{C}^2)$ with relation $S_{11}^2 = S_{11}$ leads us to .

The real action comes at $Gr(1; \mathbb{C}^2)$, with relations $S_{01}^2 = S_{01}, S_{01} S_{10} = S_{10} S_{01} = S_{10}$. Applying axioms 1 and 2 leads us to the partial fillings



We invent a new label "10" to go on the unlabeled edge, which requires only one new puzzle piece  up to rotation, finishing the fillings of all three puzzles (in which all six rotations occur).

At that point one continues to try out larger $Gr(k; \mathbb{C}^n)$ Schubert problems (using Schubert polynomials and divided difference operators to supply experimental data), and the three pieces already discovered keep supplying exactly the right number of puzzles, suggesting it's time to try to prove a theorem. (This is not precisely how Terry Tao and I discovered these three pieces, but that's a story for another book.)

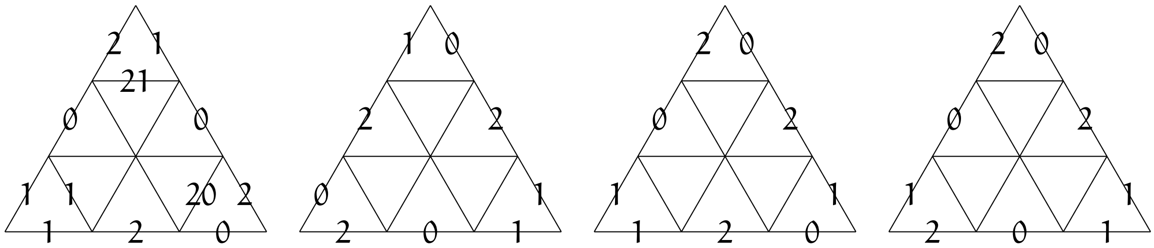
The first 2-step flag manifolds to consider are the degenerate $Fl(0, 0; \mathbb{C}^1)$, $Fl(0, 1; \mathbb{C}^2)$, $Fl(1, 1; \mathbb{C}^2)$, whose identity elements are S_2, S_{12}, S_{02} respectively. These lead us to invent the labels $2, 21, 20$ and pieces , , .

Exercise 23. Recall from exercise 15 the ring structure on $H^*(Fl(\mathbb{C}^3))$: the (left and right) identity is S_{012}, and

$$S_{102}^2 = S_{120} \qquad S_{021}^2 = S_{201} \qquad S_{102}S_{021} = S_{120} + S_{201} = S_{102}S_{021}$$

$$S_{120}S_{021} = S_{021}S_{120} = S_{021}S_{120} = S_{120}S_{021} = S_{210}$$

The first three equalities require these four puzzle boundaries (by axiom 1). Axiom 2 has been applied to the leftmost.



Find a scheme to create only two new puzzle pieces (up to rotation), while giving (unique) fillings of all these puzzles.

Exercise 24. Write a computer program to check that the puzzle pieces you found in the previous exercise do indeed correctly compute products on 2-step flag manifolds in $\mathbb{C}^n$ for $n \leq 8$. (For larger n the Schubert polynomial calculations get really slow.[9])

The particularly industrious reader can now attempt 3-step (or higher! though under some reasonable-looking assumptions, 4-step puzzles don't exist). In 1999 I performed the exercises above and invented 2-step puzzles, which were only proven in 2014 to do the job [**?**]. I had a guess about d-step puzzles in general, which was very pretty but not quite correct already at 3-step, and in despair I abandoned the approach. Thankfully Anders Buch continued experimentation based on my conjecture, far enough to discover the 3-step pieces I had missed, and we confirmed his version in 2017 [**?**]. So as not to spoil the exercises in this section, we wait to present the 3-step rule in §**??**.

3. Equivariant cohomology

3.1. **Recalling ordinary cohomology.**

We recall some basic facts about ordinary cohomology, and convolution products, before considering what properties we would want to ask about an equivariant version.

Cohomology theories are contravariant functors E^* from the category

Top = (topological spaces, continuous maps considered up to homotopy)

to the category

Ab = (pointed abelian groups, pointed group homomorphisms)

satisfying a bunch of axioms, one being that the maps $E^*(X), E^*(Y) \to E^*(X \times Y)$ induced from the projections $X \times Y \twoheadrightarrow X, Y$ extend functorially to a map $E^*(X) \otimes E^*(Y) \to E^*(X \times Y)$. This map is required compatible with $E^*(X) \cong E^*(X) \otimes 1_{E^*(Y)} \hookrightarrow E^*(X) \otimes E^*(Y) \to E^*(X \times Y)$ (where $1_{E^*(Y)}$ is the point in the pointed abelian group).

[9]Anders Buch managed to confirm the 2-step puzzle rule up to $n = 15$, before proving it in [**?**] about ten years later, but not by comparing to Schubert polynomial calculations; rather, he checked the puzzle rule *against itself* by checking that it was associative and correctly computed multiplication by some basic generating classes. This gives a sense of how much more efficient positive rules, like the puzzle rule, can be compared to cancelative rules like $c_{\pi\rho}^{\sigma} = \partial_\sigma(\mathcal{S}_\pi \mathcal{S}_\rho)$.

Since every topological space X has a diagonal inclusion $X \hookrightarrow X\times X$, one obtains a canonical map $E^*(X \times X) \to E^*(X)$; composing with the map $E^*(X) \otimes E^*(X) \to E^*(X)$ above we learn that each $E^*(X)$ is a ring Since the diagonal inclusions induce commuting squares, each map $E^*(Y) \to E^*(X)$ from $X \to Y$ is a ring homomorphism. Effectively, we could have chosen **Ring** as our target category.

Since each X comes with a canonical map $\pi_X : X \to pt$ and any triangle $\begin{smallmatrix} & \nearrow Y \searrow & \\ X & \to & pt \end{smallmatrix}$ commutes, we could first determine $E^*(pt)$ and then declare $E^*(pt)$**-Alg** to be our target category. In ordinary cohomology $E^* = H^*$, $H^*(pt) \cong \mathbb{Z}$, so $E^*(pt)$**-Alg** is just **Ring** anyway.

Ordinary cohomology – and many others, but we shift focus to ordinary now – has a corresponding homology functor H_* with a "cap product" action of $H^*(X)$ on $H_*(X)$. Unfortunately, both gradings are conventionally positive (in a perfect world cohomology would be negatively graded) so $H^i(X) \otimes H_j(X) \xrightarrow{\cap} H_{j-i}(X)$ rather than mapping to $H_{i+j}(X)$. If we map further using $H_*(X \to pt)$, this composite defines an $H_*(pt)$-valued pairing between cohomology and homology. In the case that X is an oriented (e.g. complex) manifold and is compact, this pairing is perfect.

We will frequently be dealing with noncompact complex varieties X and closed (but noncompact) subvarieties $Y \subseteq X$, so we'll also need **Borel-Moore homology** $H_*^{BM}(X)$, which is based on *locally finite* formal sums of singular chains. This provides a home for a "fundamental homology class of X" (using the fact that complex varieties are triangulable) and more generally of closed subvarieties. This functor is covariant for proper maps $f : X \to X'$, such as closed inclusions, and this map $H_*^{BM}(X) \to H_*^{BM}(X')$ is a module homorphism w.r.t. the action of the only ring that acts on both groups – the target's cohomology $H^*(X')$.

The fundamental class $[X]_* \in H_*(X)$ gives us a **Poincaré map** $\cap[X]_* : H^*(X) \to H^{BM}_{\dim_{\mathbb{R}} X - *}(X)$, which is an isomorphism when X is smooth (Poincaré duality). In particular we can often think about cohomology classes in geometric terms, as coming from cycles $Y \subseteq X$, and we reserve $[Y]$ for these. If $f : X' \to X$ is a map of oriented manifolds and $Y \subseteq X$ is an oriented cycle transverse to f, then $f^*([Y]) = [f^{-1}(Y)]$.

It will be very helpful to take that geometric point of view on the ring structure of $H^*(X)$, say in the smooth complex case. Let $Y, Z \subseteq X$ be closed subvarieties, defining (by Poincaré duality) cohomology classes $[Y], [Z]$. If Y intersects Z transversely, then $[Y][Z] = [Y \cap Z]$. Hence $[Y][Z]$ can be seen as a measure of our inability to perturb Y, Z to make them disjoint.

3.2. **Kernels and convolution.** Let A, B be smooth compact oriented manifolds, and $[K] \in H_*(A \times B)$ a homology class (usually associated to a specific cycle K for "kernel"). Define the $[K]$**-transform** (or K**-transform** when we do indeed have the cycle in hand)

$$\Phi_{[K]} : H^*(A) \to H^*(B), \qquad \alpha \mapsto \text{P.D.}\ (\pi_B)_*\big((\alpha \otimes 1) \cap [K]\big)$$

where π_A, π_B are the coordinate projections of $A\times B$, and P.D. is Poincaré dual. This should be seen as analogous to the Fourier transform $L^2(V) \to L^2(V^*)$, where K is the analogue of the Fourier kernel. A classic reference (which operates at a different level of generality than we will here) is [**?**, §??].

Exercise 25. Regard $H^*(A)$ as a nondegenerate inner product space, where the inner product on $H^*(A)$ is $\langle \alpha, \beta \rangle \mapsto \int_A \alpha\beta$; similarly $H^*(B)$ is one. A linear map $\Phi : V \to W$ from one nondegenerate inner product space to another has a **transpose** $\Phi^T : W \to V$ made by composing $W \to W^* \xrightarrow{\Phi^*} V^* \to V$. Show that the transpose $(\Phi_{[K]})^T$ is given by the homology class $[K]^T \in H_*(B \times A)$ corresponding to $[K]$ under the obvious correspondence $A \times B \cong B \times A$.

The map $(\pi_B)_*$ is effectively integration along the fibers of $\pi_B : A \times B \twoheadrightarrow B$, which is where we use A compact.

One would like to regard $(H^*, \bullet\text{-transform})$ as a transpose-respecting functor from some category of spaces and kernels, to nondegenerate inner product spaces. That requires some kind of convolution

$$\mathrm{Kernels}(A,B) \quad \times \quad \mathrm{Kernels}(B,C) \quad \to \quad \mathrm{Kernels}(A,C)$$

intertwining with the usual

$$\mathrm{Hom}(H^*(A),H^*(B)) \quad \times \quad \mathrm{Hom}(H^*(B),H^*(C)) \quad \xrightarrow{\circ} \quad \mathrm{Hom}(H^*(A),H^*(C)).$$

This is easy enough to do formally on the level of classes,

$$([K],[L]) \mapsto (\pi_{AC})_* \mathrm{P.D.}\Big(\pi_{AB}^*(\mathrm{P.D.}[K]) \cup \pi_{BC}^*(\mathrm{P.D.}[L])\Big)$$

where $\pi_{AB}, \pi_{BC}, \pi_{AC}$ are projections from $A \times B \times C$. Since this involves multiplying cohomology classes, if one works with actual kernels $K \subseteq A \times B, L \subseteq B \times C$ one runs into transversality issues. The double Poincaré dualization can be skipped as long as $K \times C, A \times L$ are transverse in $A \times B \times C$, and the convolution becomes

$$(K, L) \mapsto \pi_{AC}\left((K \times C) \cap (A \times L)\right).$$

For simplicity we work with the *categoroid*[10] $\mathcal{K}$ whose objects are compact oriented manifolds, and morphisms $\mathrm{Hom}_{\mathcal{K}}(A, B)$ are oriented cycles in $A \times B$, called "kernels". Convolution of kernels is *only defined* when they are transverse in the sense above.[11]

Exercise 26. Let A, B each be a finite set of points. Correspond the possible $[K]$ with matrices, giving linear transformations between the coordinatized vector spaces $H^*(A)$, $H^*(B)$. Show that the recipe above, interpreted this way, gives matrix multiplication.

There is of course another category **COMfld** in play here besides $\mathcal{K}$ (the categoroid just defined) and **Inner** (the category of finite-dimensional vector spaces with[12] nondegenerate inner products) which is that of compact oriented manifolds and smooth maps. This has a *covariant* functor H^* to **Inner**, which combined with transpose gives the usual contravariant functor H^*.

Lemma 7. *Let* $f : A \to B$ *be a smooth map of compact oriented manifolds. Let* $K = \mathrm{graph}(f) := \{(a, f(a)) : a \in A\} \subseteq A \times B$. *Orient it using the diffeomorphism* $K \hookrightarrow A \times B \xrightarrow{\pi_A} A$. *Then the* K*-transform* $H^*(A) \to H^*(B)$ *is* f_**, and the* K^T*-transform* $H^*(B) \to H^*(A)$ *is* f^*.

Proof. Start with $\alpha = \mathrm{P.D.}[X]$, for $X \subseteq A$ a cycle. Then

$$(\alpha \otimes 1) \cap [\mathrm{graph}(f)] = (\mathrm{P.D.}[X \times B]) \cap [\mathrm{graph}(f)] = [(X \times B) \cap \mathrm{graph}(f)] = [\{(x, f(x)) : x \in X\}]$$

[10] A groupoid is a group where not all multiplications are defined. Indeed, a category is a monoid where not all multiplications are defined, so perhaps should be a monoidoid. However, a monoid doesn't seem to be a partially defined mon.

[11] Making such a definition, even when the intersection is non-transverse, is the realm of "derived algebraic geometry" [**?**].

[12] As a category this is just **Vec**, its additional structure being the "transpose" operation on morphisms.

since the intersection $(X \times B) \cap \mathrm{graph}(f)$ is transverse. (Check: the conditions on $(\vec{v}, \vec{w}) \in T_{x,f(x)}(A \times B)$ defining the two tangent spaces intersecting at $(x, f(x)) \in (X \times B) \cap \mathrm{graph}(f)$ are given by independent conditions $\vec{v} \in T_x X$, $\vec{w} = Tf(\vec{v})$.)

The map $\iota : A \hookrightarrow A \times B$, $a \mapsto (a, f(a))$ takes $[X] \mapsto [(X \times B) \cap \mathrm{graph}(f)]$, so we can compute $(\pi_A)_*([(X \times B) \cap \mathrm{graph}(f)])$ as $(\pi_A \circ \iota)_*([X]) = f_*([X])$. This proves the first claim.

The second statement is the transpose of the first. □

Exercise 27. Show that the assignment $(f : A \to B) \mapsto (\mathrm{graph}(f) \subseteq A \times B)$ defines a functor **COMfld** $\to \mathcal{K}$, i.e. takes composition to (defined!) composition.

The above lemma and exercise together say that the *covariant* functor H^* : **COMfld** $\to$ **Inner** factors as

$$\begin{array}{ccccc} \textbf{COMfld} & \to & \mathcal{K} & \to & \textbf{Inner} \\ (f : A \to B) & \mapsto & (\mathrm{graph}(f) \subseteq A \times B) & & \\ & & (K \subseteq A \times B) & \mapsto & (K\text{-transform } \Phi_{[K]} : H^*(A) \to H^*(B)) \end{array}$$

where the second functor is transverse-respecting.

3.3. **Equivariant cohomology: properties and definition.** We now state several axioms that will characterize equivariant cohomology. Fix a topological group G.

Equivariant cohomology should be a functor H^*_G from G-**Top**, the category (topological spaces with G-actions, continuous G-equivariant maps up to G-equivariant homotopy), to pointed abelian groups (or rings, or $H^*_G(pt)$-algebras, just as in §3.1). It should satisfy:

1. If G acts on X freely, then $H^*_G(X) := H^*(X/G)$.
2. If E is a weakly contractible G-space, then $H^*_G(X) \cong H^*_G(X \times E)$. (Note that this would already be covered by the above were E *equivariantly* contractible.)

If we can find even one such G-space EG that is both free and equivariantly contractible, we then have a definition $H^*_G(X) \cong H^*_G(X \times EG) = H^*((X \times EG)/G)$ for any X. This "homotopy quotient" $(X \times EG)/G$ is called the **Borel mixing space** of X, providing the **Borel construction** of equivariant cohomology. Note that for any $G' \leq G$, the space EG serves as an EG' as well.

Example. Let $G = GL(n, \mathbb{C})$. Define EG as the space of $n \times \mathbb{N}$ matrices of full rank. Without the full-rank condition, this (vector) space is obviously contractible; since the singular matrices are a subset of infinite codimension, removing this subset doesn't create any homotopy groups. Then this space also serves as EG' for any group G' with a faithful n-dimensional representation, AKA a "linear group".

We compute the base ring $H^*_{GL(n,\mathbb{C})}(pt)$ of $GL(n, \mathbb{C})$-equivariant cohomology:

$$H^*_{GL(n,\mathbb{C})}(pt) = H^*_{GL(n,\mathbb{C})}(EG) = H^*(GL(n, \mathbb{C})\backslash EG) = H^*(Gr(n; \mathbb{C}^\infty)) = \mathbb{Z}[x_1, \ldots, x_n]^{S_n}$$

Milnor gave a construction of such a space EG for arbitrary topological groups G (subject to topological conditions like being a CW-complex) [**?**,], but we skip it, as we only need the case above of linear groups. Showing well-definedness of $H^*_G(X)$, which we also skip, requires a proof that the homotopy type of the Borel mixing space doesn't depend on the choice of EG.

Exercise 28. Let $G' \leq G$ be a subgroup such that G/G' is contractible, e.g. $U(1) \leq \mathbb{C}^\times$, $SO(n, \mathbb{R}) \leq SL(n, \mathbb{R})$, $T \leq B$. Then there is a natural isomorphism $H^*_{G'}(X) \cong H^*_G(X)$.

Our primary interest is in T-equivariant cohomology for $T \cong (\mathbb{C}^\times)^n$, called an **(algebraic) torus**.

Theorem 14. *Let* $T \cong (\mathbb{C}^\times)^n$ *be a torus, and* $T^* := \mathrm{Hom}(T, \mathbb{C}^\times) \cong \mathbb{Z}^n$ *its* **weight lattice**. *Then*

$$H^*_T(pt) \cong \mathrm{Sym}_{\mathbb{Z}}(T^*)$$

The isomorphism is induced from the map $T^* \to H^2_T(pt)$, $\lambda \mapsto c_1(\mathbb{C}_\lambda \to pt)$, *where* $\mathbb{C}_\lambda$ *is the* 1*-dimensional representation of* T *with character* λ.

This construction uses the *equivariant Chern classes* of a G-equivariant vector bundle $V \to S$, which are easy to define: apply the Borel construction to both V and S to get a vector bundle $(V \times EG)/G$ over $(S \times EG)/G$, and take its ordinary Chern classes, which live in $H^*((S \times EG)/G) =: H^*_G(S)$.

Proof. First, we use the coördinatization $T \cong (\mathbb{C}^\times)^n$, and the space $EGL_1(\mathbb{C}) = \mathbb{C}^\infty \setminus \vec{0}$ from the example above:

$$H^*_T(pt) \cong H^*_{(\mathbb{C}^\times)^n}(pt) \cong H^*(EGL_1(\mathbb{C})^n/T) \cong H^*((\mathbb{CP}^\infty)^n) \cong \mathbb{Z}[x_1^{(2)}, \ldots, x_n^{(2)}]$$

So this ring is definitely the symmetric algebra on its degree 2 part, and we certainly have a group homomorphism $T^* \to H^2_T(pt)$, where both are $\cong \mathbb{Z}^n$. It remains to show that this map is surjective (which will make it automatically $1:1$ by rank considerations).

This, we can check in coördinates. It suffices to handle the $n = 1$ case. Let $\mathbb{C}_1$ be the standard representation of $\mathbb{C}^\times$, and let $E' \subseteq EGL_1(\mathbb{C})$ be $\mathbb{C}^2 \setminus \vec{0}$, so we have the pullback of line bundles

$$\begin{array}{ccc} (\mathbb{C}_1 \times E')/\mathbb{C}^\times & \hookrightarrow & (\mathbb{C}_1 \times EGL_1(\mathbb{C}))/\mathbb{C}^\times \\ \downarrow & & \downarrow \\ (pt \times E')/\mathbb{C}^\times & \hookrightarrow & (pt \times EGL_1(\mathbb{C}))/\mathbb{C}^\times \end{array}$$

Since this diagram is a pullback, c_1 of the right-hand line bundle pulls back to c_1 of the left-hand line bundle under the induced isomorphism $H^*((pt \times EGL_1(\mathbb{C}))/\mathbb{C}^\times) \to H^*((pt \times E')/\mathbb{C}^\times)$, AKA $H^2_T(pt) \to H^2(\mathbb{CP}^1)$. The line bundle on the left is the Hopf bundle, whose c_1 generates $H^2(\mathbb{CP}^1)$, proving the surjectivity. ☐

This theorem, and the example before it, have a common generalization for connected Lie groups: $H^*_G(pt; \mathbb{Q}) \cong \mathrm{Sym}(T^* \otimes \mathbb{Q})^{W_G}$.

Exercise 29. ...prove that with hints

As in ordinary cohomology, we want to compute in equivariant cohomology geometrically, using (now G-invariant) cycles to define cohomology classes. For this we need to define equivariant (Borel-Moore) homology, and fundamental classes therein.

There is an obvious *wrong* guess: the ordinary (and Borel-Moore) homology of the Borel mixing space. The problem with this definition is that the Borel mixing space is usually infinite-dimensional so there is no degree in which its fundamental class would sit.

Exercise 30. Compute the cap-product action of $H^*(\mathbb{CP}^\infty)$ on $H_*(\mathbb{CP}^\infty)$. Awful, isn't it?

Instead, we filter EG as an increasing union of increasingly connected G-manifolds E_iG; specifically we require $\pi_j(E_iG) = 0$ for $j < i$. Then we define the **equivariant homology** groups by

$$H^G_d(X) := \lim_{i \to \infty} H_{d + \dim(E_iG)/G}((X \times E_iG)/G)$$

where the increasing connectedness of E_i causes this limit to stabilize at finite i, for any given d (and doesn't depend on the filtration). In the case of G a linear group in $GL_n(\mathbb{C})$, we can take E_iG to be $n \times \mathbb{N}$ matrices of rank n, supported in the first i columns.

Ordinary cohomology and homology of X lives in degrees $[0, \dim X]$, whereas equivariant cohomology can go up forever, in $[0, \infty)$. Equivariant homology, according to the above definition, maxes out at $\dim X$ but can go *down* forever, in $(-\infty, \dim X]$.

Equivariant homology bears a cap-product action of equivariant cohomology thanks to the maps $H^*_G(X) = H^*((X \times EG)/G) \to H^*((X \times E_iG)/G)$. Note the compatibility of this degree-subtractive action with the ranges of degrees just discussed.

If $Y \subseteq X$ is a G-invariant oriented cycle of real dimension d, and each E_iG is oriented, then $[(Y \times E_iG)/G]$ defines an element in $H^{BM}_{d+\dim_{\mathbb{R}}(E_iG)/G}((X \times EG)/G)$, which for large i is $H^{BM,G}_d(X)$. We call this class $[Y]_*$ (reserving $[Y]$ for an equivariant *co*homology class). Note that most classes in $H^G_*(X)$ are of negative degree, so cannot be of this form, an interesting departure from ordinary homology.

Now that we have fundamental classes, we can define the Poincaré map $H^*_G(X) \to H^{BM,G}_{\dim_{\mathbb{R}} X - *}(X)$, and when it is an isomorphism we can pull back Borel-Moore classes to cohomology classes. In particular we write $[Y] \in H^*_G(X)$ for the preimage of $[Y]_* \in H^{BM,G}_*(X)$.

Proposition 8. *Let V be a T-representation, with weights $\lambda_1, \ldots, \lambda_n$ (repetition allowed). Then the Poincaré map $H^*_T(V) \to H^{BM,T}_{2n-*}(V)$ is an isomorphism, and $[\vec{0}] = \prod_{i=1}^n \lambda_i$. More generally, if $W \leq V$ is a subrepresentation, then $[W]$ is the product of the T-weights in V/W.*

Proof. Write $T \cong (\mathbb{C}^\times)^r$, and let $E_iT := (\mathbb{C}^i \setminus 0)^r$. We want to compute $H^{BM,T}_{2n-d}(V)$, which is $H^{BM}_{2n-d+\dim(E_iT/T)}((V \times E_iT)/T)$ for large i. This is a $\mathbb{C}^n$-bundle over $E_iT/T = (\mathbb{CP}^{i-1})^n$, so we can contract it shifting the degree by the Gysin isomorphism, to get to the group $H^{BM}_{2n-d-2n+(2i-2)n}((\mathbb{CP}^{i-1})^n)$ and from there $H_{(2i-2)n-d}((\mathbb{CP}^{i-1})^n)$. By Poincaré duality this is dual to $H_d((\mathbb{CP}^{i-1})^n)$ which *for* i *large enough*, is isomorphic to $H_d((\mathbb{CP}^\infty)^n)$, finally, dual to $H^d((\mathbb{CP}^\infty)^n) = H^d_T(V)$. This verifies that Poincaré duality holds dimensionally. It remains to check that $H^{BM,T}_*(V)$ is torsion free as a $H^*_T(V)$-module, which is a similar calculation, again derived from Poincaré duality of E_iT/T.

no that just shows it rationally. redo this, maybe prove it for Poincaré duality spaces assuming that each E_iG/G is smooth oriented

Equivariantly coördinatize V as $\prod_{i=1}^n \mathbb{C}_{\lambda_i}$ with coördinates $z_1, \ldots, z_n$, so $\vec{0}$ is the transverse intersection of the hyperplanes $\{z_i = 0\}$. It remains to check that $[\{z_i = 0\}] = \lambda_i$. By factoring out the irrelevant directions, we reduce to the $n = 1$ case ... ☐

We get to our first interesting product calculation in equivariant cohomology: $[\vec{0}]^2 = (\prod_{i=1}^n \lambda_i)\,[\vec{0}]$. Recall our interpretation of $[A][B]$, which is that it measures our inability to move A, B to make them disjoint. In ordinary cohomology on a space of positive dimension, of course $[pt]^2 = 0$, because we can move our two points to be distinct. But in our equivariant situation, we may be unable to move the T-fixed point $\vec{0}$ *while keeping it T-fixed*. Indeed, there are nearby fixed points to move to iff V contains a trivial one-dimensional representation, iff one of the $\lambda_i = 0$, iff $\prod_{i=1}^n \lambda_i = 0$, iff $[\vec{0}]^2 = 0$.

We also get a nice source of positivity for polynomial coefficients:

Corollary 6 (theorem D from **[?]**). *If $X \subseteq V$ is a T-invariant subvariety of a T-representation with weights $\lambda_1, \ldots, \lambda_n$, then $[X]$ can be expressed as $p(\lambda_1, \ldots, \lambda_n)$ for some polynomial $p \in$*

$\mathbb{N}[x_1, \ldots, x_n]$ *with squarefree monomials. In particular, if all the* $\{\lambda_i\}$ *lie in (an open half-space and therefore lie in) an orthant* $\sum_{i=1}^n \mu_i$, *then the coefficients of* $[X]$, *written as a polynomial in the* (μ_i), *are nonnegative and* $X \neq \emptyset \implies [X] \neq 0$.

Proof sketch. Gröbner basis theory provides a T-equivariant degeneration of X to a union X' of coördinate subspaces, with scheme-theoretic multiplicities. This can be used to give a T-equivariant homology of X to X', so $[X] = [X']$. Then $[X']$ is the sum over those subspaces W, of the (natural number) multiplicity times the product of the weights in V/W.

The second statement follows straightforwardly. For the third, pick the Gröbner basis to be T-homogeneous, hence without constant term. We learn that $\vec{0}$ lies in X and in X'. Therefore the positive sum above is a nontrivial sum, of terms that cannot cancel one another. ☐

One reason to compute equivariant cohomology is that it can be *easier* than ordinary cohomology. We don't justify this claim yet (see theorem **??**) but explain how and when one may recover ordinary from equivariant.

Theorem 15. *Let* G *act on* X, *and make* $\mathbb{Z}$ *into a graded* $H^*_G(pt)$*-module in the unique way. Then there is a natural map* $\mathbb{Z} \otimes_{H^*_G(pt)} H^*_G(X) \to H^*(X)$. *If* X *enjoys Poincaré duality and* $H^{BM}_*(X)$ *is generated by* G*-invariant cycles (which is typical for* G *solvable, and otherwise atypical), then this natural map is an isomorphism over the rationals.*

Proof. Consider the principal G-bundle $G \to EG \to EG/G$, where the latter space is usually called BG. Applying the Borel construction to the G-equivariant bundle $X \to pt$ gives the X-bundle $X \to (X \times EG)/G \to BG$ – it is just the "associated bundle construction" replacing the G fibers with Xs.

This gives us a pullback square of spaces, and its cohomology:

$$H^* \begin{pmatrix} X & \to & (X \times EG)/G \\ \downarrow & & \downarrow \\ pt & \to & BG \end{pmatrix} \qquad = \qquad \begin{matrix} H^*(X) & \leftarrow & H^*_G(X) \\ \uparrow & & \uparrow \\ \mathbb{Z} & \leftarrow & H^*_G(pt) \end{matrix}$$

So we have our natural map $H^*_G(X) \to H^*(X)$. To show that it factors through the quotient $\mathbb{Z} \otimes_{H^*_G(pt)} H^*_G(X)$ we need check that the composite $H^*_G(pt) \to H^*_G(X) \to H^*(X)$ factors through $\mathbb{Z}$, but of course this is guaranteed by the commuting square.

For the isomorphism statement, pick G-invariant cycles $\{X_i\}$ giving a $\mathbb{Q}$-basis of $H^*(X; \mathbb{Q})$, then apply the Leray-Hirsch theorem to the X-bundle $(X \times EG)/G \to BG$. We learn that the $[X_i] \in H^*((X \times EG)/G)$ form a $H^*(BG)$-basis of the free $H^*(BG)$-module $H^*((X \times EG)/G) = H^*_G(X)$, hence after tensoring they form a $\mathbb{Q}$-basis of $\mathbb{Q} \otimes_{H^*_G(pt;\mathbb{Q})} H^*_G(X; \mathbb{Q})$, finally bijecting to the $\mathbb{Q}$-basis of $H^*(X; \mathbb{Q})$. ☐

Much of the literature invokes the property of an action being **equivariantly formal**, which is the statement that Leray-Hirsch applies to the X-bundle $(X \times EG)/G \to BG$. In this language, the theorem suggested above becomes the statement that a solvable group action on a variety enjoying Poincaré duality is equivariantly formal.

I guess I want to prove that pullback to fixed points is an isomorphism, H^*_T-rationally

3.4. **From** G **to** T. We explain here why we focus on T-equivariant cohomology to the exclusion of other theories $H^*_G(\bullet)$.

There is an intermediate group, the normalizer $N(T) \leq G$, with the properties that $W := N(T)/T$ is finite (and $\cong S_n$ in the case we most care about, $G = GL_n(\mathbb{C})$) and indexes the Bruhat cells on $B_-\backslash G$. Let $G \circlearrowright X$, and consider the two fiber bundles

$$(X \times EG)/T \twoheadrightarrow (X \times EG)/N(T) \twoheadrightarrow (X \times EG)/G$$

(with fibers W and $G/N(T)$ respectively) inducing the maps

$$H^*_T(X) \leftarrow H^*_{N(T)}(X) \leftarrow H^*_G(X)$$

which we call the "first map" and the "second map" in what follows. Note that W acts on $(X \times EG)/T$ from the right, giving an action on $H^*_T(X)$.

Lemma 8. *The first map* $H^*_T(X) \leftarrow H^*_{N(T)}(X)$ *induces an isomorphism* $H^*_T(X;\mathbb{Q})^W \cong H^*_{N(T)}(X;\mathbb{Q})$.

Proof. Because the map $(X \times EG)/T \twoheadrightarrow (X \times EG)/N(T)$ is a principal covering space with finite fiber, we have a map the other direction, "integration (really summation) over the fiber", that we then divide by $\#W$. (This is one reason we pass to rational coefficients, though $\frac{1}{\#W}\mathbb{Z}$ would be good enough.)

The image of $H^*_{N(T)}(X) \to H^*_T(X)$ obviously lands inside $H^*_T(X)^W$, and the composite $H^*_{N(T)}(X;\mathbb{Q}) \to H^*_T(X;\mathbb{Q}) \to H^*_{N(T)}(X;\mathbb{Q})$ is the identity. Meanwhile, the composite $H^*_T(X;\mathbb{Q})^W \to H^*_{N(T)}(X;\mathbb{Q}) \to H^*_T(X;\mathbb{Q})^W$ is the identity. These establish the lemma. □

Lemma 9. $H^*(G/N(T);\mathbb{Q}) = H^0(G/N(T);\mathbb{Q}) = \mathbb{Q}$ *for* G *a connected Lie group with maximal torus* T.

Proof. We can safely replace G by a maximal compact subgroup K if necessary, since G/K is contractible by the Iwasawa decomposition. Hereafter we assume G is compact connected.

As in the previous lemma, $H^*(G/N(T);\mathbb{Q}) \cong H^*(G/T;\mathbb{Q})^W$. Since G/T is diffeomorphic to $G^{\mathbb{C}}/B_+$ for which we have the Bruhat decomposition, we know that $H^{odd}(G/T;\mathbb{Q}) = 0$, hence $H^{odd}(G/T;\mathbb{Q})^W = 0$ and thus $H^*(G/N(T);\mathbb{Q}) = 0$. Consequently $\chi(G/N(T)) = \dim H^*(G/N(T);\mathbb{Q})$.

Meanwhile, since G/T is a $\#W$-cover of $G/N(T)$, $\chi(G/N(T)) = \chi(G/T)/\#W$. (We could deformation retract G to a maximal compact subgroup $G_{\mathbb{R}}$ to make both spaces compact, and pull back a triangulation of $G_{\mathbb{R}}/N(T_{\mathbb{R}})$ to get a triangulation of $G_{\mathbb{R}}/T_{\mathbb{R}}$, with $\#W$ as many simplices.) Again using the Bruhat decomposition, we learn $\chi(G/N(T)) = 1$.

Together we learn $\dim H^*(G/N(T);\mathbb{Q}) = 1$, giving the result. □

Theorem 16. *Let* G *act on* X, *where* G *is a connected Lie group, and* T *is a maximal torus. Then* $H^*_G(X;\mathbb{Q}) \cong H^*_T(X;\mathbb{Q})^W$.

Proof. The Leray-Hirsch theorem, applied to the $G/N(T)$-bundle $(X \times EG)/N(T) \twoheadrightarrow (X \times EG)/G$, says that since the set $\{1\}$ restricts to a $\mathbb{Q}$-basis of cohomology of the fiber (by lemma 9) it also gives a basis of $H^*_{N(T)}(X;\mathbb{Q})$ as a module over $H^*_G(X;\mathbb{Q})$. Hence the two are isomorphic. Now apply lemma 8. □

3.5. **Convolution in equivariant cohomology.** Return to the setting of §3.2, and the categoroid $\mathcal{K}$ of smooth compact oriented manifolds A and kernels (oriented cycles) $K \subseteq A \times B$. As before, we define the convolution $K \circ L$ of $K \subseteq A \times B$, $L \subseteq B \times C$ only when $K \times C$ transversely intersects $A \times L$.

Even when B is noncompact, it can happen that the projection $(K \times C) \cap (A \times L) \to A \times C$ is proper, giving a reasonable definition of $K \circ L$. The following serves as an example, which will be key:

Proposition 9. *Recall that for $X \subseteq M$ a locally closed submanifold, the* **conormal bundle** $C_X M$ *inside the cotangent bundle* $T^*M := \{(m, \vec{v}) : m \in M, \vec{v} \in T^*_m M\}$ *is* $\{(m, \vec{v}) : m \in X, \vec{v} \perp T_m X\}$.

Let $f : B \to A$ be a function, and $\iota_A : A \hookrightarrow T^*A$, $\iota_B : B \hookrightarrow T^*B$ be the inclusions of the zero sections of the cotangent bundles. Then the following square of kernels commutes:

$$\begin{array}{ccccc} & T^*A & \xrightarrow{C_{\mathrm{graph}(f)^T}(A\times B)} & T^*B & \\ \mathrm{graph}(\iota_A) & \uparrow & & \uparrow & \mathrm{graph}(\iota_B) \\ & A & \xrightarrow{\mathrm{graph}(f)^T} & B & \end{array}$$

Proof. We go from Southwest to SE to NE first:

$$\{(f(b), b, (b', \vec{v})) \in A\times B\times T^*B\} \cap \{(a, b, (b, \vec{0})) \in A\times B\times T^*B\} = \{(f(b), b, (b, \vec{0})) \in A\times B\times T^*B\}$$

whose (bijective) image under π_{A,T^*B} is $\{(f(b), (b, \vec{0})) \in A \times T^*B\}$.

To go from NW to NE we need to compute the conormal bundle to $\mathrm{graph}(f) \subseteq A \times B$. The tangent space at $(f(b), b)$ is the image of $T_b f \oplus \mathrm{Id} : T_b B \to T_{f(b)} A \oplus T_b B$. Hence its perp is the kernel of $T^*_b f + \mathrm{Id} : T^*_{f(b)} A \oplus T^*_b B \to T^*_b B$, so, pairs $\{(\vec{a}, -Tf^*(\vec{a})) \ : \ \vec{a} \in T^*_{f(b)} A\}$.

Now, going SW to NW to NE, we intersect

$$\{(a, (a, \vec{0}), (b, \vec{w})) \in A \times T^*A \times T^*B\} \quad \cap \quad \{(a, (f(b), \vec{a}), (b, -Tf^*(\vec{a}))) \ : \ b \in B, \ \vec{a} \in T^*_{f(b)} A\}$$

obtaining

$$\{(f(b), (f(b), \vec{0}), (b, -Tf^*(\vec{0}) = \vec{0}))\}$$

whose (bijective) image under π_{A,T^*B} is likewise $\{(f(b), (b, \vec{0})) \in A \times T^*B\}$. □

We would like to infer a consequence for the corresponding maps on H^*. But we cannot do so directly, because T^*A, T^*B are noncompact. The idea will be to reduce T^*A to (compact) A in some sense.

Lemma 10. *Let A be a complex manifold, so its cotangent carries a $\mathbb{C}^\times$-action scaling the fibers, where $H^*_{\mathbb{C}^\times} =: \mathbb{Z}[\hbar]$. Let $\iota_A : A \to T^*A$ be the inclusion of the zero section. Then*

$$\begin{array}{rrcll} \iota_A^* : & & H^*_{\mathbb{C}^\times}(T^*A) & \to & H^*_{\mathbb{C}^\times}(\ \ A) \ = H^*(A)[\hbar] \\ \iota^A_* : & \mathbb{Z}[\hbar^\pm] \otimes_{\mathbb{Z}[\hbar]} & H^*_{\mathbb{C}^\times}(\ \ A) & \to & \mathbb{Z}[\hbar^\pm] \otimes_{\mathbb{Z}[\hbar]} H^*_{\mathbb{C}^\times}(T^*A) \end{array}$$

are ring and $\mathbb{Z}[\hbar^\pm]$-module isomorphisms, respectively. Their composite $\iota_A^*(\iota_A)_*$ is multiplication by the equivariant Euler class $e_{\mathbb{C}^\times}(T^*A)$, whose dehomogenization $\hbar \mapsto -1$ is the total Chern class $c(A) = c(TA)$ up to a sign $(-1)^{\dim A}$.

If A, B carry T-actions (separate from the $\mathbb{C}^\times$-action on T^*A, T^*B) with isolated fixed points, then the second statement holds also in $(T \times \mathbb{C}^\times)$-equivariant cohomology if one inverts the non-zero-divisor $e_{T\times\mathbb{C}^\times}(T^*A)$.

Proof. Of course ι_A^* is an isomorphism – the inclusion ι_A is an equivariant homotopy equivalence. The composite takes $1 \mapsto e_{\mathbb{C}^\times}(T^*A)$ essentially by the definition of Euler class. Since the pushforward map $(\iota_A)_*$ is a module homomorphism (the push-pull formula), it is multiplication by this class.

As explained before

lame, $e(T^*A)|_{\hbar\mapsto 1}$ is the total Chern class of T^*A. The $\dim A$ many Chern roots of T^*A are the negatives of those of TA, so the elementary symmetric polynomials in T^*A's Chern roots are off by an alternating sign from those of TA, which accounts for the $\hbar \mapsto -1$ we use here. Finally, the total Chern class (of any complex vector bundle) should start with 1 not $\hbar^{\dim A}|_{\hbar\mapsto -1}$, so to correct the overall sign we multiply by $(-1)^{\dim A}$.

Observe that $H^{>0}(A)$ consists of nilpotents. Since $\hbar^{-\dim A} e_{\mathbb{C}^\times}(T^*A) \in 1 + H^{>0}(A)[\hbar^\pm]$, it is invertible; hence ι_*^A is an isomorphism.

This last argument doesn't quite hold in T-equivariant cohomology, because $H_T^{>0}(A)$ is no longer nilpotent, but the argument still shows that $e_{\mathbb{C}^\times}(T^*A)$ is not a zero divisor. □

We take the above as motivation for extending our categoroid $\mathcal{K}$ as follows. Let the objects in $\mathcal{K}_{\mathbb{C}^\times}$ be oriented $\mathbb{C}^\times$-manifolds M such that $M^{\mathbb{C}^\times}$ is compact, with the morphisms from M to N being $\mathbb{C}^\times$-invariant kernels $K \subseteq M \times N$. Then we get a •-transform functor $\mathcal{K}_{\mathbb{C}^\times} \to \mathbb{Z}[\hbar^\pm]$-modules, where the K-transform Φ_K is defined by

$$\alpha \mapsto \sum_{F \subseteq M^{\mathbb{C}^\times} \text{ component}} \Phi_{K \cap (F \times N)} \left(\frac{\alpha}{e_{\mathbb{C}^\times}(N_F M)} \right)$$

Each summand uses a kernel in $F \times N$, where F is compact.

justify with some Borel-Moore stuff

Exercise 31. Let M be an oriented $\mathbb{C}^\times$-manifold, and $F \subseteq M^{\mathbb{C}^\times}$ a connected component of its fixed point set. Show that F inherits an orientation.

Exercise 32. Show that •-transform is again a functor (takes convolution of transverse kernels to composition of maps on localized equivariant cohomology).

The motivation for the theorem below, which allows us to compute the pullback along $f : B \to A$ in terms of the harder-looking *but sometimes easier* $C\mathrm{graph}(f)^T$-transform, will likely seem completely obscure at the moment. Its utility will become clearer after we study quiver varieties.

Theorem 17. *Let* A, B *be smooth, compact, complex manifolds, so their cotangent bundles each carry a* $\mathbb{C}^\times$*-action scaling the fibers.*

Let $f : B \to A$ *be a function, and* $C\mathrm{graph}(f)^T$ *the conormal bundle to its graph, a kernel in* $T^*A \times T^*B$. *Then if the* $C\mathrm{graph}(f)^T$*-transform*

$$\mathbb{Z}[\hbar^\pm] \otimes_{\mathbb{Z}[\hbar]} H^*_{\mathbb{C}^\times}(T^*A) \to \mathbb{Z}[\hbar^\pm] \otimes_{\mathbb{Z}[\hbar]} H^*_{\mathbb{C}^\times}(T^*B)$$

takes $\alpha \mapsto \beta$, *we learn that*

$$f^* \left(\frac{\alpha}{e_{\mathbb{C}^\times}(T^*A)} \right) = \frac{\beta}{e_{\mathbb{C}^\times}(T^*B)}.$$

Proof. First, we observe that $\alpha = \Phi_{\mathrm{graph}\, \iota_A}(\alpha / e_{\mathbb{C}^\times}(T^*A))$ from $\mathbb{Z}[\hbar^\pm] \otimes_{\mathbb{Z}[\hbar]} H^*_{\mathbb{C}^\times}(A)$, using lemma 10. Hence

$$\begin{aligned} \beta &= \Phi_{C_{\mathrm{graph}(f)^T}(A \times B)}(\alpha) = \Phi_{C_{\mathrm{graph}(f)^T}(A \times B)} \Phi_{\iota_A} \left(\frac{\alpha}{e_{\mathbb{C}^\times}(T^*A)} \right) \\ &= \Phi_{\iota_A} \Phi_{\mathrm{graph}(f)^T} \left(\frac{\alpha}{e_{\mathbb{C}^\times}(T^*A)} \right) = \Phi_{\iota_A} f^* \left(\frac{\alpha}{e_{\mathbb{C}^\times}(T^*A)} \right) = e_{\mathbb{C}^\times}(T^*B)\, f^* \left(\frac{\alpha}{e_{\mathbb{C}^\times}(T^*A)} \right) \end{aligned}$$

where we use proposition 9 to go from the first to the second line, and lemma 10 again in the last step. □

4. Equivariant cohomology of the flag manifold

4.1. **The equivariant Borel presentation.** With equivariant cohomology, and some of our results about it, we can give a much quicker and more satisfying derivation of the Borel presentation (theorem 7). First, we consider the *equivariant* (with respect to B or T) Borel presentation, everywhere over $\mathbb{Q}$: go to B_\G

$$\begin{array}{rcll} H^*_T(G/B) \cong H^*_B(G/B) & \cong & H^*_{B\times B}(G) & \text{undividing by the free B-action} \\ & \cong & H^*_{B\times G_\Delta \times B}(G\times G) & \text{undividing } G\times G \text{ by the free } G_\Delta\text{-action} \\ & \cong & H^*_G(G/B\times G/B) & \text{dividing by the now-free } B\times B\text{-action} \\ & \cong & H^*_G(G/B)\otimes_{H^*_G(pt)} H^*_G(G/B) & \text{the equivariant Künneth theorem} \\ & \cong & H^*_{G\times B}(G)\otimes_{H^*_G(pt)} H^*_{G\times B}(G) & \text{undividing by the } B\times B\text{-action} \\ & \cong & H^*_B(pt)\otimes_{H^*_G(pt)} H^*_B(pt) & \text{dividing by the free } G\times G\text{-action} \\ & \cong & H^*_T(pt)\otimes_{H^*_G(pt)} H^*_T(pt) & \text{since } B/T \text{ is contractible} \\ & \cong & Sym(T^*)\otimes_{Sym(T^*)^W} Sym(T^*) & \end{array}$$

In the $G = GL_n(\mathbb{C})$ case the final result is

$$H^*_T(Fl(\mathbb{C}^n)) \cong \mathbb{Q}[x_1,\ldots,x_n,y_1,\ldots,y_n] \Big/ \langle p(x) = p(y)\ \forall \text{ symmetric polynomials } p\rangle$$

The map from theorem 15 is

$$\begin{array}{rcl} H^*(Fl(\mathbb{C}^n)) & \leftarrow & \mathbb{Q}\otimes_{H^*_T(pt)} H^*_T(Fl(\mathbb{C}^n)) \\ & \cong & \mathbb{Q}\otimes_{Sym(T^*)} (Sym(T^*)\otimes_{Sym(T^*)^W} Sym(T^*)) \\ & \cong & \mathbb{Q}\otimes_{Sym(T^*)^W} Sym(T^*) \quad\cong\quad Sym(T^*) \Big/ \langle p \,:\, p\in Sym(T^*)^W_+\rangle \end{array}$$

In the $G = GL_n$ case we've proven that the Schubert cycles span the cohomology. (Indeed, this spanning condition holds for other connected algebraic groups as well). Since the Schubert cycles are T-invariant, theorem 15 gives a new proof of the nonequivariant Borel presentation from theorem 7, albeit only over the rationals.

4.2. **Double Schubert polynomials.** Consider the $B_-\times B_+$-equivariant open inclusion $\iota : GL_n(\mathbb{C}) \hookrightarrow M_n(\mathbb{C})$. This induces the second map in

$$\begin{array}{rcl} H^*_{B_+}(B_-\backslash GL_n(\mathbb{C})) & \cong & H^*_{B_-\times B_+}(GL_n(\mathbb{C})) \quad \xleftarrow{\iota^*} \quad H^*_{B_-\times B_+}(M_n(\mathbb{C})) \\ & \cong & H^*_{B_-\times B_+}(pt) \quad\cong\quad H^*_{T\times T}(pt) \quad\cong\quad \mathbb{Z}[x_1,\ldots,x_n,y_1,\ldots,y_n] \end{array}$$

Define $\overline{X_\pi} := \overline{B_-\pi B_+} \subseteq M_n(\mathbb{C})$, a **matrix Schubert variety**. Since $\overline{X_\pi}$ is closed inside the smooth oriented $M_n(\mathbb{C})$, it defines an element $[\overline{X_\pi}]$ of $H^*_{B_-\times B_+}(M_n(\mathbb{C}))$, seen just above to be the polynomial ring $\mathbb{Z}[x_1,\ldots,x_n,y_1,\ldots,y_n]$. We use this to define the **double Schubert polynomial** $\mathcal{S}_\pi(x,y) := [\overline{X_\pi}] \in \mathbb{Z}[x_1,\ldots,x_n,y_1,\ldots,y_n]$.

In fact we could have skipped the B_+-equivariance just above, which (as in theorem 15) amounts to setting the $y_i \cong 0$, and obtained polynomials in the x_i alone.

Theorem 18. (1) *Let $\pi' \in S_{n+1}$ be the image of $\pi \in S_n$ under the usual inclusion. Then $\mathcal{S}_\pi(x,y) = \mathcal{S}_{\pi'}(x,y)$.*

(2) *The specialization $\mathcal{S}_\pi(x,0)$ gives the (single) Schubert polynomial from §1.10.*

(3) *Schubert polynomials have positive coefficients. Double Schubert polynomials don't, but are positive sums of monomials in the linear forms* $(x_i - y_j)$.

Proof. (1) Let $p : M_{n+1}(\mathbb{C}) \twoheadrightarrow M_n(\mathbb{C})$ be the projection that forgets the last row and column. Then $p^{-1}(\overline{X}_\pi)$ is closed, irreducible, and visibly $B_-^{n+1} \times B_+^{n+1}$-invariant. It contains the permutation matrix π' hence contains $\overline{X}_{\pi'}$. Since $\ell(\pi) = \ell(\pi')$ the two subvarieties have the same codimension so must be equal. Finally, $p^*([\overline{X}_\pi]) = [p^{-1}(\overline{X}_\pi)] = [\overline{X}_{\pi'}]$.

(2) We first show that the $\mathcal{S}_\pi(x, 0)$ serve as polynomial representatives of the ordinary Schubert classes. Since the inclusion $\iota : GL_n(\mathbb{C}) \to M_n(\mathbb{C})$ is open, it is transverse to $\overline{X}_\pi$ (and any other subvariety), so $\iota^*([\overline{X}_\pi]) = [\iota^{-1}(\overline{X}_\pi)]$, which maps to $[X_\pi] \in H^*_{B_+}(B_-\backslash GL_n(\mathbb{C}))$, and from there forgets B_+ to the ordinary Schubert class in $H^*(B_-\backslash GL_n(\mathbb{C}))$.

Now use exercise 14 and part (1).

(3) Apply corollary 6 to $\overline{X}_\pi \subseteq M_n(\mathbb{C})$, based on either the left action of T for single Schubert polynomials, or the left and right actions for double Schubert polynomials (involving inverses on the right).

$\square$

One clear advantage of this approach to Schubert polynomials is that it computes them individually, directly, rather than via a recurrence relation. Here is an example.

Exercise 33. Let $\pi \in S_n$ be a **dominant permutation**, meaning that it has all descents $\pi(i) > \pi(i+1)$ followed by all ascents, such as $6 > 4 > 2 > 1 < 3 < 5 < 7 < 8$ but not $1 < 3 > 2$.

(1) Show that there is an English partition λ with $\ell(\pi)$ boxes such that $\overline{X}_\pi$ is defined by the linear equations "$m_{ij} = 0$ for $(i, j) \in \lambda$".
(2) Show that $\mathcal{S}_\pi(x, y) = \prod_{(i,j)\in\lambda}(x_i - y_j)$. In particular $\mathcal{S}_\pi(x)$ is a single monomial.

$\boxed{\text{foreshadow §4.5}}$

4.3. **Push-pull vs. sweeping.** All of the results from §1.6 and §1.7 are based on T-invariant geometrical statements, and so extend without change to T-equivariant cohomology. The only one that needs revisiting is the "self" duality of the Schubert basis; it is no longer true that $[X_\lambda] = [X_\lambda w_0]$. Rather, the dual basis to the classes of the Schubert varieties is the basis of classes of opposite Schubert varieties.

Theorem 19. *Let* $\partial_i^x, \partial_i^y$ *be the divided difference operators in the two sets of variables. Then*

$$\partial_i^x \mathcal{S}_\pi(x, y) = \begin{cases} \mathcal{S}_{r_i \circ \pi}(x, y) & \text{if } r_i \circ \pi < \pi \\ 0 & \text{if } r_i \circ \pi > \pi \end{cases} \qquad \partial_i^y \mathcal{S}_\pi(x, y) = \begin{cases} \mathcal{S}_{\pi \circ r_i}(x, y) & \text{if } \pi \circ r_i < \pi \\ 0 & \text{if } \pi \circ r_i > \pi \end{cases}$$

Proof. Consider the classes (not yet the polynomials) in any one of

$$H^*_{B_+}(B_-\backslash GL_n(\mathbb{C})) \quad \cong \quad H^*_{B_-\times B_+}(GL_n(\mathbb{C})) \quad \cong \quad H^*_{B_-}(GL_n(\mathbb{C})/B_+)$$

The first set of relations holds on push-pull of the equivariant Schubert classes by the same proof as in theorem 5. The second set reduces to the first by using transpose. The compatibility of the divided difference operators on polynomials with the push-pull operations is the same argument as in theorem 8. Then we invoke the stability property (1) of theorem 18... $\square$

I should declare this stable characterization stuff back in single

On a fixed quotient space e.g. $B_-\backslash GL_n(\mathbb{C})$ we can only do the push-pull operations from §1.7 on the left side. How can we geometrically interpret the right side operations, directly on $B_-\backslash GL_n(\mathbb{C})$ without passage to $GL_n(\mathbb{C})/B_+$?

Consider the category $\mathcal{C}$ of right B_+-varieties Y equipped with B_+-equivariant maps $\iota : Y \to P_-\backslash GL_n(\mathbb{C})$. On this category we define functors $\overline{\partial}_i : \mathcal{C} \to \mathcal{C}$ by

$$\overline{\partial}_i(Y) := Y \times^{B_+} P_i, \qquad \begin{array}{rccc} \overline{\partial}_i(\iota) : & Y \times^{B_+} P_i & \to & P_-\backslash GL_n(\mathbb{C}) \\ & [y, p] & \mapsto & \iota(y)p \end{array}$$

Note that $\overline{\partial}_i(Y)$ is a Y-bundle over $B_+\backslash P_i \cong \mathbb{CP}^1$, so is irreducible/smooth/compact iff Y has those properties. We christen this construction the **Bott-Samelson crank**.

The basic example of such a map $\iota : Y \hookrightarrow P_i\backslash GL_n(\mathbb{C})$ is the inclusion of the unique B_--fixed point. If we then apply Bott-Samelson cranks $(\overline{\partial}_i)$ in some order, determined by a word in the simple reflections, the result is a **Bott-Samelson manifold**.

Exercise 34. Show that the image of a Bott-Samelson manifold is a B_+-orbit closure on $P_-\backslash GL_n(\mathbb{C})$.

Theorem 20. *Let* $\iota : Y \to P_-\backslash GL_n(\mathbb{C})$ *define a class* $\iota_*[Y] \in H^*_{B_+}(P_-\backslash GL_n(\mathbb{C}))$. *Then* $\overline{\partial}_i(\iota)_* \left[\overline{\partial}_i(Y)\right] = \partial_i^y(\iota_*[Y])$, *i.e. the class induced by the Bott-Samelson crank is the divided difference operator applied to the original class.*

Proof. On $\mathbb{CP}^1$ we check once and for all should do this back in the H^*_T section that $\alpha_i = [0] - [\infty] = [0] - r_i[0]$. Pulling this back to $\overline{\partial}_i Y$ along the P_i-equivariant map $\overline{\partial}_i Y \twoheadrightarrow \mathbb{CP}^1$, we get the equation $\alpha_i = [Y \times^{B_+} B_+] - r_i[Y \times^{B_+} B_+]$ as classes on $\overline{\partial}_i Y$. Finally we apply $\overline{\partial}_i(\iota)_*$ to get $\alpha_i = \iota_*[Y] - r_i\iota_*[Y]$ as classes on $P_-\backslash GL_n(\mathbb{C})$. right sign? ☐

It is typical in the literature to assert that one can use the divided difference operator technology on full flag varieties to construct all the Schubert classes starting from the point class, but not on partial flag varieties; therefore to construct their Schubert classes one should do the full-flag calculation first, and use the inclusion on cohomology. However, the Bott-Samelson crank lets us do the construction directly on the partial flag variety, *so long as* we are willing to work in equivariant cohomology.

Lemma 11 (The R-matrix for $P_-\backslash G$). *Recall* $\lambda r_i := \lambda$ *with positions* $i, i+1$ *switched, and* $\alpha_i = y_i - y_{i+1}$. *Then*

$$S^\lambda r_i = S^\lambda + \alpha_i\, S^{\lambda r_i}\, [\lambda r_i > \lambda]$$

where, given a boolean statement P, *we define* $[P]$ *to be* 1 *if* P *is true,* 0 *if false.*

the proof below seems overwrought, and fizzles out at the end

Proof. The action of r_i on *non*equivariant cohomology is trivial, because G is connected, so r_i is homotopic to the identity map. Hence $S^\lambda r_i \in S^\lambda + H_T^{>0} \cdot \{\text{Schubert classes}\}$, by theorem 15.

If $\lambda r_i \not> \lambda$ then $X^\lambda r_i = X^\lambda$, so $S^\lambda r_i = S^\lambda$ even equivariantly. Finally, if $\lambda r_i > \lambda$ then $X^\lambda r_i \subseteq X^{\lambda r_i}$. Now we apply the decomposition algorithm described after proposition 10 (upside down) to compute the coefficient on $X^{\lambda r_i}$: it is

$$\frac{(X^\lambda r_i)|_{\lambda r_i}}{X^{\lambda r_i}|_{\lambda r_i}} = \frac{r_i \cdot (X^\lambda|_\lambda)}{X^{\lambda r_i}|_{\lambda r_i}} = \alpha_i$$

□

4.4. **Point restrictions of Schubert classes.** For any topological action $G \circlearrowright M$, the inclusion of the fixed point $M^G \hookrightarrow M$ is trivially a G-equivariant map, hence induces a reverse map $H^*_G(M) \to H^*_G(M^G)$. The latter ring looks pretty silly: why compute H^*_G for a trivial action? Indeed, if G acts on N trivially, then

$$H^*_G(N) := H^*((N \times EG)/G) = H^*(N \times (EG/G)) = H^*(EG/G) \otimes H^*(N) = H^*_G(pt) \otimes H^*(N)$$

so together we get a map $H^*_G(M) \to H^*_G(pt) \otimes H^*(M^G)$ of $H^*_G(pt)$-algebras. This is not so impressive if, e.g., M is a homogeneous G-space and not a point. But something very convenient is true in the case we care about:

Proposition 10. *Let T^n act on $Fl(n_1, \ldots, n_d; \mathbb{C}^n)$ as usual. Then the map*

$$H^*_G(Fl(n_1, \ldots, n_d; \mathbb{C}^n)) \to H^*_T(pt) \otimes H^*(Fl(n_1, \ldots, n_d; \mathbb{C}^n)^T) \cong \bigoplus_{Fl(n_1,\ldots,n_d;\mathbb{C}^n)^T} H^*_T(pt)$$

is an injection of $H^*_T(pt)$-algebras. Put another way, instead of a class β we can work with the list $(\beta|_\lambda)_{\lambda\in Fl(n_1,\ldots,n_d;\mathbb{C}^n)^T}$ of polynomials in $\mathbb{Z}[y_1, \ldots, y_n]$.

These point restrictions satisfy the following triangularity results:

(1) $S_w|_v = 0$ *unless $v \geq w$ in Bruhat order*
(2) $S_w|_w \neq 0$.

Proof. We first prove (1) and (2) and then prove the injectivity from them.

Recall from §**??** that the Schubert classes are in correspondence with the T-fixed points. Consider the matrix $S_w|_v$ where w, v are T-fixed points, and $S_w|_v \in H^*_T(pt)$ is the restriction of the equivariant class S_w to the point $P_-\backslash P_- v \in P_-\backslash GL_n(\mathbb{C}) \cong Fl(n_1, \ldots, n_d; \mathbb{C}^n)$. We pick a linear extension of Bruhat order and order the rows and columns by it.

To prove (1) when $v \ngeq w$, consider the inclusions

$$P_-\backslash P_- v \quad \hookrightarrow \quad P_-\backslash GL_n(\mathbb{C}) \quad \hookleftarrow \quad P_-\backslash \overline{P_- w B_+}$$

inducing

$$H^*_T(P_-\backslash P_- v) \leftarrow H^*_T(P_-\backslash GL_n(\mathbb{C})) \ni [P_-\backslash \overline{P_- w B_+}] = S_w$$

Then, since $P_-\backslash \overline{P_- w B_+}$ intersects $P_-\backslash P_- v$ transversely (indeed, in the empty set, by $v \ngeq w$), we can compute the pullback as the class of that empty set, so 0.

For (2) we want to use corollary 6, but need first to linearize our situation. Consider the T-equivariant algebraic submersions $v^{-1} \cdot \mathfrak{n}_+ \cong v^{-1} \cdot N_+ \cong B_-\backslash B_- N_+ v \hookrightarrow B_-\backslash GL_n(\mathbb{C}^n) \twoheadrightarrow P_-\backslash GL_n(\mathbb{C}^n)$ where the first is given by say $\exp$ (which is algebraic!) or just $X \mapsto \mathrm{Id} + X$. By the assumption $v \geq w$, we know the preimage of X_w under these maps is a nonempty subvariety $X \subseteq v^{-1} \cdot \mathfrak{n}_+$. Defining the μ_i from corollary 6 to be $v^{-1} \cdot \alpha_i$, we learn $[X] \neq 0$. (In fact this argument generalizes to any $v \geq w$.)

Now observe that an upper triangular matrix with nonvanishing diagonal defines an injective map. □

In fact this injectivity is a surprisingly general phenomenon, holding e.g. for all algebraic torus actions on smooth projective varieties [**?**]! The image has been characterized nicely in [**?**] under the assumption (which holds for flag varieties) that not only T-fixed points but T-fixed *curves* are isolated (rediscovering work of [**?**], which held without this assumption). We will never have cause to check that a list of polynomials satisfies the GKM conditions, so we don't recall these conditions. should put into exercises

An algorithm for decomposing classes into Schubert classes. One consequence of the triangularity is an algorithm for decomposing a class as a combination of Schubert classes. Define the **support** of a class $\gamma \in H^*_T(Fl(n_1,\ldots,n_d;\mathbb{C}^n))$ as $supp(\gamma) := \{\lambda \,:\, \gamma|_\lambda \neq 0\}$, and the **upward support** $\overline{supp}(\gamma) := \{\lambda \,:\, \exists\lambda' \leq \lambda, \gamma|_{\lambda'} \neq 0\}$. Then to decompose γ, we look for a minimal element λ of the support, subtract off $(\gamma|_\lambda / S_\lambda|_\lambda) S_\lambda$, and recurse (shrinking the upward support as we go).

Exercise 35. If $c^w_{uv} \neq 0$ show that $w \geq u, v$ in Bruhat order.

We give our first formula (not counting the 0s in (2) of the proof above) for these point restrictions.

Proposition 11. $S_w|_v = \mathcal{S}_w(y_{v(1)},\ldots,y_{v(n)},\ y_1,\ldots,y_n)$ $\forall w, v \in S_n$, *the RHS being a specialization of the double Schubert polynomial.*

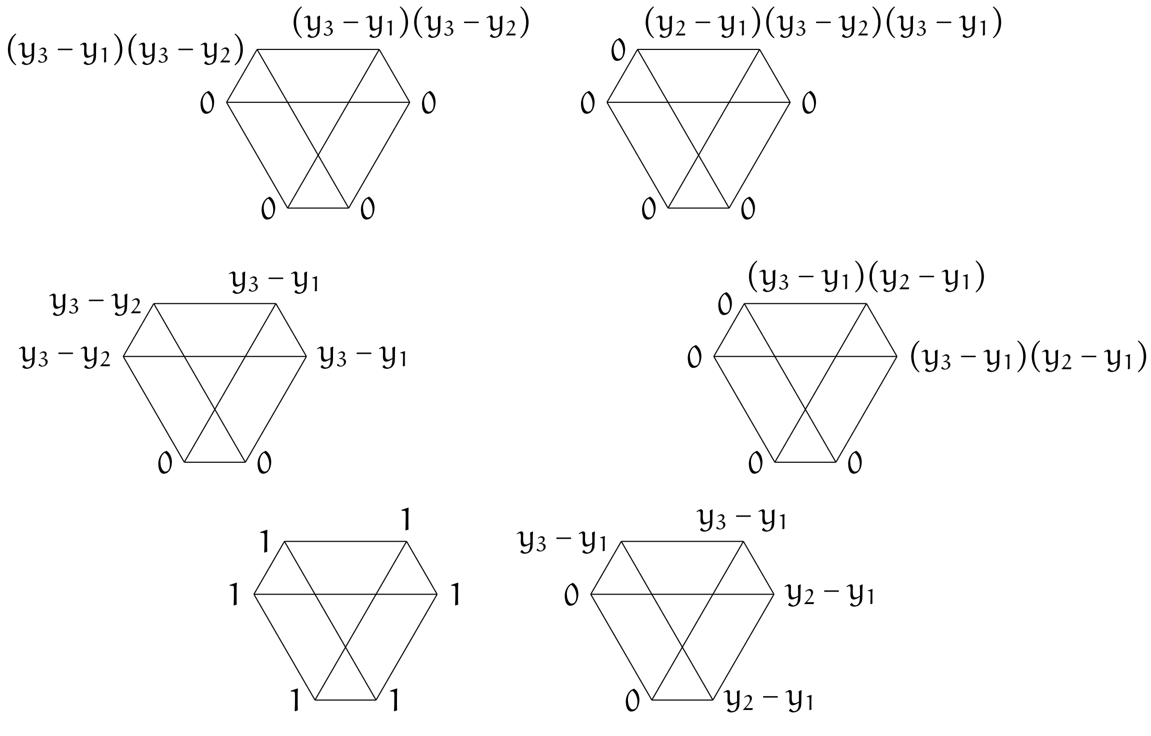
Proof. Consider the following $T \times T$-equivariant maps, and induced maps on cohomology:

$$\begin{array}{cccccccccc}
 & & B_-v & \hookrightarrow & GL_n(\mathbb{C}) & \hookrightarrow & M_n(\mathbb{C}) & \supseteq & \overline{X}_w \\
H^*_T(B_-\backslash B_-v) & \cong & H^*_{B_-\times T}(B_-v) & \leftarrow & H^*_{B_-\times T}(GL_n(\mathbb{C})) & \leftarrow & H^*_{B_-\times T}(M_n(\mathbb{C})) & \ni & [\overline{X}_w] \\
\rotatebox{90}{$\ni$} & & \wr\Vert & & & & \wr\Vert & & \\
S_w|_v & & H^*_{T\times T}(Tv) & & & & H^*_{T\times T}(pt) & & \\
 & & \wr\Vert & & & & \downarrow & & \\
 & & H^*_{T\times T}(S\backslash(T\times T)) & \cong & H^*_{T\times T\times S}(T\times T) & \cong & H^*_S(pt) & &
\end{array}$$

where $S = \{(t, v^{-1}tv) : t \in T\}$ is the $T \times T$-stabilizer on Tv.

The kernel of the downward arrow, induced from the map $(T \times T)^* \twoheadrightarrow S^*$, is then generated by $x_i - y_{v(i)}$, $i = 1 \ldots n$. $\square$

Exercise 36. Verify using the double Schubert polynomials for S_3 that the point restrictions result in the following 3!-tuples of polynomials.



Exercise 37. Show that the divided difference operators can be computed directly "at the fixed points", i.e. on these tuples of point restrictions:

$$(\partial_i \gamma)|_w = \frac{1}{w \cdot \alpha_i} (\gamma|_w - \gamma|_{wr_i}) \qquad (\partial_i^y \gamma)|_w = \frac{1}{\alpha_i} (\gamma|_w - r_i \cdot (\gamma|_{r_i w}))$$

Verify that one can use either of these to compute the pictures in the previous exercise, starting with the one in the Northeast.

Results (1), (2) from proposition 10 are quite hard to see from proposition 11, so we shouldn't be satisfied with with this way for computing $S_w|_v$. Indeed, we should hold out for a formula enjoying a certain positivity:

Proposition 12. (1) *The subgroup $N_v := N \cap v^{-1} N_- v$ is T-equivariantly isomorphic to its Lie algebra $\mathfrak{n}_v$, whose roots are $\{y_i - y_j \ : \ i < j, v(i) > v(j)\}$.* inverse?

(2) *This subgroup acts simply transitively on $X_\circ^v := B_- \backslash B_- v N_-$.*

(3) *$S_w|_v$ is a positive sum of squarefree products of those roots.*

Proof. (1) Use the (algebraic!) exponential map. The root calculation is immediate from the definition.

(2) We admit now that N_v was the group to be discovered in exercise 3.

(3) Consider the T-equivariant inclusions $B_- \backslash B_- v \hookrightarrow B_- \backslash B_- v N_- \xrightarrow{\iota} B_- \backslash GL_n(\mathbb{C})$ and pull back $[X_w]$ in stages. By Kleiman's theorem 11, X_w and the cell $X_\circ^v := B_- \backslash B_- v N_-$ are transverse, so $\iota^*[X_w] = [X_w \cap X_\circ^v]$. Now apply corollary 6.

□

Pleasingly, and in contrast to the general Schubert calculus situation, we *do* have a formula for the point restrictions $S_w|_v$ that realizes the positivity in the proposition above. Indeed, the multiplicities in the sum are all 1!

Theorem 21. [?, ???][?, ???] *Let Q be a word for v. Then*

$$S_w|_v \quad = \sum_{\substack{R \subseteq Q, \prod R = w \\ R \text{ reduced}}} \prod_Q \left(\hat{\alpha}_q^{[q \in R]} r_q \right) \cdot 1$$

where $[q \in R] := 1$ if $q \in R$ and 0 if $q \notin R$, and $\hat{\alpha}$ denotes the multiplication operator.

If we pull all the r_q operators to the right past the multiplication operators, we can rewrite as

$$S_w|_v \quad = \sum_{\substack{R \subseteq Q, \prod R = w \\ R \text{ reduced}}} \prod_Q {\beta_q}^{[q \in R]}, \qquad \beta_q := \prod_{p \text{ left of } q} r_p \cdot \alpha_q$$

and if Q is reduced, the β_q roots are roots of the group N_v defined in proposition 12.

Proof. We write the right action of W (really, of $N(T)$) on $P_- \backslash G$ as a left action, the better to fit with the usual notation of H_T^* as a left W-module.

First we claim that

$$(*) \qquad \left(\prod_Q r_q \right)^{-1} \cdot S^1 = \sum_{R \subseteq Q, \ R \text{ reduced}} \left((\prod_Q \widehat{\alpha_q}^{[q \in R]} r_q) \cdot 1 \right) S^{\prod R}$$

which we prove by induction on $\#Q$. Let $Q = Q't$, reduced. Then

$$\begin{aligned}
\left(\prod_Q r_q\right)^{-1}\cdot S^1 &= r_t\left(\prod_{Q'} r_q\right)^{-1}\cdot S^1 = r_t \sum_{R\subseteq Q',\ R \text{ reduced}} \left(\Big(\prod_Q \widehat{\alpha_q}^{[q\in R]} r_q\Big)\cdot 1\right) S^{\prod R} \\
&= \sum_{R\subseteq Q',\ R \text{ reduced}} \left(r_t\Big(\prod_Q \widehat{\alpha_q}^{[q\in R]} r_q\Big)\cdot 1\right)\left(r_t\cdot S^{\prod R}\right) \\
&= \sum_{R'\subseteq Q',\ R' \text{ reduced}} \left(r_t\Big(\prod_Q \widehat{\alpha_q}^{[q\in R']} r_q\Big)\cdot 1\right)\left(S^{\prod R} + \alpha_t\, S^{(\prod R')r_t}\left[\Big(\prod R'\Big)r_t > \prod R'\right]\right)
\end{aligned}$$

the last step by lemma 11. We can now attach the r_t to R' (if the result is reduced), or not, to make each $R \subseteq Q$ term in $(*)$.

Now we can compute

$$\begin{aligned}
S_w|_v &= \int_{G/B} S_w[P_-\backslash P_- v] = \int_{G/B} S_w(v^{-1}\cdot S^1) = \int_{G/B} S_w\left(\Big(\prod_Q r_q\Big)^{-1}\cdot S^1\right) \\
&= \int_{G/B} S_w \sum_{R\subseteq Q,\ R \text{ reduced}} \left(\Big(\prod_Q \widehat{\alpha_q}^{[q\in R]} r_q\Big)\cdot 1\right) S^{\prod R} = \sum_{\substack{R\subseteq Q,\ R \text{ reduced}\\ \prod R = w}} \Big(\prod_Q \widehat{\alpha_q}^{[q\in R]} r_q\Big)\cdot 1
\end{aligned}$$

□

The multiplicities being 1, and the Gröbner-degeneration derivation of corollary 6, suggest that $X_w \cap X^v_\circ$ might have a degeneration to a *reduced* union of coördinate spaces. Indeed this is the case [**?**].

Exercise 38.
(1) Let $v = 321 \in S_3$, with two reduced words $r_1r_2r_1$ and $r_2r_1r_2$. Compute $S_w|_v$ for each $w \in S_3$ and both reduced words.
(2) Let Pr_ir_jQ, Pr_jr_iQ be two reduced words for the same permutation v (i.e. $|i-j| > 1$). Show that the two resulting AJS/Billey formulæ for $S_w|_v$ agree term-by-term.
(3) Call a permutation v 321-**avoiding** if $\nexists\ i < j < k$, $v(i) > v(j) > v(k)$. Show that v is 321-avoiding iff any two reduced words for v differ by commuting, not braid, moves. In particular, for such v the formula $S_w|_v$ is essentially canonical.

Exercise 39. The **multiplicity** of a point p on a scheme Y, locally $\mathrm{Spec}\,(R/I_p) \hookrightarrow \mathrm{Spec}\,R$, is defined to be the degree of the normal cone $\oplus_d I_p^d/I_p^{d+1}$. It is additive over top-dimensional components and 1 for smooth points.

Characterize 321-avoiding permutations v by another condition: there exists a one-parameter subgroup $S \leq T$ with the property that S acts on $T_vX^v_\circ$ by scaling. Then show that the degree of such a $v = \prod Q$ on X_w is the *number* of reduced subwords of Q with product w. references to Ikeda, Graham

We give two applications of the AJS/Billey formula.

Proposition 13. *The equivariant Schubert structure constant c^v_{wv} can be computed as $S_w|_v$.*

Proof. We use the triangularity from proposition 10 to give an algorithm to expand any class β in Schubert classes $\sum_u d_u S_u$. Start with $\beta' := \beta$, then run through a linear extension of Bruhat order (small to big), and at stage $u \in W$ let $d_u = \frac{\beta|_u}{S_u|_u}$ and subtract d_uS_u from our running β'.

In the case at hand $\beta = S_wS_v$, so $\beta|_u = 0$ for $u \ngeq v$ (also for $u \ngeq w$). Hence β' remains β by the time we get to stage $u = v$. Then $d_v = \frac{\beta|_v}{S_v|_v} = \frac{(S_wS_v)|_v}{S_v|_v} = \frac{S_w|_v S_v|_v}{S_v|_v} = S_w|_v$. □

Theorem 22. (1) [?] *Let* $w_0w_0^P \in S_{2n}$ *be the permutation taking* $i \mapsto i + n \bmod 2n$, *and for* $v \in S_n$ *let* $v \oplus \mathrm{Id}_n \in S_{2n}$ *denote the evident permutation fixing* $n+1, \ldots, 2n$. *Then there is a* T^{2n}*-equivariant isomorphism* $M_n(\mathbb{C}) \cong X_\circ^{w_0w_0^P}$ *taking each matrix Schubert variety* $\overline{X}_v$ *to* $X_{v\oplus \mathrm{Id}_n} \cap X_\circ^{w_0w_0^P}$.

(2) $\mathcal{S}_v(x_1, \ldots, x_n, y_1, \ldots, y_n) = S_{v\oplus\mathrm{Id}_n}|_{w_0w_0^P}$ *where the left is the double Schubert polynomial.*

Proof. (1) The isomorphism is ...

(2) As in the proof of proposition 12 (3), the class $[X_{v\oplus\mathrm{Id}_n} \cap X_\circ^{w_0w_0^P}] \in H^*_{T^{2n}}(X_\circ^{w_0w_0^P})$ is $S_{v\oplus\mathrm{Id}_n}|_{w_0w_0^P}$. Then use the T-equivariant isomorphism above of contractible spaces, and our definition of double Schubert polynomial.

□

The resulting "pipe dream" formula for (double) Schubert polynomials (which doesn't really depend on the choice of reduced word for $w_0w_0^P$, in light of exercise 38) is essentially due to [?, ?], but see also [?, ?].

4.5. **The AJS/Billey formula, in terms of $\check{R}$-matrices.** It took about ten years of talking with a quantum integrable systems person (Paul Zinn-Justin) for me to understand a basic principle. When you see a formula given by a sum of products (e.g. the AJS/Billey theorem 21), you should

(1) recast it as a matrix entry in a product of matrices, and
(2) investigate the commutation relations of those matrices.

More specifically, it would be nice for the vector space to be a big tensor product, on which each of the individual matrices $\check{R}$ only acts on a few, say only two, tensor factors. So the rows and columns are *each* indexed by a pair of indices, i.e. the matrix entries are $\check{R}_{(i,j),(k,l)}$.

The same simple reflection r_i may appear multiple times in Q, with different associated factors $y_j - y_k$; this is a hint that our matrices should depend on (the difference between) two parameters. Define

$$\check{R}(a,b)_{(i,j),(k,l)} \quad := \quad [(i,j) = (k,l)] \; + \; [(i,j) = (l,k)]\,[i > j]\,(a-b)$$

where $[p] = 1$ respectively 0 if a property p is true respectively false. That is to say, $\check{R}$ is an identity matrix plus some off-diagonal entries $a - b$.

Theorem 23. *Let* λ, μ *be strings in* $0, \ldots, d$ *of the same content, and* Q *a reduced word for the minimal tie-breaking lift of* μ *to a permutation. Let* V_{y_i} *denote the vector space*

explicitly do pipe dreams as the square special case

...

Proof of lemma 6 from §1.10. By theorem 22 (2), we have

$$\mathcal{S}_{1\ 2\ldots(k-1)\ n\ k\ (k+1)\ldots(n-1)}\cdots$$

□

5. Equivariant separated-descent puzzles. Proof via YBE.

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