MATH 2240 FINAL, SPRING 2019

Name, **written slowly and legibly:** ______________________________

In each answer, write as much (on front and back) as it takes to convey your thought process; full English sentences are much easier to give credit to than bare, unmotivated scribbled formulæ. (They won't do any good if they can't be read, so *do* put effort into making them legible.)

Feel free to ask me questions during the test, especially if you need a little reminder about a definition. Worst case is I don't answer. (It's very sad to afterward hear "I didn't realize I could ask you that" — find out!)

In general the letters in part 5b refer to those introduced in 5, 5a, etc.

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1. Consider the 2-form

$$\alpha = \frac{x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy}{(x^2+y^2+z^2)^{3/2}}$$

on $\mathbb{R}^3 \setminus \vec{0}$.

1a [10]. Compute $d\alpha$.

Answer. Let's start with d of the first term:

$$d\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\, dy \wedge dz\right) = \frac{\partial}{\partial x}\frac{x}{(x^2+y^2+z^2)^{3/2}}\, dx \wedge dy \wedge dz$$

since the $\frac{\partial}{\partial y}$ term has a dy which dies when $\wedge d$ with the $dy \wedge dz$, likewise the $\frac{\partial}{\partial z}$ term. Continuing,

$$\begin{aligned} &= \frac{(x^2+y^2+z^2)^{3/2} - x(3/2)(2x)(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}\, dx \wedge dy \wedge dz \\ &= (x^2+y^2+z^2)^{1/2}\frac{(x^2+y^2+z^2) - 3x^2}{(x^2+y^2+z^2)^3}\, dx \wedge dy \wedge dz \end{aligned}$$

Now add the three cyclic rotations of this term, rotating $x \to y \to z \to x$. The only parts that change are the $-3x^2$ to $-3y^2$ and $-3z^2$, and the $dx \wedge dy \wedge dz$ to

$$dy \wedge dz \wedge dx = dz \wedge dx \wedge dy = dx \wedge dy \wedge dz$$

i.e. that factor *doesn't* change! Adding these three up, we get

$$(x^2+y^2+z^2)^{1/2}\frac{3(x^2+y^2+z^2) - 3x^2 - 3y^2 - 3z^2}{(x^2+y^2+z^2)^3}\, dx \wedge dy \wedge dz = 0$$

(This α is actually the 3-d analogue of the $d\theta$ we thought about before.)

1b [10]

Let $\Delta = \{(x, y, z) \ : \ x, y, z \geq 0, x + y + z = 1\}$ be the standard triangle in the first octant. Set up the computation of $\int_\Delta \alpha$, by pulling back along some parametrization Ψ of Δ.

Let's take $\Psi(x,y) := (x, y, 1-x-y)$, $x, y \geq 0$, $x+y \leq 1$, i.e. $z = 1-x-y$, $dz = -dx - dy$. Then

$$\begin{aligned}
\Psi^*\alpha(x,y) &= \frac{x\ dy \wedge (-dx - dy) + y\ (-dx - dy) \wedge dx + (1-x-y)\ dx \wedge dy}{(x^2 + y^2 + (1-x-y)^2)^{3/2}} \\
&= \frac{x + y + (1-x-y)}{(x^2 + y^2 + (1-x-y)^2)^{3/2}}\ dx \wedge dy = \frac{1}{(x^2 + y^2 + (1-x-y)^2)^{3/2}}\ dx \wedge dy \\
&= \frac{1}{(x^2 + y^2 + (1-x-y)^2)^{3/2}}\ dx \wedge dy
\end{aligned}$$

so the integral is

$$\int_{y=0}^{1} \int_{x=0}^{1-y} \frac{1}{(x^2 + y^2 + (1-x-y)^2)^{3/2}}\ dx \wedge dy = \int_{y=0}^{1} \int_{x=0}^{1-y} \frac{1}{(2x^2 + 2xy + 2y^2 - 2x - 2y + 1)^{3/2}}\ dx \wedge dy$$

1c [5]. Actually do the integral you just set up.

Answer. Nobody did!

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2 [5]. Let $\alpha \in A^2(\mathbb{R}^8)$. For β a form, let $F(\beta) = d\alpha \wedge \beta$.

Do the operations F and d commute, i.e. you can go either way around the square

$$\begin{array}{ccc}
A^k(\mathbb{R}^8) & \stackrel{d}{\to} & A^{k+1}(\mathbb{R}^8) \\
\downarrow F & & \downarrow F \\
A^{k+3}(\mathbb{R}^8) & \stackrel{d}{\to} & A^{k+4}(\mathbb{R}^8)
\end{array}$$

and you get the same answer?

Answer.

$$d(F(\beta)) = d(d\alpha \wedge \beta) = (dd\alpha) \wedge \beta + (-1)^3 d\alpha \wedge d\beta = -d\alpha \wedge d\beta$$

the second by Leibniz' rule, since $d\alpha$ is a 3-form, the third since $d^2 = 0$. Whereas $F(d\beta) = d\alpha \wedge d\beta$. So no, these operations usually anticommute (unless $d\alpha \wedge d\beta = 0$).

Many people put a $(-1)^2$ since α is a 2-form. But we weren't trying to pull d past α, rather, past $d\alpha$.

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3 [10]. Let $\tau : \left\{(r,\theta)\ :\ r \in [0,1], \theta \in [0, \frac{3}{4}\pi]\right\} \to \mathbb{R}^2$ take $(r,\theta) \mapsto (r\cos\theta, r\sin\theta)$, and let T be its image, 3/8 of the unit disc. (Draw a picture!)

For $f : \mathbb{R}^2 \to \mathbb{R}$ a function, write down $\int_T f\ dx \wedge dy$ as an iterated integral (i.e. with explicit bounds of integration) three different ways: over x then y, over y then x, over r then θ. (Don't evaluate it, obviously; you don't know what f is.)

Answer.

$$\int_{y=0}^{1} \int_{x=\max\left(-\sqrt{1-y^2}, -y\right)}^{\sqrt{1-y^2}} f\ dx\ dy \qquad \int_{x=-1/\sqrt{2}}^{1} \int_{y=\max(-x,0)}^{\sqrt{1-x^2}} f\ dy\ dx \qquad \int_{\theta=0}^{\frac{3}{4}\pi} \int_{r=0}^{1} f\ r\ dr\ d\theta$$

Writing them as a sum of two integrals was fine too. Lots of people did it in either way.

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4. Let $f : [0,1] \to \mathbb{R}$ be a weakly increasing function, i.e. $x \leq y \implies f(x) \leq f(y)$. It's not assumed continuous though. Extend it to $\mathbb{R}$ by $f(x) = 0$ for $x \notin [0,1]$.

4a [5]. State what you're required to check in order to show that this f is Riemann integrable (i.e. state the definition) – the easy parts and the hard parts.

Answer. Bounded, bounded support, upper limits and lower limits U_N, L_N converge to the same value.

We learned other conditions (that weren't the *definition*, but were acceptable anyway) in class: one about oscillation, the other about being continuous a.e.

4b [10]. Show that f is indeed Riemann integrable.

Answer. On $[0,1]$, f is bounded by $f(0) \leq f(x) \leq f(1)$. So $|f| \leq |f(0)| + |f(1)|$ everywhere. Its support is contained in $[0,1]$. (This wouldn't have worked if we only knew f was defined on $(0,1)$.)

Now let's compute the Nth upper and lower sums, i.e. in the Nth dyadic paving, with f taking on values $f(i/2^N), i = 0, \ldots, 2^N$. Then

$$U_N = \frac{1}{2^N} \sum_{i=1}^{2^N} f(i) \qquad L_N = \frac{1}{2^N} \sum_{i=0}^{2^N-1} f(i)$$

so $U_N - L_N = \frac{1}{2^N}(f(1) - f(0)) \to 0$ as $N \to \infty$.

4c [5]. Show that f is continuous almost everywhere.

Answer. Since it's Riemann integrable, it's continuous almost everywhere. Ta-da.

A harder question (that I didn't ask): show that the set of discontinuities is at most countable. Well, if f is discontinuous at x, the usual statement is that $\lim_{a \to x} f(a)$ doesn't exist. But $\lim_{a \to x, a < x} f(a) = \sup_{a<x} f(a)$ and $\lim_{a \to x, a > x} f(a) = \inf_{a>x} f(a)$ each exist by f increasing, so the only issue is that $\sup_{a<x} f(a) < \inf_{a>x} f(a)$ at x.

Define $S_k := \{x \in [0,1] \ : \ \sup_{a<x} f(a) + 1/k < \inf_{a>x} f(a)$, i.e. there's a jump up of $> 1/k$ at x. I claim that $\#S_k < k(f(1) - f(0))$. Otherwise the jumps around the points in S_k are already enough to get you from $f(0)$ to higher than $f(1)$.

For x to be a point of discontinuity, $\sup_{a<x} f(a) < \inf_{a>x} f(a)$, so the difference is more than some $1/k$; hence the points of discontinuity are $\bigcup_k S_k$, a countable union of finite sets hence countable.

4d [10]. Give an example of an increasing f that's discontinuous at every point of $(0,1) \cap \mathbb{Q}$.

Answer. Enumerate those rationals as $q_1, q_2, \ldots$. Let

$$f = \sum_{i=1}^{\infty} 1_{[q_i,1)}/2^i$$

This sum converges at any x since the total is at most 1. Also, f has a jump of 2^{-i} at q_i, so is discontinuous there.

There were three ways that people tended to answer this wrong. One was to define a function like $f(x) = x 1_{(0,1)\cap\mathbb{Q}}$, which isn't increasing (and is discontinuous *everywhere* on $[0,1]$, but that's beside the point).

Another was to use the phrase "the greatest rational number q less than x". There is *never* a greatest rational less than x. There are rationals, very close, none of which is the largest.

The third was to give a circular definition of f, something like

$$f(x) = \begin{cases} x & \text{if } \ldots \\ f(x-q) + 1/2^n & \text{if } \ldots \end{cases}$$

as if it were an inductive definition. Writing something like this doesn't let you actually compute f somewhere.

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5. Let M be a square matrix with characteristic polynomial $\det(t\mathbf{1} - M) = \prod_{i=1}^n (t - \lambda_i)$.

5a [5]. What is the characteristic polynomial of $-M$? Don't just write down the definition – your answer should be in terms of the (λ_i).

Answer. Let's upper-triangularize M, which doesn't change its characteristic polynomial. Now the (λ_i) are down the diagonal, and it's easy to calculate $\det(t\mathbf{1} + M) = \prod_{i=1}^n (t + \lambda_i)$.

5b [5]. What is the characteristic polynomial of M^2? Don't just write down the definition – your answer should be in terms of the (λ_i).

Answer. Same trick lets us calculate $\det(t\mathbf{1} - M^2) = \prod_{i=1}^n (t - \lambda_i^2)$.

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6. Assume A, B are two square matrices, and $\vec{v}_1, \ldots, \vec{v}_n$ a basis consisting of eigenvectors of *both* matrices.

6a [10]. Show that $AB = BA$.

Answer. Let $\vec{w}$ be any vector. Expand in the basis, so $\vec{w} = \sum_{i=1}^n c_i \vec{v}_i$. Let the eigenvalues be $(\alpha_i), (\beta_i)$ for A, B respectively. (*Not* the same for the two matrices!) Then

$$AB\vec{w} = AB \sum_{i=1}^n c_i \vec{v}_i = \sum_{i=1}^n c_i AB\vec{v}_i = \sum_{i=1}^n c_i A\beta_i \vec{v}_i = \sum_{i=1}^n c_i \beta_i A\vec{v}_i = \sum_{i=1}^n c_i \beta_i \alpha_i \vec{v}_i$$

Meanwhile

$$BA\vec{w} = BA \sum_{i=1}^n c_i \vec{v}_i = \sum_{i=1}^n c_i BA\vec{v}_i = \sum_{i=1}^n c_i B\alpha_i \vec{v}_i = \sum_{i=1}^n c_i \alpha_i B\vec{v}_i = \sum_{i=1}^n c_i \alpha_i \beta_i \vec{v}_i$$

so these are equal.

Some people instead said "let P have columns $(\vec{v}_i)$, so $P^{-1}AP$, $P^{-1}BP$, are diagonal hence commute. Now

$$AB = P(P^{-1}AP)(P^{-1}BP)P^{-1} = P(P^{-1}BP)(P^{-1}AP)P^{-1} = BA$$

6b [5]. Give an example of n, A, B such that $AB = BA$ but there doesn't exist a basis consisting of eigenvectors of both matrices.

Answer. Let $n = 2$, $A = B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $AB = A^2 = BA$, but neither matrix has two linearly independent eigenvectors.

Lots of people were unwilling to let 0 be an eigenvalue, and said things like, "if $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ then it only has one eigenvector, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. No, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is one too.

If A is diagonal – as in many answers – then the standard basis is a basis of eigenvectors. If B is diagonal too, then yep it's a basis of eigenvectors for both of them.

In fact, if A, B are both diagonal*izable* and commute, then (theorem) there exists a basis of eigenvectors. This question was only solvable by choosing A or B to not be diagonalizable.

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7 [5]. Let X be the space of 2×2 orthogonal matrices, $\{M : MM^T = \mathbf{1}\}$. Recall that this is a manifold! How many orientations are there on this manifold?

Answer. On a homework we figured out that this space had two components (one with $\det M = 1$, the other with $\det M = -1$), each of which is a circle, so has two orientations. In all there are $2^2 = 4$ orientations.

Many people figured out that $\det M = \pm 1$, so decided that specifying a point on X took two numbers, θ and a sign. True. That doesn't mean it's 2-dimensional – the dimension is the number of *continuous* parameters. X is 1-dimensional. (Which isn't relevant to answering this question – what was important was that X had two components, each orientable.)