What do puzzles really compute?

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IMPANGA 20, July 2021

Abstract
Among other things (these all since 2017),

- $K_T(2$-step flag manifolds) and $K(3$-step) [K–Paul Zinn-Justin]
- the restriction $H_T^n(Gr(k, 2n)) \rightarrow H_T^n(SpGr(k, 2n))$ [K–ZJ–Iva Halacheva]
- a bijective proof of associativity of the Grassmannian puzzle product, using 3-d puzzle pieces [H–ZJ–Hannah Perry]
- the “separated descents” restriction map, generalizing Kogan’s cases $K_T(Fl(1, \ldots, k; n)) \times K_T(Fl(k+1, \ldots, n; n)) \rightarrow K_T(Fl(n))$ [K–ZJ]
- the Euler characteristic of the $\bigcap$ of three Bruhat cells [K–ZJ]

Most of these extend to formulæ for pullbacks of motivic Segre classes, which naturally live on the cotangent bundle and generalize to $K$-theoretic stable classes on Nakajima quiver varieties. I’ll explain the geometry of this extension.

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Graph-theoretic duals of equivariant puzzles.

Recall from [K-Tao ’03] the **equivariant puzzle rule** for computing the $H^*_T \cong \mathbb{Z}[y_1, \ldots, y_n]$ structure constants of Schubert classes in $Gr(k, \mathbb{C}^n)$:

\[
(S_{0101})^2 = S_{1001} + S_{0110} + (y_2 - y_3)S_{0101}
\]

The $n$ $\Delta$s on the bottom of a puzzle shape are different from the others: they can’t occur in equivariant pieces. Let’s pair up the other triangles into vertical rhombi. Now, let’s look at the graph-theory dual of an equivariant puzzle, an overlay of $n$ $Y$s.

This one is worth $(y_1 - y_2)(y_2 - y_4)$:
The Yang-Baxter equation and algebraic sources thereof.

Observation [Zinn-Justin ’05].
Rotating the nonrotatable equivariant pieces appropriately (!?), the equivariant puzzle R-matrix satisfies the **Yang-Baxter equation:***

Let $U_q(g[z^\pm])$ be the **quantized loop algebra**; it comes with many “evaluation representations” $(V_\delta, c \in C^\times)$ taking $z \mapsto c$ then using the usual irrep $V_\delta$ of $g$.

Drinfel’d and Jimbo observed that $(V_\gamma, a) \otimes (V_\delta, b)$ is irreducible for generic $a/b$, but $\cong$ to $(V_\delta, b) \otimes (V_\gamma, a)$, and these isos are “R-matrices” (solution to YBE).

**Theorem [K-ZJ].**
1. The $d = 1$ puzzle R-matrix, acting on the $\otimes^2$ of the 3-space with basis {$\vec{0}, \vec{1}, \vec{10}$}, is a $q \to \infty$ limit of the R-matrix for $\mathfrak{sl}_3 \otimes C^3 \otimes C^3$.
2. For the $d = 2$ case and its 8 edge labels $\vec{0}, \vec{1}, \vec{2}, \vec{10}, \vec{20}, \vec{21}, 2(\vec{10}), (2\vec{1})0$, we need a $q \to \infty$ limit of the R-matrix for $\mathfrak{d}_4 \otimes \text{spin}_+ \otimes \text{spin}_-$.  
3. For the $d = 3$ case and its 27 edge labels, we need a $q \to \infty$ limit of the R-matrix for $\mathfrak{e}_6 \otimes C^{27} \otimes C^{27}$ (which one can find in the 1990s physics literature).  
4. For $d = 4$, the same tech gave a **nonpositive** rule based on $\mathfrak{e}_8 \otimes (\mathfrak{e}_8 \oplus C) \otimes^2$.

In each case, the Yang-Baxter equation (and similar “bootstrap” equation to deal with trivalent vertices) is used in a quick proof [K-ZJ ’17] of the puzzle rule, and the nonzero matrix entries in the $q \to \infty$ limit tell us the valid puzzle pieces.

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Nakajima’s geometry of some $\mathcal{U}_q(\mathfrak{g}[z^\pm])$ representations.

But why should such representations come up in studying $\text{Fl}(n_1, n_2, \ldots, n_d; \mathbb{C}^n)$? Given an oriented graph $(Q_0, Q_1)$, with some vertices declared “gauged” and the others “framed”, double it by adding a backwards arrow for every arrow. Attach a vector space $W_i$ to each framed vertex and $V_j$ to each gauged vertex.

**Definition.** A point in the quiver variety $\mathcal{M}(Q_0, Q_1, W, V)$ is a choice of linear transformation for every edge,

- such that $\sum \pm (\text{go out}) \circ (\text{come back in})$ is zero at each gauged vertex;
- every $\vec{v}$ in each $V_i$ can leak into some $W_j$ via some path;
- all is considered up to $\prod_i \text{GL}(V_i)$ change-of-bases at the gauged vertices.

Let $\mathcal{M}(Q_0, Q_1, W) := \bigsqcup_W \mathcal{M}(Q_0, Q_1, W, V)$ be the quiver scheme.

**Theorem [Nakajima ’01].** If $Q$ is ADE, then $\mathcal{U}_q(\text{its } \mathfrak{g}[z^\pm]) \circ K(\mathcal{M}(Q_0, Q_1, W))$.

**Main example.** $\mathcal{M} \left( \begin{array}{c} n \\ \uparrow \\ n_d \leftarrow n_{d-1} \leftarrow \ldots \leftarrow n_1 \end{array} \right) \cong T^*\text{Fl}(n_1, \ldots, n_d; \mathbb{C}^n)$.

For this framing the $\mathcal{U}_q(\mathfrak{sl}_{d+1}[z^\pm])$-action appears already in [Ginzburg-Vasserot 1993], and the rep is $K(\mathcal{M}(Q_0, Q_1, n\omega_1)) \cong (\mathbb{C}^{d+1})^\otimes n$, whose weight multiplicities are $(d + 1)$-nomial coefficients.
Some Lagrangian relations of quiver varieties.

On $\mathbb{C}^n \oplus \mathbb{C}^n$ we put a $\mathbb{C}^\times$-action with weights 0, 1, extending to an action on $\mathcal{M} \left( \begin{array}{c} n+n \\ n+k \\ 0 \end{array} \right); \text{then } \mathcal{M} \left( \begin{array}{c} n \\ k \\ 0 \end{array} \right) \times \mathcal{M} \left( \begin{array}{c} n \\ n+k \\ k \end{array} \right)$ is a fixed-point component.

Let $\text{attr}$ be the (closed!) attracting set, the Morse/Białynicki-Birula stratum.

Now let $\Phi_N^{-1}(1) := \{ \text{the composite } (\mathbb{C}^n \oplus 0) \rightrightarrows \mathbb{C}^{n+k} \rightrightarrows (0 \oplus \mathbb{C}^n) \text{ is the identity} \}$. Points (reps) in that set enjoy splittings of $\mathbb{C}^{n+k}$, plus coordinates on the $\mathbb{C}^n$.

**Imprecisely stated theorem [K-ZJ].** The Lagrangian relations

$$\mathcal{M} \left( \begin{array}{c} n \\ k \\ 0 \end{array} \right) \times \mathcal{M} \left( \begin{array}{c} n \\ n+k \\ k \end{array} \right) \xrightarrow{\text{attr}} \mathcal{M} \left( \begin{array}{c} n+n \\ n+k \\ k \end{array} \right) \xrightarrow{\Phi_N^{-1}(1)} \mathcal{M} \left( \begin{array}{c} k \\ n+k \\ k \end{array} \right)$$

induce the usual multiplication map on $H^*_{T \times \mathbb{C}^\times}(T^* \text{Gr}(k, \mathbb{C}^n))$, up to a scale, and by following the natural (analogues of Schubert) bases (and taking $q$, or really $\hbar$, to $\infty$) we recover Grassmannian puzzles. Specifically, the rhombus pieces compute a change-of-basis in $H^*_{T \times \mathbb{C}^\times}(\text{the middle space})$.

In the $d = 2, 3, 4$ cases, the quiver is $D_4, E_6, E_8$ respectively, and the quiver variety used in the middle is not a cotangent bundle.

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Z₂ fixed points give the restriction to \( \text{SpGr}(k, 2n) \).

For a first variant on the quiver varieties above, consider

\[
\mathcal{M}\left(\begin{bmatrix} N \\ j \\ 0 \end{bmatrix}\right) \times \mathcal{M}\left(\begin{bmatrix} N \\ N \end{bmatrix}\right) \xrightarrow{\text{attr}} \mathcal{M}\left(\begin{bmatrix} N+N \\ N+j \\ k \end{bmatrix}\right) \xrightarrow{\Phi^{-1}_N(1)} \mathcal{M}\left(\begin{bmatrix} j \\ N \\ k \end{bmatrix}\right)
\]

inducing \( H^*_{\mathbb{T} \times \mathbb{C}}(T^*\text{Fl}(j, k; \mathbb{C}^N)) \to H^*_{\mathbb{T} \times \mathbb{C}}(T^*\text{Gr}(j, \mathbb{C}^N)) \times H^*_{\mathbb{T} \times \mathbb{C}}(T^*\text{Gr}(k, \mathbb{C}^N)) \).

**Theorem [Halacheva-K-ZJ].** Index the Schubert classes on \( \text{Fl}(j, k; \mathbb{C}^N) \) by strings with content \( 0^j(10)^{k-j}1^{N-k} \). Then puzzles with Grassmannian puzzle pieces, but allowing \( k-j \) 10-labels on the South edge, compute this pullback.

Now take \( N = 2n, j = 2n - k \). Then there are compatible \( Z_2 \) actions on these spaces with fixed points

\[
T^*\text{Gr}(k, \mathbb{C}^{2n}) \xrightarrow{\text{attr}} T^*\text{OGr}(2n - k, \mathbb{C}^{4n}) \xrightarrow{\text{attr}} T^*\text{SpGr}(k, \mathbb{C}^{2n})
\]

**Theorem [H-K-ZJ].** Consider puzzles like the above, but “self-dual” in being invariant under left-right flip plus exchange \( 0 \leftrightarrow 1 \). These puzzles compute the equivariant pullback from \( \text{Gr}(k, \mathbb{C}^{2n}) \) to \( \text{SpGr}(k, \mathbb{C}^{2n}) \), extending work of [Pragacz ’98] and [Coşkun ’14].

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A pipe dream picture of puzzles.

In $M\left(\begin{array}{c}n \\ k \end{array}\right) \times M\left(\begin{array}{c}n \\ n \end{array}\right) \to M\left(\begin{array}{c}k \\ n \end{array}\right)$ the different appearances of $\text{Gr}(k, \mathbb{C}^n)$ are best studied from the weights in $\mathbb{C}^3 \otimes \mathbb{C}^3 \to \text{Alt}^2 \mathbb{C}^3 \cong (\mathbb{C}^3)^*$. This leads to a superior labeling, in which the $T$-equivariance of that map gives a weight conservation which one can interpret with pipes:

(Alternately one can label the horizontal edges by the missing number 0, 1, 2 instead of the pairs $1 \wedge 2$, $0 \wedge 2$, $0 \wedge 1$.)

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Associativity via 3-d puzzles.

Go beyond $\mathbb{C}^3 \otimes \mathbb{C}^3 \to \text{Alt}^2 \mathbb{C}^3 \cong (\mathbb{C}^3)^*$ to $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4 \to \text{Alt}^3 \mathbb{C}^4 \cong (\mathbb{C}^4)^*$:

$$
\mathcal{M}
\begin{pmatrix}
 n & 0 & 0 \\
 k & 0 & 0 \\
\end{pmatrix}
\times
\mathcal{M}
\begin{pmatrix}
 n & k & 0 \\
 n & k & 0 \\
\end{pmatrix}
\mathcal{M}
\begin{pmatrix}
 n & n & k \\
 n & n & k \\
\end{pmatrix}
$$

Associativity says that the coefficients of $S_o$ in $(S_\lambda S_\mu)S_\nu$ and $S_\lambda(S_\mu S_\nu)$ are the same. In puzzle terms, we label the front or back of a tetrahedron with bipuzzles, and should be able to biject them:

Theorem [Henriques ’04]. One can compute $c_{\lambda\mu\nu}^o$ using any lattice surface $\Sigma$ in the tetrahedron with $\partial \Sigma$ this same $(\lambda, \mu, \nu, o)$ boundary.
Proof: $\exists$ 3-d puzzle pieces giving correspondences between $\Sigma$- and $\Sigma'$-puzzles.
His very unpleasant 0, 10, 1 pieces were lost, but essentially rediscovered by [H-Perry-ZJ] in the $A_3$ formulation above.

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The newest Schubert calculus: separated descents.

**Theorem [K-ZJ].** Consider the puzzle pieces at right, and their $180^\circ$ rotations. Make size $n$ puzzles with $1, \ldots, k$ and $n - k$ blanks on NE side, $k + 1, \ldots, n$ and $k$ blanks on NW side. Then these compute the structure constants of $H^*(\text{Fl}(k, \ldots, n; \mathbb{C}^n)) \otimes H^*(\text{Fl}(1, \ldots, k; \mathbb{C}^n)) \to H^*(\text{Fl}(\mathbb{C}^n))$, and with two more pieces we get the $K_T$-version.

[Kogan '01], the previous state-of-the-art for general $H^*(\text{Fl}(\mathbb{C}^n))$ calculations (extended to $K$-theory in [K-Yong '04]), assumed that one of the two factors was a Grassmannian (and was algorithmic, and nonequivariant).

"Proof".

\[
\begin{align*}
\mathcal{M} & \left( \begin{array}{cccc}
 n & n & \ldots & n \ 
n & k & k - 1 & \ldots & 1 \end{array} \right) \times \mathcal{M} \left( \begin{array}{cccc}
 n & n - 1 & \ldots & 0 \ 
n & n - 2 & \ldots & 0 \end{array} \right) \\
\text{attr} \mathcal{M} & \left( \begin{array}{cccc}
 n + n & 2n - 1 & 2n - 2 & \ldots & n + k \ 
 2n - 1 & 2n - 2 & \ldots & n + k \ 
 & n - 1 & 2n - 2 & \ldots & n + k \ 
 & & n - 1 & 2n - 2 & \ldots & n + k \end{array} \right) \sim T^*\text{Fl}(\mathbb{C}^n)
\end{align*}
\]

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Example. A separated-descents puzzle.
Finite $\hbar$ application: Euler characteristics of triple intersections.

The elements of the natural basis of $H^*_{T \times \mathbb{C}^\times}(T^*GL_n/P)$ arise in three essentially different ways:

- by following $B \cdot wL/L$ under Grothendieck-Springer’s $GL_n/L \leadsto T^*GL_n/P$
- as characteristic cycles of the $\mathcal{D}_{G/P}$-modules associated to Bruhat cells
- as Chern-Schwartz-MacPherson classes associated to Bruhat cells

The latter’s connection to Chern classes and Euler characteristics gives rise to the following theorem, statable without explicit reference to cotangent bundles:

**Theorem [K-ZJ].** Take $g, h \in GL_n$ generic, and $M := X_\lambda^o \cap (g \cdot X_\mu^o) \cap (h \cdot X_\nu^o)$. Then $(-1)^{\dim M} \chi_c(M)$ is nonnegative, counted by ordinary puzzles in which one also allows 10-10-10 pieces (both $\Delta$s and $\nabla$s).

For single and double intersections these numbers are 1 and 0 (unless $\lambda = \mu^c$).

We have similar results for 2, 3, 4-step (though the 4-step isn’t positive), prompting the question:

Is $(-1)^{\dim M} \chi_c(M) \geq 0$ for triple intersections $M$ inside general $G/P$?

The puzzle calculation naturally extends to $K$-theory, where the 10-10-10 pieces are worth $q, q^{-1}$ for $\Delta, \nabla$ respectively. Do these (times some power of $q$) have a point-counting-over-$\mathbb{F}_q$ interpretation?

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Other people’s results, unrelated (so far) to quiver varieties.

Consider usual Grassmannian puzzle pieces, but in a parallelogram, with boundary strings \( \lambda, \alpha, \mu, \beta \) clockwise from NW. Then it’s easy to show that \( \lambda, \mu \) have the same content, and likewise \( \alpha, \beta \). Call the number of these puzzles \( c_{\lambda\alpha\mu\beta} \).

Obviously \( c_{\lambda\alpha\mu\beta} = c_{\mu\beta\lambda\alpha} \), by rotating the puzzles \( 180^\circ \). But more is true:

**Theorem [P. Anderson].** \( c_{\lambda\alpha\mu\beta} = c_{\lambda\beta\mu\alpha} \), as each can be interpreted as the same integral over a *product* of two Grassmannians.

Consider \( K_* \left( \text{Gr}(a, a + b) \times \text{Gr}(c, c + d) \rightarrow \text{Gr}(a + c, a + c + b + d) \right) \), inducing a bigraded ring structure on \( \bigoplus_{a,b} K_* \left( \text{Gr}(a, a + b) \right) \).

**Theorem [Pylyavskyy-Yang].** This \( K \)-homology product can be computed by puzzles with one extra hexagonal piece.

We don’t know a Yang-Baxter equation interpretation of this rule. Of course a first step would be an equivariant extension.

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