MATH 4340 FINAL, SPRING 2020

DUE 11:59PM FRIDAY MAY 22, EASTERN TIME

Once you look at p2 of this exam, you may only make reference to your class notes and to Dummit & Foote, no other sources (books, web pages, people, etc.)

I'd *prefer* if your final were composed in LaTeX, and promise you that (1) if you're not yet good at LaTeX that as a math person you will need to get there, and (2) the time spent on learning that for this final will be minuscule on top of actually doing the math. I have put the source for this final exam on the Canvas page by way of random example, so if you've never TeXed before just copy that LaTeX file, skip past the preamble, change the `\title`, and edit in between the `\maketitle` and `\end{document}`. If you need to find commands for symbols, Google `detexify`. All that said, if you still aren't willing to try LaTeX, I will accept alternate formats, e.g. scans of handwritten answers.

In each answer, write as much as it takes to convey your thought process; full English sentences are much easier to give credit to than bare, unmotivated scribbled formulæ. (They won't do any good if they can't be read, so *do* put effort into making them legible.)

If two questions have the same number, they concern the same object. For example, the relation in 4b is the one defined in 4. [5π] indicates that a problem is worth 5π points.

Feel free to ask me questions (by email) during the test, especially if you need a little reminder about a definition. Worst case is I don't answer. (It's very sad to afterward hear "I didn't realize I could ask you that" — find out!) Likewise, check back on the Canvas page to see if there are updates to the questions.

Turn over only when you're ready to foresake all other sources except for your class notes and Dummit & Foote...

1. Let $G \leq GL(4, \mathbb{C})$ be the set of upper triangular matrices

$$\begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & k \end{bmatrix}$$

1a [2]. What are the conditions on $a, \ldots, k$ such that this matrix is invertible?

1b [3]. Let $T \leq G$ be the subgroup of diagonal matrices.
Find another subgroup $F \trianglelefteq G$ such that G is a semidirect product $T \ltimes F$.

1c [5]. Find at least three other ways to see G as a semidirect product.

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2 [5]. Let G be a potentially infinite group, but $H \leq G, H \ntrianglelefteq G$ an abnormal subgroup with G/H finite. Show that H contains a proper[1] normal subgroup $K \triangleleft H$ of finite index.

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3 [10]. How many automorphisms does the group $\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$ have?

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4. For $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k) \in \mathbb{N}^k$, and p prime, let G_λ denote the finite abelian group $\prod_{i=1}^k \mathbb{Z}_{(p^{\lambda_i})}$, e.g. $G_{3,1} = \mathbb{Z}_{p^3} \times \mathbb{Z}_p$.

Consider "short exact sequences"

$$1 \to A \to B \to C \to 1$$

of groups, meaning, A is (corresponded with) a normal subgroup of B and C with the quotient group B/A.

4a [3]. If $1 \to G_\lambda \to G_\mu \to G_\nu \to 1$ is a short exact sequence, show that $|\mu| = |\lambda| + |\nu|$.

4a [7]. If $1 \to G_\lambda \to G_\mu \to G_\nu \to 1$ is a short exact sequence, show that $\mu_1 \leq \lambda_1 + \nu_1$.

Fun fact (you may not use!): there exact such sequences iff there exist Hermitian matrices H_λ, H_μ, H_ν with those natural numbers as eigenvalues, and $H_\mu = H_\lambda + H_\nu$.

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5. Let $GL_n(\mathbb{F}_p)$ be the group of $n \times n$ invertible matrices over $\mathbb{F}_p$. If you find this problem hard, try $n = 2$.

5a [3]. Looking one row at a time, show that

$$\#GL_n(\mathbb{F}_p) = (p^n - p^0)(p^n - p^1)(p^n - p^2) \cdots (p^n - p^{n-1})$$

5b [2]. How big are its Sylow p-subgroups?

5c [5]. Find a Sylow p-subgroup of $GL_n(\mathbb{F}_p)$.

5d [5]. How many p-Sylow subgroups are there?

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6 [10]. Let G be the group of 3×3 complex matrices, where each row and column has two zeros and a ± 1, e.g. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$. Describe the conjugacy classes in this group – how many are there, of what sizes. (You may use a computer for this question, if that helps.)

[1] meaning, $K \neq H$

7. Let $\mathbb{F}_4 = \{0, 1, a, a+1\}$ where $a^2 = a + 1$.

7a [10]. Find an irreducible quadratic polynomial $p(x) \in \mathbb{F}_4[x]$. (Prove it's irreducible.)

7b [5]. Find a generator of the multiplicative group of $\mathbb{F}_4[x]/\langle p(x)\rangle$.

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8 [10]. Let $p(x) = (x^2-2)^2 - 3 \in \mathbb{Q}[x]$. Find its splitting field E, its Galois group $Gal(E, \mathbb{Q})$, and all intermediate subfields.

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9 [5]. Let G be finite, and V a finite-dimensional complex representation over $\mathbb{C}$. Show that $V^{\otimes n} := V \otimes V \otimes \cdots \otimes V$ is irreducible for all $n > 0$ iff V is 1-dimensional. (Recall our definition of $V \otimes W$ for column vectors, considered as $k \times 1$ matrices, using the Kronecker product of matrices.)

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10. A mysterious group G has the following partial character table (one column for each conjugacy class, one row for each character of irrep, possibly in nontraditional order):

	I	II	III	IV
χ_1	1	???	1	1
χ_2	1	1	???	???
χ_3	???	1	ω	???
χ_4	-1	3	???	0

10a [2]. Which column is that of the identity element (and why)?

10b [2]. What is the order of the group (and why)?

10c [4]. How big is conjugacy class I (and why)?

10d [2]. How big are the other conjugacy classes (and why)?

If along the way you figure out some entries in the table, say so.