What do puzzles *really* compute?

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Abstract

Among other things (these all since 2017),

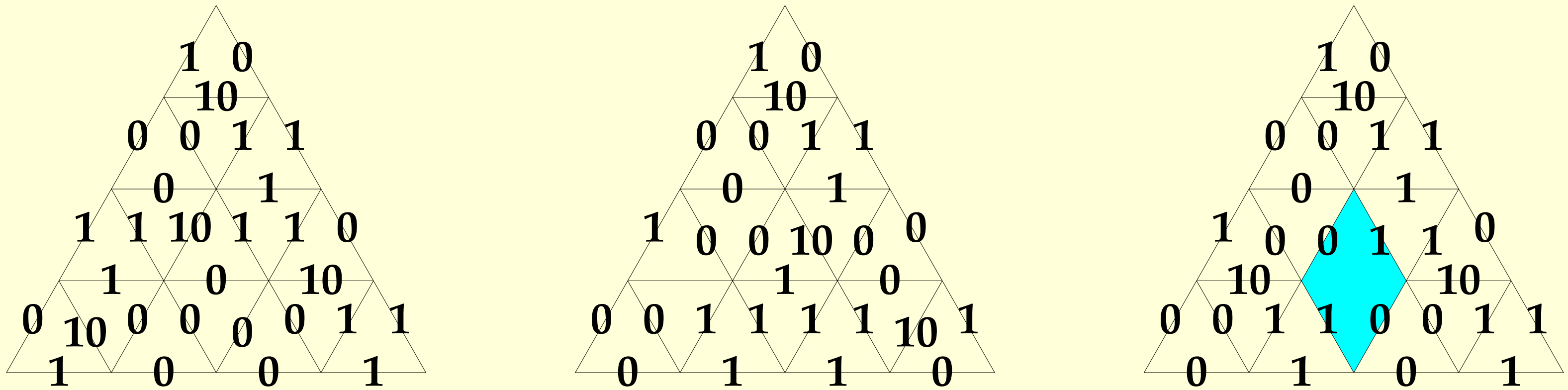
- K_T(2-step flag manifolds) and K(3-step) [K–Paul Zinn-Justin]
- the restriction $H_{T^n}(Gr(k, 2n)) \to H_{T^n}(SpGr(k, 2n))$ [K–ZJ–Iva Halacheva]
- a bijective proof of associativity of the Grassmannian puzzle product, using 3-d puzzle pieces [H–ZJ–Hannah Perry]
- the "separated descents" restriction map, generalizing Kogan's cases
$$K_T(Fl(1, \ldots, k;\ n)) \times K_T(Fl(k+1, \ldots, n;\ n)) \to K_T(Fl(n)) \qquad \text{[K–ZJ]}$$
- the Euler characteristic of the $\bigcap$ of three Bruhat cells [K–ZJ]

Most of these extend to formulæ for pullbacks of *motivic Segre classes*, which naturally live on the cotangent bundle and generalize to K-*theoretic stable classes* on Nakajima quiver varieties. I'll explain the geometry of this extension.

These transparencies are available at `http://math.cornell.edu/~allenk/`

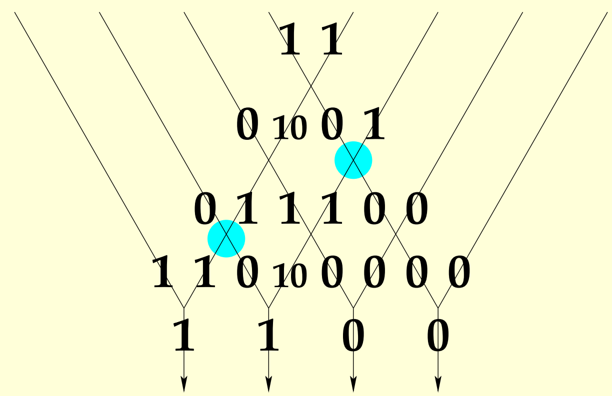
Graph-theoretic duals of equivariant puzzles.

Recall from [K-Tao '03] the **equivariant puzzle rule** for computing the $H_T^* \cong \mathbb{Z}[y_1, \ldots, y_n]$ structure constants of Schubert classes in $Gr(k, \mathbb{C}^n)$:



$$(S_{0101})^2 \quad = \quad S_{1001} \quad + \quad S_{0110} \quad + \quad (y_2 - y_3) S_{0101}$$

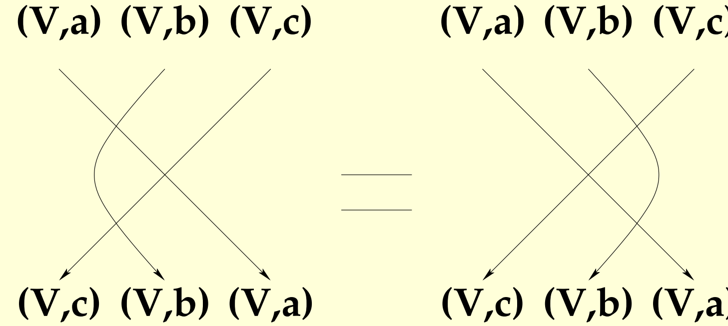
The n Δs on the bottom of a puzzle shape are different from the others: they can't occur in equivariant pieces. Let's pair up the other triangles into vertical rhombi. Now, let's look at the graph-theory dual of an equivariant puzzle, an overlay of n Ys.



This one is worth $(y_1 - y_2)(y_2 - y_4)$:

The Yang-Baxter equation and algebraic sources thereof.

Observation [Zinn-Justin '05].
Rotating the nonrotatable equivariant pieces appropriately (!?), the equivariant puzzle R-matrix satisfies the **Yang-Baxter equation:**



Let $U_q(\mathfrak{g}[z^{\pm}])$ be the **quantized loop algebra**; it comes with many "evaluation representations" $(V_\delta, c \in \mathbb{C}^\times)$ taking $z \mapsto c$ then using the usual irrep V_δ of $\mathfrak{g}$.

Drinfel$'$d and Jimbo observed that $(V_\gamma, a) \otimes (V_\delta, b)$ is irreducible for generic a/b, but $\cong$ to $(V_\delta, b) \otimes (V_\gamma, a)$, and these isos are "R-matrices" (solution to YBE).

Theorem [K-ZJ]. 1. The $d = 1$ puzzle R-matrix, acting on the $\otimes^2$ of the 3-space with basis $\{\vec{0}, \vec{1}, \vec{10}\}$, is a $q \to \infty$ limit of the R-matrix for $\mathfrak{sl}_3 \circlearrowright \mathbb{C}^3 \otimes \mathbb{C}^3$.
2. For the $d = 2$ case and its 8 edge labels $\vec{0}, \vec{1}, \vec{2}, \vec{10}, \vec{20}, \vec{21}, 2(\vec{10}), (\vec{21})0$, we need a $q \to \infty$ limit of the R-matrix for $\mathfrak{d}_4 \circlearrowright spin_+ \otimes spin_-$.
3. For the $d = 3$ case and its 27 edge labels, we need a $q \to \infty$ limit of the R-matrix for $\mathfrak{e}_6 \circlearrowright \mathbb{C}^{27} \otimes \mathbb{C}^{27}$ (which one can find in the 1990s physics literature).
4. For $d = 4$, the same tech gave a **nonpositive** rule based on $\mathfrak{e}_8 \circlearrowright (\mathfrak{e}_8 \oplus \mathbb{C})^{\otimes 2}$.

In each case, the Yang-Baxter equation (and similar "bootstrap" equation to deal with trivalent vertices) is used in a quick proof [K-ZJ '17] of the puzzle rule, and the nonzero matrix entries in the $q \to \infty$ limit tell us the valid puzzle pieces.

Nakajima's geometry of some $U_q(\mathfrak{g}[z^\pm])$ representations.

But why *should* such representations come up in studying $Fl(n_1, n_2, \ldots, n_d; \mathbb{C}^n)$?

Given an oriented graph (Q_0, Q_1), with some vertices declared "gauged" and the others "framed", double it by adding a backwards arrow for every arrow. Attach a vector space W_i to each framed vertex and V_j to each gauged vertex.

Definition. A point in the **quiver variety** $\mathcal{M}(Q_0, Q_1, W, V)$ is a choice of linear transformation for every edge,

- such that $\sum \pm$ (go out) $\circ$ (come back in) is zero at each gauged vertex;
- every $\vec{v}$ in each V_i can leak into some W_j via *some* path;
- all is considered up to $\prod_i GL(V_i)$ change-of-bases at the gauged vertices.

Let $\mathcal{M}(Q_0, Q_1, W) := \coprod_W \mathcal{M}(Q_0, Q_1, W, V)$ be the **quiver scheme**.

Theorem [Nakajima '01]. If Q is ADE, then $U_q(\text{its } \mathfrak{g}[z^\pm]) \circlearrowright K(\mathcal{M}(Q_0, Q_1, W))$.

Main example. $$\mathcal{M}\begin{pmatrix} \boxed{n} & & & & \\ \uparrow & & & & \\ n_d & \leftarrow & n_{d-1} & \leftarrow \ldots \leftarrow & n_1 \end{pmatrix} \cong T^*Fl(n_1, \ldots, n_d;\ \mathbb{C}^n).$$

For this framing the $U_q(\mathfrak{sl}_{d+1}[z^\pm])$-action appears already in [Ginzburg-Vasserot 1993], and the rep is $K(\mathcal{M}(Q_0, Q_1, n\omega_1)) \cong (\mathbb{C}^{d+1})^{\otimes n}$, whose weight multiplicities are $(d+1)$-nomial coefficients.

Some Lagrangian relations of quiver varieties.

On $\mathbb{C}^n \oplus \mathbb{C}^n$ we put a $\mathbb{C}^\times$-action with weights $0, 1$, extending to an action on $\mathcal{M}\begin{pmatrix} \boxed{n+n} & \\ n+k & k \end{pmatrix}$; then $\mathcal{M}\begin{pmatrix} \boxed{n} & \\ k & 0 \end{pmatrix} \times \mathcal{M}\begin{pmatrix} \boxed{n} & \\ n & k \end{pmatrix}$ is a fixed-point component. Let attr be the **(closed!)** attracting set, the Morse/Białynicki-Birula stratum.

Now let $\Phi_N^{-1}(\mathbf{1}) := \{$the composite $(\mathbb{C}^n \oplus 0) \searrow \mathbb{C}^{n+k} \nearrow (0 \oplus \mathbb{C}^n)$ is the identity$\}$. Points (reps) in that set enjoy splittings of $\mathbb{C}^{n+k}$, plus coordinates on the $\mathbb{C}^n$.

Imprecisely stated theorem [K-ZJ]. The Lagrangian relations

$$\mathcal{M}\begin{pmatrix} \boxed{n} & \\ k & 0 \end{pmatrix} \times \mathcal{M}\begin{pmatrix} \boxed{n} & \\ n & k \end{pmatrix} \xrightarrow{\mathrm{attr}} \mathcal{M}\begin{pmatrix} \boxed{n+n} & \\ n+k & k \end{pmatrix} \xrightarrow{\Phi_N^{-1}(\mathbf{1})} \mathcal{M}\begin{pmatrix} & \boxed{n} \\ k & k \end{pmatrix}$$

induce the usual multiplication map on $H^*_{T\times\mathbb{C}^\times}(T^*Gr(k, \mathbb{C}^n))$, up to a scale, and by following the natural (analogues of Schubert) bases (and taking q, or really $\hbar$, to ∞) we recover Grassmannian puzzles. Specifically, the rhombus pieces compute a change-of-basis in $H^*_{T\times\mathbb{C}^\times}$(the middle space).

In the $d = 2, 3, 4$ cases, the quiver is D_4, E_6, E_8 respectively, and the quiver variety used in the middle is not a cotangent bundle.

Z_2 fixed points give the restriction to $SpGr(k, 2n)$.

For a first variant on the quiver varieties above, consider

$$\mathcal{M}\left(\begin{array}{cc}\boxed{N} & \\ j & 0\end{array}\right) \times \mathcal{M}\left(\begin{array}{cc}\boxed{N} & \\ N & k\end{array}\right) \xrightarrow{\text{attr}} \mathcal{M}\left(\begin{array}{cc}\boxed{N+N} & \\ N+j & k\end{array}\right) \xrightarrow{\Phi_N^{-1}(\mathbf{1})} \mathcal{M}\left(\begin{array}{cc} & \boxed{N} \\ j & k\end{array}\right)$$

inducing $H^*_{T\times\mathbb{C}^\times}(T^*Fl(j,k;\mathbb{C}^N)) \to H^*_{T\times\mathbb{C}^\times}(T^*Gr(j,\mathbb{C}^N)) \times H^*_{T\times\mathbb{C}^\times}(T^*Gr(k,\mathbb{C}^N))$.

Theorem [Halacheva-K-ZJ]. Index the Schubert classes on $Fl(j,k;\mathbb{C}^N)$ by strings with content $0^j(10)^{k-j}1^{N-k}$. Then puzzles with Grassmannian puzzle pieces, but allowing $k-j$ 10-labels on the South edge, compute this pullback.

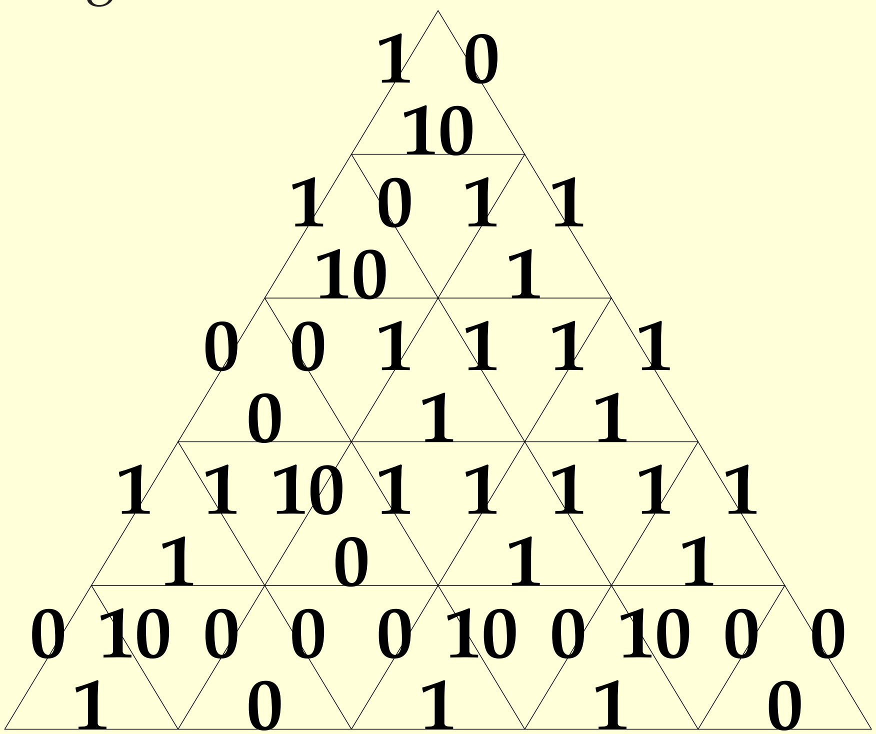
Now take $N=2n$, $j=2n-k$. Then there are compatible Z_2 actions on these spaces with fixed points

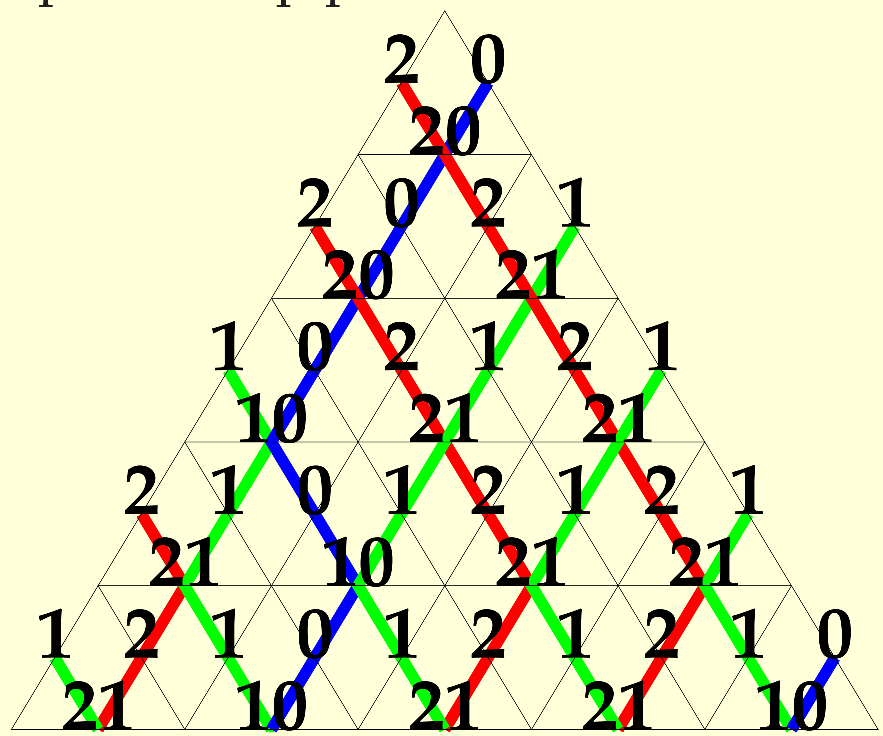
$$T^*Gr(k,\mathbb{C}^{2n}) \xrightarrow{\text{attr}} T^*OGr(2n-k,\mathbb{C}^{4n}) \xrightarrow{\text{attr}} T^*SpGr(k,\mathbb{C}^{2n})$$

Theorem [H-K-ZJ]. Consider puzzles like the above, but "self-dual" in being invariant under left-right flip plus exchange $0 \leftrightarrow 1$. These puzzles compute the equivariant pullback from $Gr(k,\mathbb{C}^{2n})$ to $SpGr(k,\mathbb{C}^{2n})$, extending work of [Pragacz '98] and [Coşkun '14].

A pipe dream picture of puzzles.

In $\mathcal{M}\begin{pmatrix}\boxed{n} & \\ k & 0\end{pmatrix} \times \mathcal{M}\begin{pmatrix}\boxed{n} & \\ n & k\end{pmatrix} \to \mathcal{M}\begin{pmatrix} & \boxed{n} \\ k & k\end{pmatrix}$ the different appearances of $Gr(k, \mathbb{C}^n)$ are best studied from the weights in $\mathbb{C}^3 \otimes \mathbb{C}^3 \to Alt^2\mathbb{C}^3 \cong (\mathbb{C}^3)^*$. This leads to a superior labeling, in which the T-equivariance of that map gives a weight conservation which one can interpret with pipes:





(Alternately one can label the horizontal edges by the missing number $0, 1, 2$ instead of the pairs $1 \wedge 2, 0 \wedge 2, 0 \wedge 1$.)

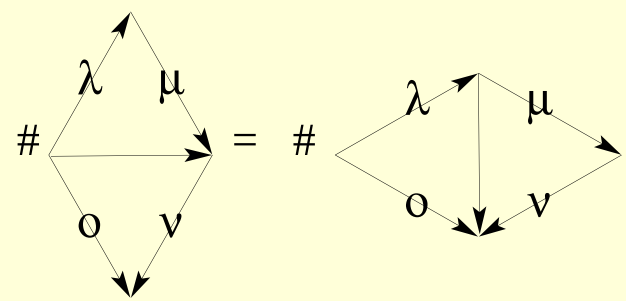
Associativity via 3-d puzzles.

Go beyond $\mathbb{C}^3 \otimes \mathbb{C}^3 \to \mathrm{Alt}^2\mathbb{C}^3 \cong (\mathbb{C}^3)^*$ to $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4 \to \mathrm{Alt}^3\mathbb{C}^4 \cong (\mathbb{C}^4)^*$:

$$\mathcal{M}\begin{pmatrix} \boxed{n} & & \\ k & 0 & 0 \end{pmatrix} \times \mathcal{M}\begin{pmatrix} \boxed{n} & & \\ n & k & 0 \end{pmatrix} \mathcal{M}\begin{pmatrix} \boxed{n} & & \\ n & n & k \end{pmatrix}$$

$$\xrightarrow{\text{attr}} \mathcal{M}\begin{pmatrix} \boxed{n+n+n} & & \\ 2n+k & n+k & k \end{pmatrix} \xrightarrow{\Phi_N^{-1}(\mathbf{1})} \mathcal{M}\begin{pmatrix} & & \boxed{n} \\ k & k & k \end{pmatrix}$$

Associativity says that the coefficients of S_o in $(S_\lambda S_\mu)S_\nu$ and $S_\lambda(S_\mu S_\nu)$ are the same. In puzzle terms, we label the front or back of a tetrahedron with bipuzzles, and should be able to biject them:

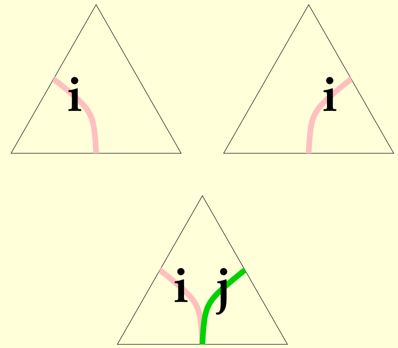


Theorem [Henriques ∼'04]. One can compute $c^{o}_{\lambda\mu\nu}$ using any lattice surface Σ in the tetrahedron with $\partial\Sigma$ this same (λ, μ, ν, o) boundary.
Proof: $\exists$ 3-d puzzle pieces giving correspondences between Σ- and Σ'-puzzles.

His very unpleasant $0, 10, 1$ pieces were lost, but essentially rediscovered by [H-Perry-ZJ] in the A_3 formulation above.

The newest Schubert calculus: separated descents.

Theorem [K-ZJ]. Consider the puzzle pieces at right, and their 180° rotations. Make size n puzzles with $1, \dots, k$ and $n-k$ blanks on NE side, $k+1, \dots, n$ and k blanks on NW side. Then these compute the structure constants of $H^*(Fl(k, \dots, n; \mathbb{C}^n)) \otimes H^*(Fl(1, \dots, k; \mathbb{C}^n)) \to H^*(Fl(\mathbb{C}^n))$, and with two more pieces we get the K_T-version.



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[Kogan '01], the previous state-of-the-art for general $H^*(Fl(\mathbb{C}^n))$ calculations (extended to K-theory in [K-Yong '04]), assumed that one of the two factors was a Grassmannian (and was algorithmic, and nonequivariant).

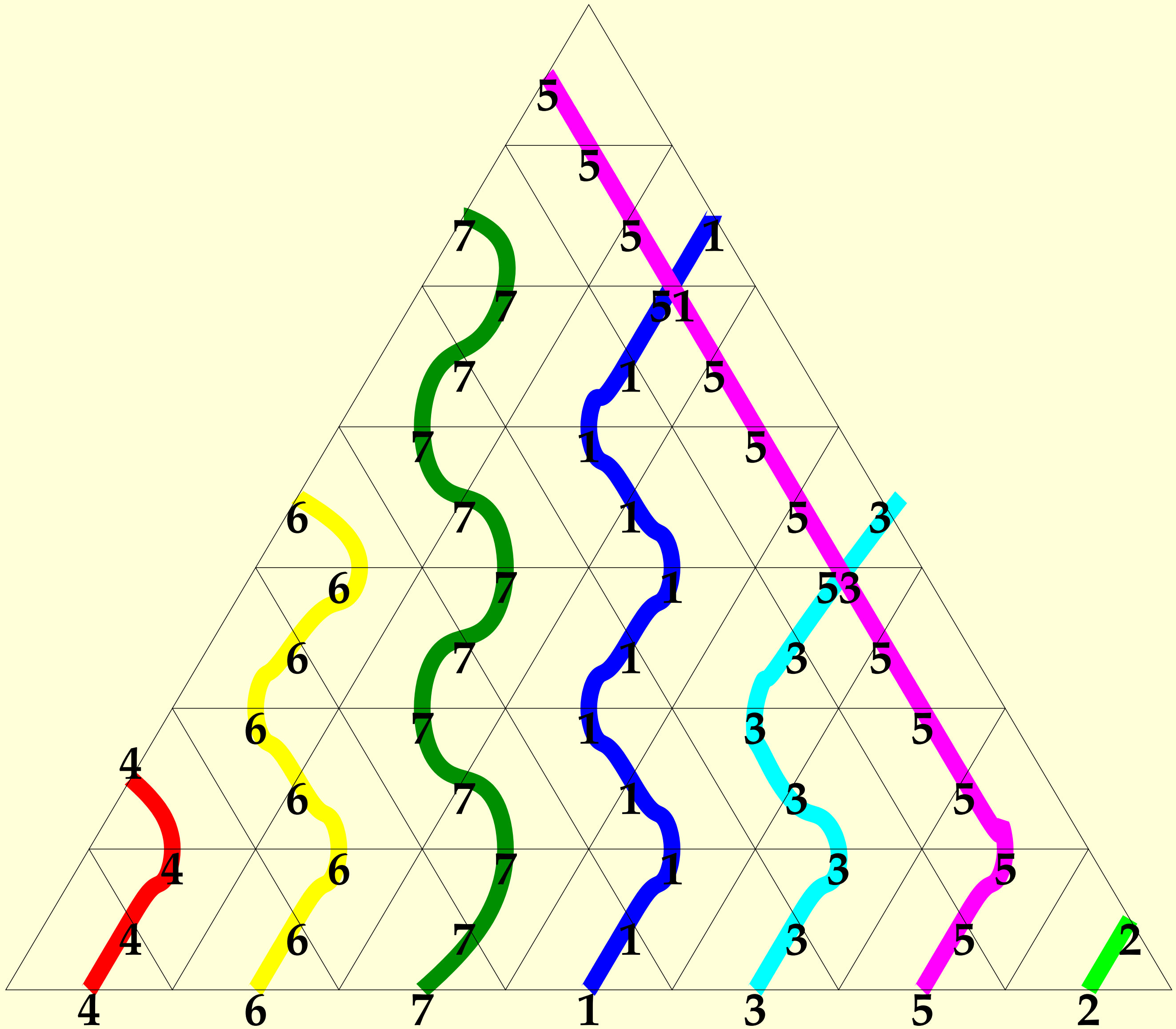
"Proof".

$$\mathcal{M}\left(\begin{matrix} \boxed{n} & & & & & \\ n & n\dots n & k & k-1 & \dots & 1 \end{matrix}\right) \times \mathcal{M}\left(\begin{matrix} \boxed{n} & & & & & \\ n-1 & n-2\dots k & 0 & 0 & \dots & 0 \end{matrix}\right)$$

$$\xleftrightarrow{\mathrm{attr}} \mathcal{M}\left(\begin{matrix} \boxed{n+n} & & & & & & \\ 2n-1 & 2n-2 & \dots & n+k & k & k-1 & \dots & 1 \end{matrix}\right)$$

$$\xleftrightarrow{\Phi_N^{-1}(\mathbf{1})} \mathcal{M}\left(\begin{matrix} & \boxed{n} & & & & & & \\ n-1 & 2n-2 & \dots & n+k & k & k-1 & \dots & 1 \end{matrix}\right) \cong T^*Fl(\mathbb{C}^n)$$

Example. A separated-descents puzzle.



Finite $\hbar$ application: Euler characteristics of triple intersections.

The elements of the natural basis of $H^*_{T\times\mathbb{C}^\times}(T^*GL_n/P)$ arise in three essentially different ways:

- by following $B_- wL/L$ under Grothendieck-Springer's $GL_n/L \rightsquigarrow T^*GL_n/P$
- as characteristic cycles of the $\mathcal{D}_{G/P}$-modules associated to Bruhat cells
- as Chern-Schwartz-MacPherson classes associated to Bruhat cells

The latter's connection to Chern classes and Euler characteristics gives rise to the following theorem, statable without explicit reference to cotangent bundles:

Theorem [K-ZJ]. Take $g, h \in GL_n$ generic, and $M := X^\circ_\lambda \cap (g \cdot X^\circ_\mu) \cap (h \cdot X^\circ_\nu)$. Then $(-1)^{\dim M}\chi_c(M)$ is nonnegative, counted by ordinary puzzles in which one also allows 10-10-10 pieces (both Δs and ∇s).

For single and double intersections these numbers are 1 and 0 (unless $\lambda = \mu^c$).

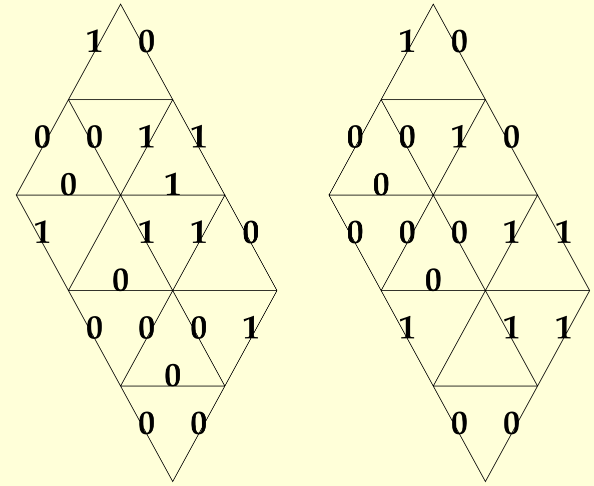
We have similar results for $2, 3, 4$-step (though the 4-step isn't positive), prompting the question:

$$\text{Is } (-1)^{\dim M}\chi_c(M) \geq 0 \text{ for triple intersections } M \text{ inside general } G/P?$$

The puzzle calculation naturally extends to K-theory, where the 10-10-10 pieces are worth q, q^{-1} for Δ, ∇ respectively. Do these (times some power of q) have a point-counting-over-$\mathbb{F}_q$ interpretation?

Other people's results, unrelated (so far) to quiver varieties.

Consider usual Grassmannian puzzle pieces, but in a parallelogram, with boundary strings $\lambda, \alpha, \mu, \beta$ clockwise from NW.
Then it's easy to show that λ, μ have the same content, and likewise α, β.
Call the number of these puzzles $c_{\lambda\alpha\mu\beta}$.



Obviously $c_{\lambda\alpha\mu\beta} = c_{\mu\beta\lambda\alpha}$, by rotating the puzzles 180°. But more is true:

Theorem [P. Anderson]. $c_{\lambda\alpha\mu\beta} = c_{\lambda\beta\mu\alpha}$, as each can be interpreted as the same integral over a *product* of two Grassmannians.

Consider $K_*\Big(Gr(a, a+b) \times Gr(c, c+d) \to Gr(a+c, a+c+b+d)\Big)$, inducing a bigraded ring structure on $\bigoplus_{a,b} K_*(Gr(a, a+b))$.

Theorem [Pylyavskyy-Yang]. This K-homology product can be computed by puzzles with one extra hexagonal piece.

We don't know a Yang-Baxter equation interpretation of this rule. Of course a first step would be an equivariant extension.