MATH 2240 MIDTERM, SPRING 2019

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1. Let M be the $n \times n$ matrix of all 1s.

a [10 pts]. What are its eigenvalues?

Answer. There are two ways to find them: the roots of the characteristic polynomial (which is quickly seen to be very difficult to compute), and directly from the eigenvector/eigenvalue equation:

$$\lambda \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = M \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n a_i \\ \sum_{i=1}^n a_i \\ \vdots \\ \sum_{i=1}^n a_i \end{bmatrix} = \left(\sum_{i=1}^n a_i \right) \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

In scalar equations, we learn $\lambda a_1 = \ldots = \lambda a_n = \sum_{i=1}^n a_i$.

Now there are two cases: $\lambda = 0$ and therefore $\sum_{i=1}^n a_i = 0$, or $\lambda \neq 0$ in which case $a_1 = \ldots = a_n$, hence $\lambda a_1 = n \sum_{i=1}^n a_i = n a_1$. If $a_1 = 0$ then all $a_i = 0$ which would mean that our eigenvector is $\vec{0}$, which isn't allowed, so we divide by a_1 and learn $\lambda = n$.

In all: there are at most two eigenvalues, 0 and n. For the 0 to actually occur we need $\sum_{i=1}^n a_i = 0$, which for $n = 1$ means our eigenvector is $\vec{0}$, which isn't allowed; the eigenvalue 0 only occurs for $n > 1$.

A couple of dumb checks: since the rows of M are linearly dependent (for $n > 1$) the determinant is 0, which is the product of the eigenvalues, so one of them is 0. The trace of M is n, the sum of the eigenvalues (with multiplicity).

1b (continuing 5a) [15 pts]. Find a list of n eigenvectors (for various eigenvalues) forming a basis.

Answer. We found an eigenvector with eigenvalue n already: the all-ones vector. For 0 we just need $\sum_{i=1}^n a_i = 0$, and people's favorite choice was the $n-1$ vectors

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

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2. Let f be continuous and g integrable.

a [15 pts]. If $f \circ g$ has bounded support, show that it is integrable.

Most of the errors people made in this question came from mixing up bounded vs. bounded support.

Answer. Since g is integrable, it is continuous away from a measure zero set Δ. For $x \notin \Delta$, g is continuous at x and f is continuous at $g(x)$, so $f \circ g$ is continuous at x. Hence $f \circ g$ is continuous away from Δ. (It can happen that $f \circ g$ is continous at some $x \in \Delta$, e.g. if $f = 0$ everywhere, but to apply the criterion it's not important that we find the exact set of discontinuities.)

We also need to show that $f \circ g$ is bounded, to be able to apply our integrability criterion. Since g is bounded, its values lie in some interval $[-B, B]$. Since f is continuous, it takes on a maximum and minimum on that compact interval. Hence $f \circ g$ is bounded.

In all $f \circ g$ is bounded, has bounded support by assumption, and is continuous away from a set of measure zero, so is integrable.

2b (continued from 2, not 2a) [10 pts]. Give an example of f, g where $f \circ g$ doesn't have bounded support.

"Continued from 2" means that as assumed in 2, f is continuous and g is integrable. It was specified also on the board that f should be bounded with bounded support.

Answer. Let f be a continous function with bounded support and $f(0) \neq 0$. Some examples:

$$f(x) = \begin{cases} 1 - |x| & x \in [-1, 1] \\ 0 & \text{otherwise} \end{cases} \qquad f(x) = \begin{cases} \sqrt{1 - x^2} & x \in [-1, 1] \\ 0 & \text{otherwise} \end{cases} \qquad f(x) = \begin{cases} \cos x & x \in [-\pi/2, \pi/2] \\ 0 & \text{otherwise} \end{cases}$$

Whatever. *But it's required to actually write down a specific* f.

Then g could be any function with bounded support, i.e. there exists a B such that $g(x) = 0$ for $|x| > B$. Now $(f \circ g)(x) = f(0) \neq 0$ for $|x| > B$, i.e. $f \circ g$ doesn't have bounded support.

Again, *it's required to actually write down a specific* g. My favorite is $g(x) = 0 \ \forall x$.

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3. Let $M_{2,2}(\mathbb{C})$ denote the space of 2×2 matrices.

Let $V := \{f : M_{2,2}(\mathbb{C}) \to \mathbb{C} \mid f \text{ is bilinear in the rows}\}$. (Hint: V is a 4-dimensional vector space.)

a [15 pts]. Write down "a general element" (meaning, write down *every* element) f of V.

Answer. To write down an element of V is to write down an f. Of course the most famous such f is the one we've been studying recently, $f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$.

What do we know about more general f?

$$\begin{aligned} f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) &= f\left(\begin{bmatrix} a & 0 \\ c & d \end{bmatrix}\right) + f\left(\begin{bmatrix} 0 & b \\ c & d \end{bmatrix}\right) \\ &= f\left(\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}\right) + f\left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right) + f\left(\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}\right) + f\left(\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}\right) \\ &= ac\, f\left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right) + ad\, f\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) + bc\, f\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) + bd\, f\left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right) \end{aligned}$$

by linearity in the first, then in the second, row.

Now we're stuck; without further assumptions we don't know anything about those four numbers. But if we just put anything there, say W, X, Y, Z, we get a function

$$f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = Wac + Xad + Ybc + Zbd$$

that is bilinear in the two rows.

3b [10 pts]. Use this to show that V a 4-dimensional vector space.

Answer. The easiest thing is to list a 4-element basis, i.e. a list of four f such that every f is uniquely a linear combination of those.

From the analysis in 3a, the obvious list is ac, ad, bc, bd. A couple of people hit upon $ad \pm bc$, $ac \pm bd$, which seems pretty odd to me but is perfectly fine.

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4 [25 pts]. Let $A = \{(x, x^2) : x \in [0, 1]\}$. Show that this set is pavable i.e. $vol_2(A)$ is defined, **directly from** the "U_N, L_N tend to the same limit as $N \to \infty$" definition of integration.

Many people decided (either explicitly, or implicitly in their calculation) to integrate the function $f(x) = x^2$ on $\mathbb{R}^1$, but that's not relevant to the problem. Rather, the set $A \subseteq \mathbb{R}^2$ is pavable if the function 1_A is integrable. The 2 in the $vol_2(A)$ should serve as a reminder what kind of integral to be attempting here.

Answer. Since 1_A is nonnegative, obviously $L_N \geq 0$. Actually $L_N = 0\ \forall N$, but we don't need to prove that directly; once we show $U_N \to 0$ that'll force U_N, L_N to go to the same limit since $U_N \geq L_N \geq 0$.

$$U_N = \sum_{C \in D_N(\mathbb{R}^2)} sup_C(1_A) vol_2(C) = \frac{1}{2^{2N}} \sum_{C \in D_N(\mathbb{R}^2)} sup_C(1_A) = \frac{\#\{C \in D_N(\mathbb{R}^2) : C \cap A \neq \emptyset\}}{2^{2N}}$$

Now we need to count how many dyadic squares C meet this curve A. Most people summed over the 2^N horizontal intervals $[k/2^N, (k+1)/2^N)$, and asked for each one, how many squares in that column meet A. Since the slope of $y = x^2$ on $[0, 1]$ is between 0 and 2, from $k/2^N$ to $(k+1)/2^N$ the function $y = x^2$ increases by at most $2\frac{1}{2^N}$, so goes through at most **three** squares. (Not two.) Therefore $U_N \leq \frac{3 \cdot 2^N}{2^{2N}} = 3/2^N \to 0$.

A couple of people had an ingenious alternate argument for $U_N \to 0$. When we increment N by 1, each of the D_N-squares that meets A gets cut into four D_{N+1}-squares, like a windowpane. But of those 4 *at most three* can meet A, since A is the graph of an increasing function. Hence $U_N \leq 3^N/2^{2N} \to 0$.

(This second bound upper bound is much looser than the previous, but who cares? It still goes to 0 as $N \to \infty$.)
